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Applications of Krasnoselskii-Dhage Type Fixed point Theorems to Fractional Hybrid Differential Equations

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Dedication

Yousra

Praise be to God who made beginnings easy, completed the endings, and allowed us to reach our goals. Praise be to God—no effort is accomplished except by His help, and no journey is completed except by His grace. Praise be to God for granting me this knowledge and aiding me in completing it. All praise, love, and gratitude to Him. With all my love, I dedicate the fruit of my graduation to my strong self, who endured every stumble despite the hardships

To the one whose forehead was crowned with the sweat of hard work, who taught me that success comes only through patience and determination. To the one whose name is forever linked with mine. To the one who was my source of support and giving. To my first supporter, my backbone, and my strength after God. To my pride and my honor—my dear father, may God have mercy on him.

To my guardian angel,To the one beneath whose feet Paradise lies, To the one who lit my path during the darkest nights, To the one whose prayers were the secret behind my success— My beloved mother.

To my steady support and the safety of my days, To those who strengthened me and held me up, To the best and dearest part of my life—My beloved brothers and sisters.

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Hoping from Allah, the Almighty, that He benefits me with what He has taught me, teaches me what I do not know, and makes it a proof for me, not against me.

Ouissal

It is most fitting to begin with praise to Allah, the Almighty, the Exalted. The best expression of gratitude before addressing people is to offer thanks to the Lord of all creation. We thank Him for His countless blessings and for His generous support. O Allah, to You belongs all praise — praise that has no beginning nor end. May peace and blessings be upon the noblest of humankind, the intercessor of the Ummah, Muhammad ibn Abdullah (peace and blessings be upon him), and upon his family and companions in abundance.

To my dear parents, who have always been a constant source of inspiration and support. To those who instilled in me a love for knowledge and learning, who never hesitated to offer guidance and wise counsel, and who stood by me through every step of my academic and professional journey — may God bless and protect you always .

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The fruit of our patience and struggle was inspired by the core of our cause, by the soldiers of our Al-Aqsa, and by the school of steadfastness in Gaza, Palestine, voiced by the one behind the famous keffiyeh.
And you will have the clear victory.

Contents

The Primary Notations Employed

We will use the following notations throughout this work:

Sets :

$\mathbb{R}$: The set of real numbers.

$\mathbb{R}_+$: The set of positive real numbers.

$\mathbb{N}$: The set of natural numbers.

$\mathbb{C}$: The set of complex numbers.

$C^0[a,b] = C[a,b]$: The space of continuous functions on $[a,b]$ with real values.

$AC^n[a,b] = \{g : [a,b] \to \mathbb{C} \quad \text{and} \quad g^{(n-1)} \in AC([a,b])\}$

$\mathcal{C}(E, \Lambda)$: The vector space of all continuous functions from E to Λ.

$\mathbb{L}^P([a,b])$: The space of measurable functions u on $[a,b]$ satisfying $(\int_a^b |u(t)|^p dt < \infty)$.

$B(x_0, r_0)$: The open ball centered at x_0 with radius r_0 .

$B_c(x_0, r_0)$: The closed ball centered at x_0 with radius r_0 .

$S(x_0, r_0)$: The sphere centered at x_0 with radius r_0 .

$\delta(F)$: The diameter of the set F .

$co(A)$: The intersection of all convex sets that contain A .

D : $\{,\}$ Set of D-type functions .

L : $\{,\}$ Set of L-type functions .

$HFPTs$: hybrid fixed point theorems .

Functions

$\Gamma(s)$: The Gamma function

$B(p,q)$: The Beta function.

$\mathcal{L}f(t)$: The Laplace transform of the function f.

Operators integrals and derivatives

I^{α}_{a+}: The Riemann-Liouville fractional integral of order α.

D^{α}_{a+}: The Riemann-Liouville fractional derivative of order α

R-L: Riemann-Liouville.

Introduction

Fixed point theory is fundamental to nonlinear analysis, as it provides the essential tools for establishing existence and uniqueness theorems in a wide variety of nonlinear problems. Let E be a set and $T : E \to E$ a mapping. A solution for the equation

$T(x) = x$ is called a fixed point of T. The origin of fixed point theory constitutes a significant branch of nonlinear functional analysis, dating back to the latter part of the 19th century, and relies largely on the method of successive approximations for proving the existence and uniqueness of solutions. Most natural phenomena in physics, chemistry, economics, biology, and mechanics exhibit nonlinear behavior, and such problems are often mathematically formulated in terms of nonlinear differential equations.

Fixed point theorems serve as fundamental mathematical tools to demonstrate the existence of solutions to various types of equations. Among these theorems are those of Banach, Brouwer, Schauder, and Krasnoselskii-Dhage.

In this thesis, we apply the Krasnoselskii-Dhage fixed point theorem, a significant result in fixed point theory, which is based on the Banach and Schauder fixed point theorems. The contraction mapping theorem, proven by Banach in 1922, states that a contraction on a complete metric space.

In 1955, for the first time, Krasnoselskii developed his own fixed point theorem.

He combined the results of Banach and Schauder to formulate a theorem stating that, in a convex and compact set, any operator that can be written as the sum of two mappings-one being a contraction and the other compact and continuous-admits a fixed point. This theorem is highly useful in solving nonlinear differential equations, integral equations, and integro-differential equations.

This work is divided into three chapters: **The first chapter** is devoted to some

definitions and preliminaries results that will be used throughout the thesis.

In the second chapter, we present several fixed point theorems, particularly

Banach's and Schauder's fixed point theorems, and study the Krasnoselskii-Dhage fixed point theorem in detail.

In the third chapter, we propose to study the existence of solutions for applications of Krasnoselskii-Dhage type Fixed -point theorems to fractional hybrid differential equations.

$$\begin{cases} D^{\alpha}\left[x(t) - f(t, x(t))\right] = g(t, x(t)) \quad \text{a.e} \quad t \in J \\ x(t_0) = x_0, \end{cases}$$

where $f, g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

We consider the fractional hybrid differential equation that involving the Riemann-Liouville differential and integral operators of orders $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^{\alpha}\left[x(t) - f(t, x(t))\right] = g(t, x(t), I^{\beta}(x(t))) \quad \text{a.e} \quad t \in J, \beta > 0 \\ x(t_0) = x_0, \end{cases}$$

where $J = [t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for a fixed t_0, $a \in \mathbb{R}$ with $a > 0$, and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

Chapter 1

Preliminary

In this chapter, we review the definitions, concepts, and theories that are essential and form a reference framework for the rest of this study.

1.1 Topological space

Definition 1.1.1. [**20**]

Let E be a non-empty set. A topology τ on E is a subset of $P(E)$ that satisfies the following conditions :

1. $\phi, E \in \tau$.
2. If $w_1, w_2, ..., w_n \in \tau$ for any $n \in \mathbb{N}$, then their intersection $\bigcap_{i=1}^{n} w_i \in \tau$.
3. $\forall (w_i)_i \in \tau, \bigcup_{i \in I} w_i \in \tau$.

The paire (E, τ) is called a topological space, and the elements of τ are referred to as open sets.

Definition 1.1.2. [**20**] Let (E, τ) be a topological space. A subset $U \subset E$ is called open if its complement U^c is closed, meaning that $U^c \in \tau$.

Definition 1.1.3. [**17**] Let (E, τ) be a topological space and let $x \in E$.
A neighborhood V of x is any subset contains an open $U \in \tau$ such that
$x \in U \subset V$.
The collection of all neighborhoods of x is denoted by $R(x)$.

Definitions : [**20**]

Let (E, τ) be a topological space, and let $B \subseteq E$. Let $x \in E$.

1. **Closure of B :**

   The closure of B, denoted by $\overline{B}$, is the smallest (with respect to the subset relation " $\subseteq$ ") closed set containing B. Equivalently it is the intersection of all closed subsets that contain B.

2. **Interior of B :**

   The interior of B, denoted by B°, is the largest (with respect to the subset relation " $\subseteq$ ") open set contained in B. Equivalently, it is the union of all open sets contained in B.

3. **Exterior of B :**

   The exterior of B, denoted by $\mathbf{ext}\,(\mathbf{B})$, is the interior of the complement of B, i.e , $\mathbf{ext}\,(\mathbf{B}) = (\mathbf{B^c})^{\circ}$.

4. **Limit point of B :**

   A point x is called a limit point of B if :

   $\forall\, U \in R\,(x) : U \cap B - \{x\} \neq \phi$.

   The collection of limit point of B is denoted by B'.

5. **Isolated point of B :**

   If a point x is not a limit point, meaning there exists a neighborhood V of x such that $U \cap B = \{x\}$, then x is called an isolated point.

Proposition 1.1.4. [**20**] Let (E, τ) be a topological space, and let $B \subseteq E$. Then :

1. B is closed if and only if $B = \overline{B}$.

2. B is open if and only if $B = B^{\circ}$.

3. $x \in \overline{B}$ if and only if for every neighborhood $U \in R\,(x)\,,\, U \cap B \neq \phi$.

4. $\overline{B} = B' \cup B$, where B' is the set of limit point of B.

Definition 1.1.5. [**20**] Let (E, τ) be a topological space, and let $B \subseteq E$. We say that B is dense in E if $\overline{B} = E$.

Remark 1.1.6. [**20**] We may also use the term ”everywhere dense” instead of ”dense.”

Definition 1.1.7. [**20**] A topological space (E, τ) is said to be separable if it contains a countable and everywhere dense subset.

Definition 1.1.8. [**20**] Let E be a topological space. A basis (or base)for E is a collection D of open subset of E such that every open set U in E can be expressed as the union of elements of D.

Proposition 1.1.9. [**20**] A collection D of open subsets of a topological space E is a basis for E if and only if for every point x in an open set U, there exists some $G \in D$ such that $x \in G \subseteq U$.

The following is a criterion for a collection of sets to form a basis.

1.2 Metric Space

Definition 1.2.1. [**20**] Let E be a non-empty set. A distance on E is a function $d : E \times E \longrightarrow \mathbb{R}^+$ satisfying the following properties :

1. $d(x, y) = 0$ if and only if $x = y$.
2. $\forall x, y \in E: \; d(x, y) = d(x, y)$ (Symmetry) .
3. $\forall x, y, z \in E: \; d(x, z) \leqslant d(x, y) + d(y, z)$ (Triangle Inequality).

The pair (E, d) is called a metric space .

Example 1.2.2.

$$d(x, y) = |\frac{1}{x} - \frac{1}{y}|, \quad x, y \in E = \mathbb{R}^*$$

1. $d(x, y) = 0 \Leftrightarrow |\frac{1}{x} - \frac{1}{y}| = 0 \Leftrightarrow \frac{1}{x} = \frac{1}{y} \Leftrightarrow x = y$.
2. $d(x, y) = |\frac{1}{x} - \frac{1}{y}| = |(-1)(\frac{1}{y} - \frac{1}{x})| = |-1||\frac{1}{y} - \frac{1}{x}| = d(y, x)$.
3. $d(x, z) = |\frac{1}{x} - \frac{1}{z}| = |\frac{1}{x} - \frac{1}{y} + \frac{1}{y} - \frac{1}{z}| \leq |\frac{1}{x} - \frac{1}{y}| + |\frac{1}{y} - \frac{1}{z}|$.

Then, $\quad d(x,z) \leq d(x,y) + d(y,z)$.

Proposition 1.2.3. [**20**] (Second Triangle Inequality) Let (E,d) be a metric space. Then, for all $x,y,z \in E$

$$\mid d\,(x,z) - d\,(y,z) \mid \leqslant d\,(x,y)\,.$$

By repeatedly applying the triangle inequality, we can derive a generalized triangle inequality.

Proposition 1.2.4. [**23**] Two metrics d_1 and d_2 on the same set E are said to be equivalent if there existe constants k_1, k_2 such that

$$\forall x,y \in E, \quad k_1 d(x,y) \leq \mid d'(x,y) \mid \leq k_2 d(x,y).$$

Definition 1.2.5. [**20**] Let (E,d) be a metric space. A sequence (x_n) in (E,d) is bounded if

$$\exists M > 0, \quad \exists a \in E, \quad \forall n \in \mathbb{N} : \; d\,(x_n, a) < M.$$

Definition 1.2.6. [**20**] Let (E,d) be a metric space.

An open ball centered at $x_0 \in E$ with radius $r_0 > 0$, denoted by $B\,(x_0, r_0)$, is defined as :

$$B\,(x_0, r_0) = \{x \in E : d\,(x, x_0) < r_0\}.$$

A closed ball centered at $x_0 \in E$ with radius $r_0 > 0$, denoted by $B_c\,(x_0, r_0)$, is defined as :

$$B_c\,(x_0, r_0) = \{x \in E : d\,(x, x_0) \leqslant r_0\}.$$

A sphere centered at $x_0 \in E$ with radius $r_0 > 0$, denoted by $S\,(x_0, r_0)$, is given by :

$$S\,(x_0, r_0) = \{x \in E : d\,(x, x_0) = r_0\}.$$

Definition 1.2.7. [**20**] Let (E, d) be a metric space and let $F \subseteq E$, define :

$$d(F) = \sup_{x,y \in A} d(x, y).$$

Then d(F) is called the diameter of F.

Definition 1.2.8. [**20**] Let (E, d) be a metric space , and let $B \subseteq E$ be a non-empty set. The distance between a point $x \in E$ and B , denoted by $d(x, B)$, is defined as :

$$d(x, B) = \inf_{t \in B} d(x, t).$$

Definition 1.2.9. [**20**](Convergence).
Let (E, d) be a metric space . A sequence (x_n) in E converges to $x \in E$ if :

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \in \mathbb{N} : \; n \geq N \Longrightarrow d(x_n, x) < \epsilon.$$

Definition 1.2.10. [**13**] (Suite de Cauchy).
Let (E, d) be a metric space , and let (x_n) ba a sequence in E. We say that (x_n) is cauchy if :

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N}, \quad \forall n, m \in \mathbb{N} : \; n, m \geq N \Longrightarrow d(x_n, x_m) < \epsilon.$$

Proposition 1.2.11. [**13**]
A convergent sequence in a metric space is cauchy.

Definition 1.2.12. [**20**] (Complete metric space).
A metric space (E, d) is said to be complete if every cauchy sequence in (E, d) converges in (E, d).

Definition 1.2.13. [**20**] (Continuity in Metric Space).

Let (E, d) and (E', d') be two metric spaces, and let $g : (E, d) \to (E', d')$ be a function. We say that g is continuous at $a \in E$ if :

$$\forall \epsilon > 0, \quad \exists \alpha > 0, \quad \forall x \in E : \; d(x, a) < \alpha \Longrightarrow d'(g(x), g(a)) < \epsilon.$$

We say that g is continuous on E if it is continuous at each point $a \in E$.

Definition 1.2.14. [**20**] Let (E, d) be a metric space.
A function $h : (E, d) \to (E, d)$ is called a contraction if :

$$\exists K : 0 < K < 1 : d(h(x), h(y)) \leqslant Kd(x, y), \forall x, y \in E.$$

Remark 1.2.15. If K is positive but not necessarily smaller one , then we may say that the function satisfies the Lipschitz condition.

Definition 1.2.16. [**20**] Let (E, d) and (E', d') be two metric spaces, and let $Z : (E, d) \to (E', d)$ be a function. We say that Z is uniformly continuous if :

$$\forall \epsilon > 0 \ , \ \exists \delta > 0 \ \forall x, x' \in E : d(x, x') < \delta \Rightarrow d'(Z(x), Z(x')) < \epsilon.$$

Definition 1.2.17. [**20**] Let (E, τ) be a topological space. We say that τ is metrizable if there exists a metric d on E that induces the topology τ.

Proposition 1.2.18. [**20**] Let (E, d) be a metric space.
If there exists an uncountable subset $B \subseteq E$ and some $r > 0$ such that for all $x, y \in B$ with $x \neq y$, the distance $d\,(x, y) > r$, then E is not separable.

1.3 Banach Space

Definition 1.3.1. [**21**] (Norm) . A norm on a set E is a continuous function satisfying the following properties :

1. $\forall x \in E, \quad \| x \| \geq 0, \quad \| x \| = 0$ if and only if $x = 0$. (Separation)

2. $\forall \lambda \in \mathbb{K},\ x \in E, \quad \| \lambda x \| = | \lambda | \| x \|$, where $| \lambda |$ denotes the absolute value if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. (Homogeneity)

3. $\forall x, y \in E, \quad \| x + y \| \leqslant \| x \| + \| y \|$. (Triangle Inequality)

Example 1.3.2.
On $E = \mathbb{R}^N$, the functions

$$\| x \|_p = \left(\sum_{i=1}^{N} | x_i |^p \right)^{\frac{1}{p}}, \quad \| x \|_\infty = \max_{1 \leq i \leq N} |x_i|.$$

are norms.

Definition 1.3.3. [**21**] A normed space $(E, \| \, . \, \|)$ is a space E over the filed $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ equipped with a norm .

Definition 1.3.4. [**13**] A Banach space is a complete normed vector space.

Definition 1.3.5. [**21**] A subset C of normed $(E, \| \, . \, \|)$ is said to be bounded if there exists a constant $p > 0$ such that $\| x \| \leqslant p$ for all $x \in C$.

A function f is called bounded if its image is a bounded subset of E .

1.3.1 Typical Examples Of Normed Vector Spaces

In the following, $\Lambda = \mathbb{R}$ or $\mathbb{C}$.

1) **The space** ℓ^α [**13**]

   For $\alpha \geq 1$, we consider

$$\ell^\alpha = \{x = (x_i)_i \in \Lambda^{\mathbb{N}} \text{ telle que} \sum_{i=0}^{\infty} | x_i^\alpha | < \infty\}$$

   For all $\alpha \geq 1$, ℓ^α equipped with the norm

$$x \to \| x \| = (\sum_{i=0}^{\infty} | x_i^\alpha |)^{\frac{1}{\alpha}}$$

   Is a Banach space.

2) **The space** $B(E, \Lambda)$ [**13**]

   Let E be any set

   The vector space $B(E, \Lambda)$ of bounded functions $g : E \to \Lambda$, equipped with the norm

$$g \to \sup_{x \in E} | g(x) |$$

is a Banach space.

3) **The space $\mathcal{C}(E, \Lambda)$**

Let E be a topological space, and $\mathcal{C}(E, \Lambda)$ denote the vector space of all continuous functions from E to Λ.

Corollary 1.3.6. [**13**] The space $\mathcal{C}(K, \Lambda)$ of continuous functions from a compact topological space K into (Λ), equipped with the norm

$$g \to \sup_{x \in E} \mid g(x) \mid$$

It is a Banach space.

1.4 Compact Space

Definition 1.4.1. [**15**] In a metric space (E, d), we say that E is compact if every sequence of points in E has a convergent subsequence.

Definition 1.4.2. : [**15**] A subset M of a metric space (E, d) is called compact if every sequence $(x_n)_{n\in\mathbb{N}}$ in M has a subsequence that converges to a limit within M.

Example : Every closed and bounded subset of $\mathbb{R}^n$ is compact.

Theorem 1.4.3. [**20**] (Heine-Borel) : Every closed and bounded interval $[a, b]$ in the usual topology of $\mathbb{R}$ is compact .

Proposition 1.4.4. [**20**] Let (E, d) be a metric space , and let $A \subseteq E$ be compact . Then A is bounded .

Definition 1.4.5. [**20**] Let E be a topological space , and let B be a subspace of E . We say that B is relatively compact if its closer $\overline{B}$ is compact .

Theorem 1.4.6. [**20**] Let $g : E \to E'$ be a continuous map between two topological spaces . If E is compact , then its image $g(E)$ is also compact .

Corollary 1.4.7. [**20**] Let (E, τ) be a compact topological space , and let (E', d) be a metric space . If $g : E \to E'$ is continuous , then g is bounded .

The Arzela-Ascoli Theorem 1.4.8.

Let T be an interval in $\mathbb{R}$ and let g_n be a sequence of functions with $g_n : T \to \mathbb{R}^p$. Let $|\cdot|$ be an arbitrary norm in $\mathbb{R}^p$.

Definition 1.4.9. [**7**] $\{t_1, t_2\}$ is uniformly bounded on T if there exists a $M > 0$ such that $| g_n(t) | \leq M$ for all n and $t \in T$.

Definition 1.4.10 [**7**] The sequence g_n is called equicontinuous if , for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $t_1, t_2 \in [a, b]$ with $| t_1 - t_2 | < \delta$, we have $| g_n(t_1) - g_n(t_2) | < \epsilon$ for all n.

Theorem 1.4.11.(Ascoli-Arzela)[**5**] If g_n is a uniformly bounded and equicontinuous sequence of real function on an interval $[a, b]$, then there exists a subsequence that converges uniformly on $[a, b]$ to a continuous function .

1.5 Lebesgue Spaces

Definition 1.5.1. [**16**] The Lebesgue space $\mathbb{L}^P(\Omega)$, $1 \leqslant p \leqslant \infty$, consists of functions $W : \Omega \to \mathbb{R}$ with a finite $\| W \|_{\mathbb{L}^p(\Omega)}$, where

$$\| W \|_{\mathbb{L}^p(\Omega)} = \begin{cases} \left(\int\limits_{\Omega} | W |^p \ dx \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \text{ess } \sup | \mathrm{W} |_{\Omega}, & \text{if } p = \infty \end{cases}$$

A function $W : \Omega \to \mathbb{R}$ is said to be in $\mathbb{L}^p_{loc}(\Omega)$, if for any compact subset $\omega \subset (\Omega)$, there holds $W \in \mathbb{L}^p(\omega)$.

Theorem 1.5.2. [**3**] (Lebesgues Dominated Convergence Theorem)
Let (g_n) be a sequence of functions in $\mathbb{L}^1$. Assume that

1. $g_n \to g(x)$ a.e on Ω.
2. There exists a function $f \in \mathbb{L}^1$ such that for each n , $| g_n(x) | \leqslant f(x)$ a.e on Ω.

   Then $g \in \mathbb{L}^1(\Omega)$ and $\| g_n - g \|_{\mathbb{L}^1} \to 0$.

Definition 1.5.3. [**2**] Let $\mathbb{R}$ be the real line , and let $T = [0, D)$ be a bounded interval in $\mathbb{R}$ for some $D \in \mathbb{R}$. Let $C(T \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $g : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and let $\mathcal{C}(T \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ denote the class of functions $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

1. The map $s \mapsto f(s, x, y)$ is measurable for each $x, y \in \mathbb{R}$.
2. The map $x \mapsto f(s, x, y)$ is continuous for each $x \in \mathbb{R}$.
3. The map $x \mapsto f(s, x, y)$ is continuous for each $y \in \mathbb{R}$

The class $\mathcal{C}(T \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of function on $T \times \mathbb{R} \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on T.

1.6 Spaces of continuous and absolutely continuous functions

Definition 1.6.1. [**17**] Let $\Omega = [a, b]$, where $(-\infty < a < b < \infty)$, and let $n \in \mathbb{N}$. We denote by $C^n[a, b]$ the space of functions g that are n times continuously differentiable on $[a, b]$, equipped with the norm :

$$\| g \|_{C^n} := \sum_{k=0}^{n} \| g^{(k)} \|_C$$

In particular, for $n = 0$, $C^0[a, b] = C[a, b]$ is the space of continuous functions g on $[a, b]$, equipped with the norm

$$\| g \|_C := \max_{x \in \Omega} | g(x) |$$

Definition 1.6.2. [**17**] Let $\Omega = [a, b]$ be a finite interval with $(-\infty < a < b < \infty)$, and let $AC[a, b]$ denote the space of functions that are absolutely continuous on $[a, b]$. It is known that $AC[a, b]$ coincides with the space of primitives of Lebesgue integrable functions:

$$AC([a, b]) := \{g / \ \exists \phi \in \mathbb{L}([a, b]) : \ g(x) = c + \int_a^x \phi(t) dt\}$$

Definition 1.6.3. [**17**] For $n \in \mathbb{N}$, we denote by $AC^n[a,b]$ the space of complex-valued functions $g(x)$ that have continuous derivatives up to order $(n-1)$ on $[a,b]$, such that $g^{(n-1)}(x) \in AC[a,b]$:

$$AC^n[a,b] = \{g : [a,b] \to \mathbb{C} \quad \text{and} \quad g^{(n-1)} \in AC([a,b])\}$$

In particular, $AC^1([a,b]) = AC([a,b])$.

1.7 The Cauchy Problem

Definition 1.7.1. [**6**] Let E be a Banach space, $H \subseteq E$ an open set , $I \subseteq \mathbb{R}$ and $g : I \times H \to E$ a continuous function.

The Cauchy problem is given by the system of equations :

$$\begin{cases} x'(t) = g(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1.1}$$

Solving the Cauchy problem involves finding an interval $j \subseteq I$ containing $t_0 \in I$ and solution φ of class C^1 on j that satisfy (1.1) .

Proposition 1.7.2. [**6**] A function $\varphi : I \to H$ is a solution to the Cauchy problem if and only if :

1. The function φ is continuous , and $\forall t \in I, (t, \varphi(t)) \in I \times H$.

2. $$\varphi(t) = x_0 + \int_{t_0}^{t} g(s, \varphi(s))ds.$$

Definition 1.7.3. [**17**] A function g is said to be k-Lipschtz for x if there exists $k > 0$ such that

$$\| g(t,x_1) - g(t,x_2) \| \leqslant k \| x_1 - x_2 \| .$$

for all $(t,x_1),(t,x_2) \in I \times H$.

Definition 1.7.4. [**17**] A function g is said to be locally Lipschitz if, for every point $(t_0,x_0) \in I \times H$ there exists a neighborhood V of (t_0,x_0) in $I \times H$ and a constant $k > 0$ such that

$$\| g(t,x_1) - g(t,x_2) \| \leqslant k \| x_1 - x_2 \| .$$

for all $(t,x_1),(t,x_2) \in V$.

Theorem 1.7.5. [**6**] (Existence Theorem) Let g be a continuous function on H with values in E and locally Lipschitz with respect to x. Then :

for every $(t_0,x_0) \in I \times H$, there exists $\epsilon > 0$ such that

$$\varphi \in C^1([t_0 - \epsilon, t_0 + \epsilon], E)$$

is a solution to (1.1).

Theorem 1.7.6. [**6**] (Uniqueness Theorem) Let $H \subseteq \mathbb{R} \times E$ and let $g : H \to E$ be a continuous function that is k-Lipschitz in x. If $\varphi_1, \varphi_2 : I \to E$ are two solutions to the differential equation

$$\frac{dx}{dt} = g(t,x).$$

and $\varphi_1(t_0) = \varphi_2(t_0)$ (with $t_0 \in I$).
Then the function φ_1 and φ_2 are identical on the interval I.

Theorem 1.7.7. [**6**] suppose $g : I \times E \to E$ is continuous and Lipschitz with respect to x. Then $\forall (t_0,x_0) \in I \times E$, there exists a unique solution $\varphi \in C^1(I,E)$ to the problem (1.1).

Chapter 2

Krasnoselskii-Dhage fixed point theorem

In this chapter, we will present some results from fixed point theory. We will begin with the best known of these: **Banach's fixed point theorem** for contracting maps, followed by **Schauder's theorem** for compact maps.

Finally, we will present **Krasnoselskii's-Dhage theorem,** which is a fixed point existence theorem concerning maps that can be written as the sum of two maps, one of which is continuous and compact and the other of which is contracting.

Definition [14]

Let X be a topological space, and let f be a function from X, or from a subset of X, into X. A point $x \in X$ is said to be a **fixed point** of f if $f(x) = x$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$, and is referred to as the **fixed point set** of f.

2.1 Principle of Banach contraction

This principle guarantees the existence of a unique fixed point for any contracting map of a complete metric space into itself.

Theorem 2.1.1 :[24]**(Banach fixed point theorem (1922))**

Let (M, d) be a complete metric space and $T : M \to M$ a contracting map with contraction constant k, then T has a unique fixed point in M.

Proof :

<u>**Existence:**</u>

Let $y \in M$ be an arbitrary point in M. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ given by:

$$\begin{cases} x_0 = y \\ x_n = T(x_{n-1}), n \geq 1 \end{cases}$$

We must prove that $(x_n)_n$ is a Cauchy sequence in M.

For $m < n$, we use the triangular inequality:

$$d(x_m, x_n) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

since T is a contraction, we have:

$$d(x_p, x_{p+1}) = d(Tx_{p-1}, Tx_p) \leqslant kd(x_{p-1}, x_p) \ \textbf{pour}\ p \geq 1$$

Repeating this inequality, we obtain:

$$\begin{aligned} d(x_m, x_n) &\leqslant (k^m + k^{m+1} + \cdots + k^{n-1})d(x_0, x_1) \\ &\leqslant k^m(1 + k + \cdots + k^{n-m-1})d(x_0, x_1) \\ &\leqslant k^m(1-k)^{-1}d(x_0, x_1) \end{aligned}$$

We deduce that $(x_n)_n$ is Cauchy in M which is complete, so $(x_n)_n$ converges to x in M

Moreover, since T is continuous, we have:

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) = Tx$$

So x is a fixed point of T(i.e. $Tx = x$)

Uniqueness:

Suppose $x = Tx$ and $y = Ty$, then :

$$d(x, y) = d(Tx, Ty) \leqslant kd(x, y)$$

which implies that $d(x; y) = 0$ i.e $x = y$(since $k < 1$).

Noticed 2.1.2.

If T is a Lipschitz map (not necessarily a contraction) but one of its iterates T^p is a contraction, then T still has a fixed point and is unique.

This results from uniqueness.

Indeed, let x be the unique fixed point of T^p, we have

$T^p(T(x)) = T(T^p(x)) = T(x)$

which means that $T(x)$ is also a fixed point of T^p and thanks to the uniqueness

$T(x) = x.$

So this result is valid for all types of contractions that ensure the uniqueness of the fixed point.

The local version of Banach's theorem

It may be that f is not a contraction over the entire space E, but only in the neighborhood of a given point. In this case, we have the following result.

Theorem 2.1.3 :[**1**] Let (E, d) be a complete metric space and let

$$B(x_0, r) = \{x \in E : d(x, x_0) < r\} \textbf{ or } x_0 \in E \textbf{ and } r > 0$$

Suppose that $f : B(x_0, r) \to E$ is contracting with contraction constant k, with

$$d(f(x_0), x_0) < (1 - k)r$$

Then, f has a unique fixed point in $B(x_0, r)$.

Proof :

There exists r_0 with $0 < r_0 < r$, such that $d(f(x_0), x_0) \leqslant (1 - k)r_0$. We show that

$$f : B_c(x_0, r_0) \to B_c(x_0, r_0)$$

Let $x \in B(x_0, r_0)$ then

$$d(f(x), x_0) \leqslant d(f(x); f(x_0)) + d(f(x_0), x_0)$$

$$\leqslant kd(x, x_0) + (1 - k)r_0$$

$$\leqslant kr_0 + (1 - k)r_0$$

$$\leqslant r_0$$

Therefore, the application $f : B_c(x_0, r_0) \to B_c(x_0, r_0)$ is contracting with $B_c(x_0, r_0)$ is a complete space. Consequently, the application of Banach's contraction theorem ensures that it admits a unique fixed point in $B(x_0, r)$.

2.2 Fixed point theorems for compact applications

Definition 2.1: [**1**] Let E and E' be two Banach spaces and $\Omega \subset E$ be an open set.

A continuous mapping $f : \Omega \to E'$ is said to be compact if $\overline{f(\Omega)}$ is compact. It is said to be completely continuous if the image of any bounded set is relatively compact.

Theorem 2.2.1.:[12] (Brouwer 1912) Let C be a compact, convex non-empty map of $\mathbb{R}^n$ and $f : C \to C$ a continuous map. Then f has at least one fixed point in C.

Proposition 2.2.2. [**12**] Let B be the open unit ball of a normed space E. Then:

$dimE < +\infty \Leftrightarrow$ any continuous application $f : B \to B$ admits at least one fixed point.

Proposition 2.2.3. [**12**] Let E be a normed vector space. We have the following equivalences:

$dimE < \infty \Leftrightarrow$ the closed ball $B_c(0, 1)$ is compact

$\Leftrightarrow$ the boundary $\vartheta B(0, 1)$ is compact

$\Leftrightarrow$ from any sequence of $B(0, 1)$, we can extract a convergent subsequence.

Example 1

A continuous function $f : [a, b] \longrightarrow [a, b]$ has a fixed point $x \in [a, b]$. Below is another variant of the Brower Fixed-Point Theorem (in Zeidler's book)..

Example 2 (Counter Examples)

The following counter examples show the essentials of each assumption in the Brower Fixed-Point Theorem (version 2).

- $S = [0, 1]$ compact, convex and nonempty, but $A : S \to S$ not continuous and the graph $y = A(x)$ does not cross the diagonal $y = x$. No fixed point.

- $S = R$ and $A : S \to S$,$A(x) = x + 1.A$ is continuous, S is convex, nonempty, but not compact. No fixed point.
- Let S be a closed annulus and $A : S \to S$ is a rotation of the annulus around the center. A proper rotation is fixed-point free. In this case, S is compact, nonempty but not convex.

2.2.1 Schauder's fixed point theorem

Schauder extended Brouwer's fixed point theorem to the infinite-dimensional case, using the fact that a compact infinite-dimensional map is approachable by continuous maps of finite rank.

Definition 2.2.4.[**1**]
Let E and E' be two normed vector spaces.
A mapping $F : E \to E'$ is said to be compact if $F(E)$ is contained in a compact subset of E' .
A compact mapping $F : E \to E'$ is said to be of finite rank if $F(E)$ is contained in a finite-dimensional vector subspace of E'.

Remark 2.2.5.

(a) In general, a compact map is not necessarily continuous .

(b) Any compact linear map is continuous; the converse is true if f is of finite rank (the rank is the dimension of the image space).

Theorem 2.2.6 . [**1**]
Let C be a convex subset of a normed vector space, and
$A = \{a_1, \cdots , a_n\} \subseteq C$. If P_ε denotes the Schauder projection, then:

(i) P_ε is a compact and continuous map from A_ε to $co(A) \subseteq C$,

(ii) $\|x - P_\varepsilon\| < \varepsilon$, for all $x \in A$.

Theorem 2.2.7.:[**1**]**(Schauder's approximation theorem)**

Let C be a convex subset of a normed vector space E and $F : E \to C$ a compact and continuous map. Then for each $\varepsilon > 0$, there exists a finite set
$A = \{a_1, \cdots , a_n\}$ in $F(E)$ and a continuous map $F_\epsilon : E \to C$ of finite rank with the following properties:

(i) $\|F_\epsilon(x) - F(x)\| < \epsilon$, $\forall x \in E$

(ii) $F_\epsilon(x) \subseteq co(A) \subseteq C$

Before proving Schauder's theorem, we introduce the notion of ϵ fixed point.

Definition 2.2.8.[**3**] Let E be a Banach space, $C \subset E$ a non-empty, closed convex subset and $F : C \to C$ a map. We say that F admits a $\epsilon-$fixed point in C if the following condition is satisfied:

$$\forall \epsilon > 0,\ \exists x_\epsilon \in C :\ \ \|F(x_\epsilon) - x_\epsilon\| \leqslant \epsilon.$$

Theorem 2.2.9.[**1**] Let D be a closed subset of a normed vector space E and $F : D \to E$ a compact and continuous application. Then F has a fixed point if and only if it has an $\epsilon-$fixed point

Theorem 2.2.10.(Schauder's fixed point theorem)[**1**]

Let C be a non-empty closed convex subset of a normed vector space E. Then every compact and continuous application $F : C \to C$ has at least one fixed point.

Proof :

By Theorem 2.2.9. , with $D = C$, it suffices to show that F has a ϵ fixed point for each $\epsilon > 0$. If $\epsilon > 0$, Theorem 2.2.7. Guarantees the existence of a continuous map of finite rank $F_\epsilon : C \to C$ with:

$$\|F_\epsilon(x) - F(x)\| < \epsilon, \forall x \in C \tag{2.1}$$

and $F_\epsilon(C) \subseteq co(A) \subseteq C$ for a finite set $A \subseteq C$. Since $co(A)$ is closed and bounded and $F(co(A)) \subseteq co(A)$, we can apply Theorem 2.2.1 (Brouwer fixed point theorem) to deduce that there exists $x_\epsilon \in co(A)$ with $x_\epsilon = F(x)$. Furthermore (2.1) gives

$$\|x_\epsilon - F_\epsilon(x_\epsilon)\| = \|F_\epsilon(x_\epsilon) - F(x_\epsilon)\| < \epsilon$$

Corollary 2.2.11.[**12**]

Let C be a convex, compact, nonempty subset of a Banach space E and $f : C \to C$ a continuous map. Then f has at least one fixed point.

Corollary 2.2.12.[**12**]

Let C be a convex, closed, nonempty subset, C not necessarily bounded, of a Banach space E and $f : C \to C$ a continuous map such that $f(C)$ is included in a compact set of C. Then f has at least one fixed point.

2.2.2 Leary-Schauder Alternative

Theorem 2.2.13.[**12**] Let Ω be an open, bounded in E, E Banach space and $f : E \to E$ be a compact map. Then

(i) f has a fixed point in Ω .

(ii) There exists $x \in \vartheta\, \Omega$, $\exists t \in [0, 1] : x = tf(x)$

Theorem 2.2.14.[**12**] **(Schaefer's Theorem)**

Let E be a Banach space and $K : E \to E$ a compact map. We then have the following alternative:

Either, the equation $tK(x) = x$ has a solution for all $t \in [0; 1]$.

Or, the set $S = \{x \in E : \exists t \in [0, 1], tK(x) = x\}$ is unbounded.

2.3 Krasnoselskii fixed point theorem

Theorem 2.3.1.[**24**] Let M be a closed and nonempty convex set of a Banach space $(E, \|.\|)$. Suppose that A and B are two mappings from M to E such that

i) $Ax + By \in M, \forall x, y \in M$

ii) A is compact and continuous

iii) B is a contraction of constant $\alpha < 1$.

Then there exists $x \in M$, such that $Ax + Bx = x$.

Noticed 2.3.2.

If $A = 0$, the theorem is reduced to Banach's theorem. If $B = 0$, the theorem is none other than Schauder's theorem.

Proof :

By condition (iii), we have

$$\begin{aligned}\|(I-B)(x)-(I-B)(y)\| &= \|(x-y)-(Bx-By)\| \\ &\leqslant \|x-y\|+\|Bx-By\| \\ &\leqslant \|x-y\|+\alpha\|x-y\| \\ &\leqslant (1+\alpha)\|x-y\|\end{aligned}$$

On the other hand,

$$\begin{aligned}\|(I-B)(x)-(I-B)(y)\| &= \|(x-y)-(Bx-By)\| \\ &\geq \|x-y\|-\|Bx-By\| \\ &\geq \|x-y\|-\alpha\|x-y\| \\ &\geq (1-\alpha)\|x-y\| \\ &> 0\end{aligned}$$

To summarize,

$$(1-\alpha)\|x-y\| \leqslant \|(I-B)(x)-(I-B)(y)\| \leqslant (1+\alpha)\|x-y\|.$$

This inequality shows that $(I-B) : M \to (I-B)M$ is continuous.

Let us show that $(I-B)^{-1}$ exists and is continuous:

We have $\|(I-B)x-(I-B)y\| \neq 0 \Rightarrow (I-B)x \neq (I-B)y$

We conclude that $(I-B)$ is injective, and since $(I-B) : M \to (I-B)M$ then:

$\forall y \in (I-B)M, \exists x \in M$ such as $(I-B)x = y$, hence $(I-B)$ is surjective, and therefore $(I-B)^{-1}$ exists. It remains to show that $(I-B)^{-1} : (I-B)M \to M$ is continuous.

Since $(I-B)^{-1}$ therefore exists,

$$\|(I-B)(x)-(I-B)(y)\| \geq (1-\alpha)\|x-y\|$$

$$\Leftrightarrow \|x'-y'\| \geq (1-\alpha)\|(I-B)^{-1}x'-(I-B)^{-1}y'\|$$

so, $\|(I-B)^{-1}x'-(I-B)^{-1}y'\| \leqslant \dfrac{1}{(1-\alpha)}\|x'-y'\| \quad \text{with}(1-\alpha) \neq 0$

Hence, $(I-B)^{-1}$ is Lipschitzian and therefore continuous.

Let $U := (I-B)^{-1}A$. It is clear that U is a compact map, since U is a composition of a continuous map with a compact map, so according to Schauder's theorem, U

admits a fixed point, i.e.

$$\exists x \in M, \text{such as} (I - B)^{-1} Ax = x$$

This is equivalent to saying $Ax + Bx = x$.

2.3.1 Krasnoselskii's theorem for nonlinear D-contractions

In this section, we establish our principal hybrid fixed point theorems involving the sum and product of two operators in a Banach space and a Banach algebra, respectively. To proceed, we introduce the following key definition.

Definition 2.3.3. [**8**] Let E be a Banach space.

• An application $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a dominant function or **D- function** if it is continuous and non-decreasing satisfying $\psi(0) = 0$

• An apperator $Q : E \to E$ is said to be **D-Lipschitzian** if there is a **D-function** $\psi : \mathbb{R}_+ \to \mathbb{R}_+$satisfying:

$$\|Q\phi - Q\xi\| \leqslant \psi\left(\|\phi - \xi\|\right) \quad \text{for everything } \; \phi, \xi \in E$$

Definition 2.3.4.[**11**] A function $\psi : \mathbb{R} \to \mathbb{R}$ is called an **L-function** if $\psi(0) = 0$

and for every $s > 0$, there exists a $\delta > 0$ such that

$$0 < \psi(r) \leq s \quad \text{for all } r \in [s, s + \delta).$$

Similarly, ψ is called a **strictly L-function** if $\psi(0) = 0$ and for every $s > 0$, there exists a $\delta > 0$ such that

$$0 < \psi(r) < s \quad \text{for all } r \in [s, s + \delta).$$

Definition 2.3.5.[**11**]

A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *DL-function* if it is both a D-**function** and a strictly L-function. The class of all such functions is denoted by DL.

Remark 2.3.6.

It is evident that if $\phi \in DL$, then $\phi \in D \cap L$. However, the converse does not necessarily hold.

2.3.2 Krasnoselskii-Type HFPTs

The classical form of the Krasnoselskii hybrid fixed point theorem [**18**] is notable for combining the Schauder and Banach fixed point theorems to address the sum of two operators in a Banach space. It is stated as follows:

Theorem 2.3.7.[**11**]

Let S be a nonempty, closed, convex, and bounded subset of a Banach space E. Let $A : E \to E$ and $B : S \to E$ be two operators satisfying the following conditions:

(a) A is a contraction,

(b) B is completely continuous,

(c) $Ax + By \in S$ for all $x, y \in S$.

Then the operator equation

$$Ax + Bx = x \tag{2.2}$$

admits at least one solution in S.

The relevance of the Krasnoselskii HFPT in applications is well-established. Krasnoselskii himself employed this framework to demonstrate the existence of solutions to nonlinear perturbed differential equations. Numerous generalizations of Theorem 5.2 have emerged over time. Burton [**4**] relaxed condition (c), whereas Dhage [**10** , **11**] refined condition (a), resulting in a broader class of hybrid fixed point theorems concerning the sum of two operators in a Banach space E.

Remark 2.3.8.

Any Lipschitzian D-map Q is Lipschitzian with, $\psi(r) = kr, k > 0$.

Furthermore, if $\psi(r) < r$, then Q is said to be a nonlinear D-contraction and the function is said to be a D-function of Q in E.

Remark 2.3.9.

Any D-contracting map is contracting.

Theorem 2.3.10. [**8**]

Let S be a closed, convex, and bounded subset of a Banach space E and $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ two operators such that

(a) $\mathcal{A}$ is a nonlinear D-contraction.

(b) $\mathcal{B}$ is compact and continuous.

(c) $x = \mathcal{A}x + \mathcal{B}y$ for all $y \in S$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution in S.

Clarification:
Since A is a D-contraction on S, it satisfies the contraction condition:

$$\|Ax - Ay\| \leq \phi\left(\|x - y\|\right)$$

for all $x, y \in E$, where $\phi \in DL$
Let $y \in S$ be a fixed element, and define the mapping $\mathcal{A} : E \to E$ by:

$$\mathcal{A}(x) = Ax + By.$$

We show that $\mathcal{A}$ is a D-contraction on E. Let $x_1, x_2 \in E$. Then we have:

$$\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\| \leq \|Ax_1 - Ax_2\| \leq \phi\left(\|x_1 - x_2\|\right).$$

Chapter 3

Application to fractional hybrid differential equation

In this chapter, we propose to study the existence of solutions for applications of Krasnoselskii-Dhage type Fixed -point theorems to fractional hybrid differential equations

3.1 Fractional calculus

3.1.1 The Gamma Function

Definition 3.1.1. [**17**] The Gamma function, denoted as $\Gamma(s)$, is typically defined by the integral :

$$\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt, \quad \mathbf{Re}(s) > 0.$$

This function is analytic in the right half of the complex plane, denoted as $\mathbb{C}^+ = \{s \in \mathbb{C} : \mathbf{Re}(s) > 0\}$.

Proposition 3.1.2. [**17**] We have the following relation :

1. $\Gamma(s+1) = s\Gamma(s)$
2. $\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}$.

3.1.2 The Beta Function

Definition 3.1.3. [**17**] A closely related, the Beta function, denoted as $B(p,q)$ for $p,q>0$, is defined by the integral :

$$B(p,q)=\int_0^1 (1-s)^{p-1}s^{q-1}ds.$$

Proposition 3.1.4. [**17**] The Beta function is related to the Gamma function by the following relation for all $p,q>0$ we have :

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

3.1.3 Laplace Transform

Definition 3.1.5. [**17**] The Laplace transform of a function $f(t)$ of a real variable $t\in\mathbb{R}^+$ is defined by

$$(\mathcal{L}f)(s)=\mathcal{L}[f(s)]=\int_0^\infty e^{-st}f(t)dt \quad s\in\mathbb{C}.$$

Definition 3.1.6. [**17**] The inverse Laplace transform is given for $x\in\mathbb{R}^+$ by the formula :

$$(\mathcal{L}^{-1}h)(x)=\mathcal{L}^{-1}[h(x)]=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{sx}h(s)ds \quad (\beta=\mathbb{R}(s)>\sigma_\varphi).$$

Definition 3.1.7. [**17**] The Laplace convolution operator of two function $g(s)$ and $f(s)$, given on $\mathbb{R}^+$, is defined for $x\in\mathbb{R}^+$ by the integral

$$(g*f)(x)=\int_0^x g(x-s)f(s)ds.$$

3.1.4 Fractional integral

Definition 3.1.8. [**17**] For any function $g \in \mathbb{L}^1$, the left-sided Riemann-Liouville fractional integral of order $\alpha > 0$, denoted by $I_a^\alpha g$ is defined as :

$$(I_a^\alpha g)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)\, dt.$$

Proporietes 3.1.9. [**17**]

1. $I_0^0 g(x) = g(x)$
2. $I_0^\alpha I_0^\beta g(x) = I_0^{\alpha+\beta} g(x)$
3. The integral operator I_0^α is linear.

3.1.5 Fractional derivatives

3.1.5.1 Riemann-Liouville fractional derivatives

Definition 3.1.10. [**17**] Let g be an integrable function on $[a,b]$. The fractional derivative of order α with $(n-1 \leqslant \alpha < n)$ in the Riemann-Liouville sense is defined as:

$${}^{RL}D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} g(s) d(s) = \frac{d^n}{dt^n} (I^{n-\alpha} g(t))$$

where $n = [\alpha] + 1$

Property 3.1.11. [**17**]

1. **Composition with the fractional integral :**

i) The fractional derivative operator in the sense of Riemann-Liouville is a left inverse of the fractional integral operator:

$${}^{RL}D^\alpha (I^\alpha g(t)) = g(t)$$

Generally, the composition satisfies:

$${}^{RL}D^{\alpha_1} (I^{\alpha_2} g(t)) = {}^{RL}D^{\alpha_1 - \alpha_2} g(t)$$

and if $\alpha_1 - \alpha_2 < 0$, then:

${}^{RL}D^{\alpha_1-\alpha_2}g(t) = I^{\alpha_2-\alpha_1}g(t)$

ii) In general, fractional differentiation and fractional integration do not commute:

$$^{RL}D^{-\alpha_1}\left({}_a^{\mathrm{RL}}D_t^{\alpha_2}g(t)\right) = {}^{RL}D^{\alpha_2-\alpha_1}g(t) - \sum_{k=1}^{m}\left[{}^{RL}D_t^{\alpha_2-k}g(t)\right]_{t=a}\frac{(t-a)^{\alpha_1-k}}{\Gamma(\alpha_1-k+1)}$$

where $m-1 \leq \alpha_2 < m$.

2) **Composition with derivatives of integer order :**

Fractional differentiation and ordinary (integer order) differentiation commute only if:

$g^{(k)}(a) = 0 \quad \text{for all} \quad k = 0, 1, 2, \ldots, n-1$

then

$$\frac{d^n}{dt^n}({}^{RL}D^{\alpha}g(t)) = {}^{RL}D^{n+\alpha}g(t)$$

but in general:

$$^{RL}D^{\alpha}\left(\frac{d^n}{dt^n}g(t)\right) = {}^{RL}D^{n+\alpha}g(t) - \sum_{k=0}^{n-1}\frac{g^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)}$$

3) For the Riemann-Liouville fractional differential operator, the following properties hold :

$$\lim_{\alpha\to n}{}^{RL}D^{\alpha}g(t) = g^{(n)}(t)$$

$$\lim_{\alpha\to n-1}{}^{RL}D^{\alpha}g(t) = g^{(n-1)}(t)$$

4) **Linearity :**

The fractional differentiation operation is linear For $\alpha \in \mathbb{R}^+, \lambda \in \mathbb{C}$

$$^{RL}D^{\alpha}(\lambda g(t) + h(t)) = \lambda{}^{RL}D^{\alpha}g(t) + {}^{RL}D^{\alpha}h(t)$$

5) **Non-commutativity :**

The Riemann-Liouville operator is also non-commutative, meaning:

$$D^{\alpha}D^{m}g(t) = D^{\alpha+m}g(t) \neq D^{m}D^{\alpha}g(t)$$

6) **The relationship between the derivative and the integral :**

Lemma 3.1 : [**17**, **19**] Let $0 < \alpha < 1$ and $g \in \mathbb{L}^1(0,1)$. Then

1. The equality $D^{\alpha}I^{\alpha}f(x) = f(x)$ holds
2. The equality

$$I^{\alpha}D^{\alpha}f(x) = f(x) - \frac{[D^{\alpha-1}f(x)]_{x=0}}{\Gamma(\alpha)}x^{\alpha-1}$$

holds for almost everywhere on $J = [t_0, t_0 + a]$.

3.1.5.2 Caputo Fractional Derivative

Definition 3.1.12. [**17**] For a given function g on the interval $[a, b]$ the Caputo fractional derivative of g of order $\alpha > 0$ is defined as:

$$^{c}D^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}g^{(n)}(s)d(s)$$

where $n = [\alpha] + 1$

Property 3.1.13. [**17**]

1. **Relationship with the Riemann-Liouville derivative :**

Suppose $n - 1 < \alpha < n$, with $m, n \in \mathbb{N}^*, \alpha \in \mathbb{R}$ and the functions $^{c}D^{\alpha}g(t)$ and $^{RL}D^{\alpha}g(t)$ exist, then

$$^{c}D^{\alpha}g(t) = {}^{RL}D^{\alpha}g(t) - \sum_{k=0}^{n-1}\frac{g^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}$$

It follows that if $g^{(k)}(a) = 0$ for $k = 0, 1, 2, ..., n-1$, then:

$$^{c}D^{\alpha}g(t) = {}^{RL}D^{\alpha}g(t)$$

2. **Composition with the fractional integral operator :**

   If g is a continuous function, then:

$$^{c}D^{\alpha}I^{\alpha}g = g \text{ and } I_{a}^{\alpha c}D^{\alpha}g(t) = g(t) - \sum_{k=0}^{n-1}\frac{g^{(k)}(a)(t-a)^{k}}{k!}$$

3. **Non-commutativity :**

   Suppose $n-1 < \alpha < n$, with $m, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and let $g(t)$ be a function for which the Caputo fractional derivative $^{c}D^{\alpha}g(t)$ exists , then:

$$^{c}D^{\alpha}D^{m}g(t) = {}^{c}D^{\alpha+m}g(t) \neq D^{m}\ {}^{c}D^{\alpha}g(t).$$

4. **Limits as α approaches integers :**

   Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $g(t)$ such that $^{c}D^{\alpha}g(t)$ exists, then:

$$\lim_{\alpha \to n} {}^{c}D^{\alpha}g(t) = g^{(n)}(t)$$

$$\lim_{\alpha \to n-1} {}^{c}D^{\alpha}g(t) = g^{(n-1)}(t) - g^{(n-1)}(0)$$

5. **Linearity :**

   The Caputo fractional derivative operator is linear.

   Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, $\alpha, \lambda \in \mathbb{C}$, and functions $g(t)$ and $h(t)$, then:

$$^{c}D^{\alpha}(\lambda g(t) + h(t)) = \lambda^{c}D^{\alpha}g(t) + {}^{c}D^{\alpha}h(t).$$

3.2 Position of the problem

Problem:[**2**] We consider the fractional hybrid differential equation that involving the Riemann-Liouville differential and integral operators of orders $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^{\alpha}\left[x(t) - f(t, x(t))\right] = g(t, x(t), I^{\beta}(x(t))) \quad \text{a.e} \quad t \in J, \beta > 0 \\ x(t_0) = x_0 \end{cases} \tag{3.1}$$

Where $J = [t_0, t_0 + a]$, for a fixed value $t_0 \in \mathbb{R}$ and $a > 0$ and $f \in C(J \times \mathbb{R}, \mathbb{R})$ and $g \in \mathcal{C}(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. We establish the existence of a solution to equation (3.1) given specific conditions on f and g.

3.3 Existence results

Hypotheses:[**2**]

Now, we assume that :

(H1) The function $F_t : \mathbb{R} \to \mathbb{R}$ defined as $F_t(x) := x - f(t, x)$ is strictly increasing in $\mathbb{R}$ for all $t \in J$.

(H2) The function $f(t, .)$ meets the specified weak contraction condition

$$| f(t, x) - f(t, y) | \leq tanh(| x - y |)$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(H3) There exists a continuous function $h \in C(J, \mathbb{R})$ such that

$$| g(t, x, y) | \leq h(t)$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

Theorem 3.1 : [**2**] Assume that hypotheses (H1)-(H3) hold . Then FHDE (3.1) has a solution in $C(J, \mathbb{R})$.

Lemma 3.2 : [**22**] Assume that hypothesis (H1) holds . Then , for any $y \in C(J, \mathbb{R})$ and $\alpha \in (0, 1)$ the function $x \in C(J, \mathbb{R})$ is a solution of FHDE

$$D^{\alpha}[x(t) - f(t, x(t))] = y(t), \quad t \in J, \tag{3.2}$$

with the initial condition

$$x(t_0) = x_0 \tag{3.3}$$

if and only if $x(t)$ satisfies the hybrid integral equation (HIE)

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} y(s) ds, \quad t \in J. \tag{3.4}$$

Proof : [**22**] Let x be a solution to the Cauchy problem (3.2) and (3.3). Since the Riemann-Liouville fractional integral I^α is a monotone operator, we apply I^α to both sides of equation (3.2). By Lemma 3.1, we obtain the following result

$$I^\alpha D^\alpha [x(t) - f(t, x(t))] = x(t) - f(t, x(t)) - \frac{I^{1-\alpha}[x(t) - f(t, x(t))]_{t=t_0}}{\Gamma(\alpha)} (t-t_0)^{\alpha-1} = I^\alpha y(t)$$

then by (3.3) we get

$$x(t) - f(t, x(t)) = I^\alpha y(t) + \frac{I^{1-\alpha}[x(t) - f(t, x(t))]_{t=t_0}}{\Gamma(\alpha)} (t-t_0)^{\alpha-1} = x_0 - f(t_0, x_0) + I^\alpha y(t)$$

a.e

$$\begin{aligned} x(t) &= x_0 - f(t_0, x_0) + f(t, x(t)) + I^\alpha y(t) \\ &= x_0 - f(t_0, x_0) + f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} y(s) ds, \quad t \in J. \end{aligned}$$

Thus (3.4) holds .

Conversely, suppose that x satisfies HIE (3.4). Then, by applying D^α to both sides of (3.4), equation (3.3) is satisfied. Additionally, by substituting $(t = t_0)$ into (3.4), we obtain the following result.

$$x(t_0) - f(t_0, x(t_0)) = x_0 - f(t_0, x_0)$$

Is the function $x \to x - f(t, x)$ increasing in $\mathbb{R}$ for all $t \in J$ Since this function is injective in $\mathbb{R}$, it follows that $x(t_0) = x_0$.

Now, we are prepared to prove the following existence theorem for the fractional hybrid differential equation (FHDE)(3.1).

Proof of the Main Theorem 2.3.10. [2]

Let $E = C(J, \mathbb{R})$ and $\parallel . \parallel$ be the uniform norm on E, given by

$\parallel x \parallel = \max\limits_{t \in J} \mid x(t) \mid$. It is clear that $(E, \parallel . \parallel)$ is a Banach space. Consider the subset $(S \subseteq E)$ defined as follows.

$$S = \{x \in E : \parallel x \parallel \leq M\}$$

Defin $M = \mid x_0 - f(t_0, x_0) \mid +1 + L + \frac{|J|^\alpha}{\Gamma(\alpha+1)} \parallel h \parallel$, where $L = \max_{t \in J} f(t, 0)$. S is a closed, convex, and bounded subset of the Banach space E. Let $y \in S$. To demonstrate the existence of a solution to equation (3.1), consider the following generalized fractional hybrid differential equation involving the Riemann-Liouville differential and integral operators of orders $0 < \alpha < 1$ and $\beta > 0$:

$$\begin{cases} D^\alpha \left[x(t) - f(t, x(t))\right] = g(t, y(t), I^\beta(y(t))) \quad \text{a.e} \quad t \in J, \beta > 0 \\ x(t_0) = x_0, \end{cases} \tag{3.5}$$

Let $J = [t_0, t_0 + a]$, where $t_0, a \in \mathbb{R}^+$ is fixed and $f \in C(J \times \mathbb{R}, \mathbb{R})$,

$g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Suppose that f and g satisfy the hypotheses $(H1) - (H3)$. From Lemma 3.2 , it follows that equation (3.5) is equivalent to the nonlinear hybrid integral equation (HIE).

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^\beta(y(s))) ds. \quad t \in J. \tag{3.6}$$

Let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be the operators defined as follows:

$$\mathcal{A}x(t) = x_0 - f(t_0, x_0) + f(t, x(t)), \quad t \in J, \tag{3.7}$$

and

$$\mathcal{B}y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^\beta(y(s))) ds, \quad t \in J. \tag{3.8}$$

It is straightforward to observe that the HIE (3.6) can be converted into the following operator equation:

$$x(t) = \mathcal{A}x(t) + \mathcal{B}y(t), \quad t \in J. \tag{3.9}$$

We will demonstrate that the operators $\mathcal{A}$ and $\mathcal{B}$ fulfill all the conditions of Theorem 2.3.10. First, we establish that the operator $\mathcal{A}$ is a nonlinear D-contraction with $\psi_{\mathcal{A}}(t) = tanh(t)$. Let $x, y \in E$. According to hypothesis (H2), we have:

$$\mid \mathcal{A}x(t) - \mathcal{A}y(t) \mid = \mid f(t, x(t)) - f(t, y(t)) \mid \leq \tanh(\mid x(t) - y(t) \mid) \leq \tanh(\parallel x - y \parallel).$$

By taking the maximum over t on the left-hand side, we obtain:

$$\parallel \mathcal{A}x - \mathcal{A}y \parallel \leq \tanh(\parallel x - y \parallel).$$

Concider the function

$$k(r) = (r-1)e^r + \frac{r+1}{e^r}.$$

It is easily observed that $k'(t) = r(e^r - e^{-r})$ when $r > 1$. Therefore, the function $k(r)$ is strictly increasing, with $k(r) > 0$ for $r > 0$. This indicates that

$$r - \psi_{\mathcal{A}}(r) = r - \tanh(r) = \frac{k(r)}{e^r + e^{-r}} > 0.$$

For $r > 0$. Therefore, the operator $\mathcal{A}$ is a nonlinear contraction. Next, we aim to prove that the operator $\mathcal{B}$ is continuous and compact from S into E. First, we will show that $\mathcal{B}$ is continuous on S. Consider a sequence $\{y_n\}$ in S that converges to a point $y \in S$. Using the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}y_n &= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y_n(s), I^{\beta}(y_n(s)))ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} lim_{n\to\infty} g(s, y_n(s), I^{\beta}(y_n(s)))ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds = \mathcal{B}y(t)
\end{aligned}$$

For every $t \in J$, therefore the operator $\mathcal{B}$ is continuous. Next, we aim to prove that $\mathcal{B}$ is a compact operator on S . To achieve this, it is enough to show that the set $\mathcal{B}S$ is both uniformly bounded and equicontinuous in S.

According to hypothesis (H3), we conclude that

$$\begin{aligned}
\mid \mathcal{B}y(t) \mid &= \mid \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))ds \mid \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mid g(s, y(s), I^{\beta}(y(s))) \mid ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} h(s)ds \leq \frac{\mid J \mid^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel
\end{aligned}$$

For every $t \in J$, by taking the maximum over t on the left-hand side, we arrive at.

$$\parallel \mathcal{B}y \parallel \leq \frac{\mid J \mid^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel .$$

Therefore, the operator $\mathcal{B}$ is uniformly bounded on S. Let $t_1, t_2 \in J$ with $t_1 \leq t_2$. For any $x \in S$, it follows that

$$\begin{aligned}
|\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))\, ds - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} (t_2-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(\\
&\leq \Bigg| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] g(s, y(s), I^{\beta}(y(s)))\, ds \Bigg| \\
&\quad + \Bigg| \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))\, ds \Bigg| \\
&\leq \frac{\|h\|}{\Gamma(\alpha+1)} \left[|(t_2-t_0)^{\alpha} - (t_1-t_0)^{\alpha} + (t_2-t_1)^{\alpha}| + (t_2-t_1)^{\alpha}\right]
\end{aligned}$$

Therefore, for any ε, there exists a $\delta > 0$.

$$\mid t_1 - t_2 \mid \leq \delta \Rightarrow \mid \mathcal{B}x(t_1) - \mathcal{B}x(t_2) \mid \leq \varepsilon$$

For all $t_1, t_2 \in J$ and $x \in S$, this indicates that $\mathcal{B}S$ is an equicontinuous set in S. Therefore, $\mathcal{B}S$ is a set that is both uniformly bounded and equicontinuous in E, making it compact by the Arzelà-Ascoli theorem. Finally, we confirm that hypothesis (c) of Theorem 2.3.10 is met. Assume $x \in X$ and $y \in S$ satisfy the equation $x = \mathcal{A}x + \mathcal{B}y$. By hypothesis (H2)

$$
\begin{aligned}
|x(t)| &= |\mathcal{A}x(t) + \mathcal{B}y(t)| \leq |\mathcal{A}x(t)| + |\mathcal{B}y(t)| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t))| + \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))\, ds \right| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, y(s), I^{\beta}(y(s)))\, ds \right| \\
&\leq |x_0 - f(t_0, x_0)| + \tanh(\parallel x \parallel) + L + \frac{\mid J \mid^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel \\
&\leq \|x_0 - f(t_0, x_0)\| + 1 + L + \frac{\mid J \mid^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel .
\end{aligned}
$$

By considering the maximum value of t on the left-hand side, we arrive at

$$
\parallel x \parallel \leq \mid x_0 - f(t_0, x_0) \mid + 1 + L + \frac{\mid J \mid^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel = M.
$$

Hence, $x \in S$. Therefore, all the conditions of Theorem 2.3.10 are satisfied, which implies that the operator $\mathcal{A} + \mathcal{B}$ has a fixed point in S that is, there exists $z^* \in S$ such that $\mathcal{A}z^* + \mathcal{B}z^* = z^*$. Consequently, the FHDE (3.1) admits a solution in S .

3.4 Example 1

Let $J = [0,1]$, and let E be the collection of continuous, non-negative functions $f : J \to [0,\infty)$. In the space E , we examine the following fractional hybrid differential equation:

$$\begin{cases} D^{\frac{1}{2}}\left[x(t) + \frac{t}{t+2}\tanh(x(t)+3)\right] = t^2\ln(1+t^2)\sin(x(t))\frac{5I^{\beta}(x(t))+2}{6I^{\beta}(x(t))+2} & t \in J, \beta > 0 \\ x(0) = 0, \end{cases} \tag{3.10}$$

This equation is considered a special case of the fractional hybrid differential equation (3.1)

Step 1: Definition of the functions

$$f(t,x(t)) = -\frac{t}{t+2}\tanh(x+3)$$

$$g(t,x(t),I^{\beta}(x(t))) = t^2\ln(1+t^2)\sin(x(t))\frac{5I^{\beta}(x(t))+2}{6I^{\beta}(x(t))+2}$$

We now proceed to prove that equation (3.10)admits a solution .

Step 2: Checking the hypotheses (H1)-(H3)

Hypothesis (H1)

$$F_t(x) := x - f(t,x) = x + \frac{t}{t+2}\tanh(x+3)$$

$$\frac{\partial F_t}{\partial x} = 1 + \frac{t}{t+2} \quad \sinh^2(x(t)+3)$$

Since all terms are positive, $t, x \geq 0$ it follows that $\frac{\partial F_t}{\partial x} \geq 0$ and thus $F_t(x)$ is strictly increasing on $\mathbb{R}^+ \cup \{0\}$ for all $t \in J$.

Therefore, the hypothesis (H1) holds true .

Hypothesis (H2)

$$\begin{aligned} |f(t,x)-f(t,y)| &= \frac{t}{t+2}|\tanh(y+3)-\tanh(x+3)| \\ &= \frac{t}{t+2}|\tanh(y-x)(1-\tanh(y+3)\tanh(x+3))| \\ &\leq |\tanh(y-x)| \leq \tanh(|x-y|) \end{aligned}$$

The function f satisfies hypothesis (H2).

Hypothesis (H3)

We take

$$p(t) = t^2 \ln(1+t)$$

It is clear that $|sinx| \leq 1$, $\frac{5I^\beta(x)+2}{6I^\beta(x)+2} \leq 1$ therefore

$$|g(t,x(t),I^\beta(x(t)))| = t^2\ln(1+t^2)|\sin(x(t))|\frac{5I^\beta(x)+2}{6I^\beta(x)+2} \leq t^2\ln(1+t^2)$$

Thus

$$|g(t,x(t),I^\beta(x(t)))| \leq p(t).$$

Hence , the function g fulfills the conditions of hypothesis (H3).

Result :

Since the functions and satisfy hypotheses (H1)-(H3), it follows from Theorem (3.1) that the fractional hybrid differential equation (3.10) has a solution on the interval J.

Conclusion

In this thesis, we explored the classical fixed point theorem of Krasnoselskii and Dhage in Banach spaces, which we applied to investigate the existence of solutions to certain types of differential equations.

This theorem also holds for general topological vector spaces.

Numerous extensions of this theorem exist, such as the Krasnoselskii-Dhage theorems for condensing operators.

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Summary

In this thesis, we explored the classical fixed point theorem of Krasnoselskii and Dhage in Banach spaces, which we applied to investigate the existence of solutions to certain types of differential equations.

This theorem also holds for general topological vector spaces.

Numerous extensions of this theorem exist, such as the Krasnoselskii-Dhage theorems for condensing operators.

Résumé

Dans cette thèse, nous avons étudié le théorème classique du point fixe de Krasnoselskii et Dhage dans les espaces de Banach, que nous avons appliqué pour étudier l'existence de solutions à certains types d'équations différentielles.

Ce théorème est également valable pour des espaces vectoriels topologiques généraux.

De nombreuses extensions de ce théorème existent, telles que les théorèmes de Krasnoselskii-Dhage pour les opérateurs condenses.

ملخص

في هذه الرسالة، استكشفنا نظرية النقطة الثابتة الكلاسيكية لكراسنوسيلسكي وداج في فضاءات بناخ، والتي طبقناها لدراسة وجود حلول لأنواع معينة من المعادلات التفاضلية. كما أن هذه النظرية تنطبق أيضًا على فضاءات متجهية طوبولوجية عامة. توجد العديد من التوسيعات لهذه النظرية، مثل نظريات كراسنوسيلسكي-داج للعمليات المكثفة.