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Study of nonlinear differential inclusions with generalized order and its applications

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ملخص

في هذه الأطروحة، قمنا بدراسة مجموعة من الجمل التفاضلية الإحتوائية تحت شروط ابتدائية و حدية باستخدام إستراتيجية النقطة الصامدة.
في البداية أثبتنا وجود حلول للإحتواءات التفاضلية الكسرية في فضاءات لوبيغ و سوبوليف، و في الأخير ناقشنا وجود و وحدانية الحلول للإحتواءات التفاضلية الكسرية تحت شروط ابتدائية في ميادين غير محدودة.
ترتكز الإثباتات في كل جملة على تشكيل دالة متعددة القيم عناصر قيمها معادلات تكاملية و التي هي في الأصل معادلات تفاضلية كسرية طرفها الثاني دالة من الدالة المتعددة القيم الموجودة في الطرف الثاني من اللإحتواءات التفاضلية الكسرية، بعد ذلك إثبات أنها تملك نقاط صامدة.

كلمات مفتاحية: تكامل و إشتقاق ريمان ليوفيل، دالة متعددة القيم، الجمل ذات القيم الحدية والإبتدائية، اختيار قاب للقياس، نظرية النقطة الصامدة، الإحتواءات التفاضلية الكسرية.

Abstract

In this thesis, we have studied a set of fractional differential inclusions problems under boundary and initial conditions, using strategy of fixed point theorems.
In the beginning, we proved the existence of solutions for nonlinear fractional differential inclusions in Banach spaces, after that we studied the existence and uniqueness of the solution and the study of the topological structure of the solutions of fractional differential inclusions in Lebesgue and Sobolev spaces, and finally we discussed the existence and uniqueness of solutions for fractional differential inclusions under initial conditions in unbounded domains.
The proof in each problem are based on the formation of a multifunction who whose elements are integral equations, which are originally fractional differential equations, the second side is a function of the set valued mapping present in the second side of the fractional differential inclusions, after that it is prove that it has a fixed points

Key words: Riemann-Liouville integral, Riemann-Liouville derivative, set valued mapping, boundary and initial value problems, measurable selection, fixed point theorem, fractional differential inclusion.

Résumé

Dans cette thèse, nous avons étudié un ensemble de problèmes d'inclusions différentielles fractionnaires sous conditions aux limites et initiales, en utilisant la stratégie des théorèmes du point fixe.
Au début, nous avons prouvé l'existence de solutions pour les inclusions différentielles fractionnaires non linéaires dans les espaces de Banach, après cela nous avons étudié l'existence et l'unicité de la solution et l'étude de la structure topologique des solutions d'inclusions différentielles fractionnaires dans les espaces de Lebesgue et Sobolev, et enfin nous avons discuté de l'existence et de l'unicité des solutions pour les inclusions différentielles fractionnaires sous conditions initiales dans des domaines non bornés.
La preuve dans chaque problème est basée sur la formation d'une multifonction dont les éléments sont des équations intégrales, qui sont à l'origine des équations différentielles fractionnaires, le deuxième côté est une fonction de la multifonction présente dans le deuxième côté des inclusions différentielles fractionnaires, après cela il est prouvé qu'il possède un point fixe.

Mots clés : Intégrale de Riemann-Liouville, dérivée de Riemann-Liouville, fonction multivoque, problèmes aux limites et aux valeurs initiales, sélection mesurable, théorème du point fixe, inclusion différentielle fractionnaire.

Contents

Introduction

Fractional calculus is a branch of mathematical analysis dealing with pseudo- differential operators, interpreted as integrals and derivatives of non order. The fractional derivative calculus is an old subject appeared in the 17^{th} century when the scientist Leibniz introduced the concept $\frac{d^n y}{dx^n}$, when $n \in \mathbb{N}$, so H'ospital wrote his question to Leibniz : " what if $n = \frac{1}{2}$?". Further the fractional (derivative & integral) it has been mentioned in some contexts for instance Euler.L (1730), Lagrange.J.L (1772) Laplace.P.S (1812), Lacroix.S.F (1819), Fourier.J.B (1822), Abel.N.H (1823 – 1826), Liouville.J (1832), Riemann.B (1847), Green.E.A (1859), Holmgren.H (1865), Grunwald.A. K (1867), Letnikov.A.B (1868), Sonin.N.Ya (1869), Laurent.H (1884), Nekrassow.P.A (1888), Krug.A (1890), Hadamard.J (1892), Heaviside.O (1892 -1912), Pincherle.S (1902), Hardy.G.H and Littlewood.J.E (1917 -1928), Weyl.H (1917), Levy.P (1923), Marchaud.A (1927), Davis.H.T (1924 -1936), Zygmund.A (1935–1945), Love.E. R (1938-1996), Erdelyi.A (1939-1965), Kober.H (1940), Widder.D.V (1941) and Riesz. M (1949).

The fractional differential equation is generalized to ordinary differential equation to arbitrary non integer odrer and the fractional differential inclusion is generalized to ordinary differential inclusion to arbitrary non integer odrer when the solution is a function belong in multifuction (set valued map). Their importance appears in the fields physics, chemistry, engineering, biology, economics,...

Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed.

In the last years many researchers interested in studying the existence and uniqueness solutions for initial and boundary value problems fractional in Banach space, let's cite the examples [1, 2, 3, 4, 5, 6, 7, 14, 17, 25, 26, 39, 43], and the other researchers interested in studying the existence and uniquencess solutions for initial and boundary value problem on unbounded domain let's cite the examples [8, 16, 38, 41, 45], also some researchers studied the existence and uniqueness solutions for fractional problem in Sobolev spaces [10, 15, 28, 32].

This thesis consists of four chapters. In the first chapter we give topological definition and properties about the set valued mapping, some fixed point theorem used in the existence of the solutions, after that we touch the defintions integral and derivative of Riemann-Liouville and properties of this definitions, finally some theorems and properties used to prove theories in the following chapters.

In the second chapter we study the existence results for the fractional differential inclusion Let

$$\begin{cases} \mathcal{D}_{0+}^{\varsigma} z(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{A}_1 z(\omega), \dots, \mathcal{A}_n z(\omega)), & \text{for almost } \omega \in [0,1], \\ z(0) = 0, \\ \mathcal{D}_{0+}^{\nu} z(1) = p I_{0+}^{\mu} g(1, z(1)), \end{cases} \tag{1}$$

where $1 < \varsigma < 2, p, \mu > 0$ and $0 \leq \nu \leq \varsigma - 1$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}(\mathbb{R})$ is a set valued map, I_{0+}^{μ} is Riemann-Liouville fractional integral of order μ, $\mathcal{A}_j$ is an operator bounded or Lipschitzean (generally not necessary linear) defined on the space of continuous functions on $[0,1]$ into itself for all $1 \leq j \leq n$. But when $\mathcal{A}_j$ is non bounded and non Lipschitzean, this is my publication in the second section, in this section we study the existence solution in Banach space
$\mathbb{X} = \{z : z, \mathcal{D}_{0+}^{\gamma_1} z, \dots, \mathcal{D}_{0+}^{\gamma_n} z \in \mathcal{C}([0,1], \mathbb{R})\}$, for the problem

$$\begin{cases} \mathcal{D}_{0+}^{\varsigma} z(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{D}_{0+}^{\gamma_1} z(\omega), \dots, \mathcal{D}_{0+}^{\gamma_n} z(\omega)), & \text{for almost } \omega \in [0,1], \\ z(0) = 0, \\ \mathcal{D}_{0+}^{\nu} z(1) = p I_{0+}^{\mu} g(1, z(1)), \end{cases} \tag{2}$$

with $1 < \varsigma < 2, p, \mu \geq 0$, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and
$\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}(\mathbb{R})$ is a set-valued map, $0 = \gamma_0 < \gamma_1 < \dots < \gamma_n < 1$, $\varsigma > \gamma_n + 1$ and $0 \leq \nu < \gamma_n$, by using fixed point theorems, this result is published in [31].

In the third chapter we study the existence and uniqueness for boundary value problem of fractional differential inclusion

$$\begin{cases} \mathcal{D}^{\varsigma}_{0^+} z(\omega) \in \mathcal{H}(\omega, z(\omega)), \quad \text{for almost } \omega \in [0,1], \\ z(0) = 0, \\ z(1) = pI^{\mu_1}_{0^+} k_1(\xi_1, z(\xi_1)) + qI^{\mu_2}_{0^+} k_2(\xi_2, z(\xi_2)), \end{cases} \tag{3}$$

where $1 < \varsigma < 2$,$p, q \geq 0$, $\mu_1, \mu_2 \geq 1$, $0 < \xi_1, \xi_2 \leq 1$, $k_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $1 \leq j \leq 2$, and $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set valued map function, when the set valued map $\mathcal{H}$ has a convex valued by using Kakutani fixed point theorem, and we study the topological structure for the set of solutions in $\mathrm{L}^1([0,1];\mathbb{R})$ space, after that we study the existence the solution for that problem in $W^{1,1}([0,1];\mathbb{R})$ space when the set valued map $\mathcal{F}$ has a nonconvex valued by using Covitz-Nadler fixed point theorem, and in third section we study the existence for the problem (3) without using fixed point method, but by using strategy of iterative method in sobolev fractional spaces $\mathbb{X} = \{z \in \mathrm{L}^2([0,1];\mathbb{R}); D^{\beta}_{0^+} z \in \mathrm{L}^2([0,1];\mathbb{R})\}$ with $0 < \beta < \varsigma - 1$, after that we study the uniqueness for this problem through the resultat of the existence.

In the fourth chapter we study the existence and uniqueness for initial value problem of fractional differential inclusion

$$\begin{cases} \mathcal{D}^{\varsigma}_{0^+} z(\omega) \in \mathcal{H}(\omega, z(\omega)), \quad \text{for almost } \omega \in J = [0, +\infty[\\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0^+} z(0) = I^{\beta}_{0^+} z(\xi), \end{cases} \tag{4}$$

where $1 < \varsigma < 2$, $0 < \xi < 1, \beta \geq 1$, and $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a set valued map. In the first section we study the existence and the uniqueness for this problem in the space defined by $\mathbb{X} = \{z \in \mathcal{C}([0, +\infty[;\mathbb{R});\ \sup\limits_{\omega \geq 0} \dfrac{|z(\omega)|}{(1 + \omega^{\varsigma-1})^2} < +\infty\}$, by using fixed point theorems. In the second section we use other technique for the existence solution for the problem (4) in $\mathcal{C}([0, +\infty);\mathbb{R})$, for this we define the auxiliary problem

$$\begin{cases} \mathcal{D}^{\varsigma}_{0^+} z(\omega) \in F(\omega, z(\omega)), \quad \text{for almost } \omega \in [0,1] \\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0^+} z(0) = I^{\beta}_{0^+} z(\xi), \end{cases} \tag{5}$$

where $F(\omega, x) = \mathcal{H}(\omega, x)$ on $\omega \in [0, 1]$ and $x \in \mathbb{R}$. After that we study the existence for the problem 5 by using fixed point theorem and we construct an element in $\mathcal{C}([0, +\infty); \mathbb{R})$ which depend on the solution for the auxiliary problem.

Finally in the conclusion we present the aim of this work in addition to future work in this field.

Chapter 1

Preliminary

1.1 Notations

Let $a, b \in \mathbb{R}$, $J = [a, b]$ and $(\mathrm{X}, \|.\|_{\mathrm{X}})$ be a Banach space.

$\mathcal{C}(J, \mathrm{X})$ the space of all continuous functions from J into X associated the norm

$$\|x\| = \max_{a \leq \omega \leq b} \|x(\omega)\|_{\mathrm{X}}.$$

$\mathrm{L}^1(J, \mathrm{X})$ the space of all Bochner integrable functions from J into X associated the norm

$$\|x\|_1 = \int_a^b \|x(\omega)\|_{\mathrm{X}} d\omega.$$

$\mathrm{L}^\infty(J, \mathrm{X})$ the space of all Bochner functions from J into X bounded for almost everywhere associated the norm

$$\|x\|_\infty = \inf\{M > 0; \|x(\omega)\|_{\mathrm{X}} \leq M \ \text{ a.e.on } J\}.$$

For a given Banach space X, let

$\mathcal{P}(\mathrm{X})$ be the set of all subsets of X.

$\mathcal{P}_b(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ bounded in } \mathrm{X}\}$.

$\mathcal{P}_{cl}(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ closed in } \mathrm{X}\}$.

$\mathcal{P}_{cp}(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ compact in } \mathrm{X}\}$.

$\mathcal{P}_{c,cp}(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ convex and compact in } \mathrm{X}\}$.

1.2 Set valued map

Definition 1.2.1. *[23, 35] Let* X *and* Y *two sets.*
A set valued map (multivalued) from X *into* $\mathcal{P}(\mathrm{Y})$ *is an application, i.e. each* $x \in \mathrm{X}$ *associate subset in* Y.

Definition 1.2.2. *Let* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map.*

(a) The domain of F *is the set* $Dom(F)$ *where*

$$Dom(F) = \{x \in \mathrm{X}; F(x) \neq \emptyset\}.$$

(b) If $\mathrm{Y} = \mathrm{X}$, *the point* $x \in \mathrm{X}$ *is called a fixed point of* F *if* $x \in F(x)$.

(c) For all $M \subseteq \mathrm{X}$

$$F(M) = \underset{x \in M}{\cup} F(x).$$

(d) The graph of F *is defined to be the set*

$$G(F) = \{(x, y) \in \mathrm{X} \times \mathrm{Y}; y \in F(x)\}.$$

(e) For all $A \subseteq \mathrm{Y}$

$$F^{-1}(A) = \{x \in \mathrm{X}; F(x) \subseteq A\}.$$

Proposition 1.2.1. *Let* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map,* $(A_i)_{i \in I} \subseteq \mathcal{P}(\mathrm{Y})$. *Then*

$$F^{-1}(\underset{i \in I}{\cap} A_i) = \underset{i \in I}{\cap} F^{-1}(A_i).$$

Definition 1.2.3. *[35] Let* X, Y *are topological spaces,* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map. We say that* F *is an upper semicontinuous if for every open set* O *in* Y *the set* $F^{-1}(\mathrm{O})$ *is an open set in* X.

Example 1. *Let* $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ *continuous such that* $f_1 \leq f_2$, *then the set valued map* $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ *defined by*

$$F(x) = \big[f_1(x), f_2(x)\big]$$

is upper semicontinuous.

Proposition 1.2.2. *[35] Let* X, Y *are two topological spaces and* $\mathrm{A} \subseteq \mathrm{X}$*,* $F : A \to \mathrm{Y}$ *be a set valued map.*
If F *is upper semicontinuous, then for every* $x_0 \in \mathrm{A}$ *and* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that*

$$F(x) \subseteq F(x_0) + B_\varepsilon(0) \quad on\ \mathrm{A} \cap B_\delta(x_0),$$

where $B_\delta(x_0)$ *the ball with centered at a point* x_0 *and the radius* δ*. The converse is true if* F *has a compact values.*

Definition 1.2.4. *[9] Let* X, Y *are two normed spaces,* $F : \mathrm{X} \to \mathrm{Y}$ *be a set valued map. We say that* F *is Lipschitzean if there exist* $L > 0$ *such that*

$$F(x) \subseteq F(y) + B(0, L\|x - y\|) \quad for\ each\ x, y \in \mathrm{X}.$$

If the constant $L < 1$*, we say that the set valued map* F *is contraction.*

Proposition 1.2.3. *Let* X, Y *are two normed spaces,* $F : \mathrm{X} \to \mathrm{Y}$ *be a set valued map with compact values.*
If F *is Lipschitzean then* F *is upper semicontinuous.*

Proof. Let $x_0 \in \mathrm{X}$ and $\varepsilon > 0$, then there exist $\delta = \frac{\varepsilon}{L}$ such that

$$F(x) \subseteq F(x_0) + B_\varepsilon(0) \ \text{ for all } \ x \in B_\delta(x_0).$$

□

Proposition 1.2.4. *Let* X, Y *are two metric spaces and* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map.* F *has a closed graph if and only if for each* $(x_n) \subseteq \mathrm{X}$ *converge to* $x \in \mathrm{X}$ *and for each* (y_n) *such that* $y_n \in F(x_n)$ *converge to* y *we have* $y \in F(x)$*.*

Proof. the proof of this proposition is obvious. □

Definition 1.2.5. *Let* X, Y *be two topological spaces and* $F : \mathrm{X} \to \mathrm{Y}$ *be a set valued map.*

(i) F *is called completely continuous if for each* $B \in \mathcal{P}_b(\mathrm{X})$ *the set* $F(B)$ *is relatively compact in* Y*.*

(ii) F *is called compact if the set* $F(\mathrm{X})$ *is relatively compact in* Y*.*

Proposition 1.2.5. *[33] Let $F : \mathrm{X} \to \mathcal{P}_{cp}(\mathrm{Y})$ be a set valued map such that F is completely continuous. Then F is upper semicontinuous if and only if F has a closed graph.*

Definition 1.2.6. *[13] Let (X, d) be a metric space and $A, B \in \mathcal{P}_{cl}(\mathrm{X})$.*
The Hausdorff metric is an application $H_d : \mathcal{P}_{cl}(\mathrm{X}) \times \mathcal{P}_{cl}(\mathrm{X}) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$H_d(A, B) = \begin{cases} \max\{\sup_{a\in A} d(a, B); \sup_{b\in B} d(A, b)\} & \text{if} \quad \sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b) \in \mathbb{R}_+, \\ +\infty & \text{else.} \end{cases} \tag{1.1}$$

Proposition 1.2.6. *If $A, B \in \mathcal{P}_{cp}(\mathrm{X})$, then $H_d(A, B) \in \mathbb{R}_+$.*

Proof. For all $A, B \in \mathcal{P}_{cp}(X)$, the application $a \in A \mapsto d(a, B)$ is continuous. Then by theorem 1.13-6 in [21] there exist $a_0 \in A$ and $b_0 \in B$ such that

$$\sup_{a\in A} d(a, B) = d(a_0, B) = d(a_0, b_0) \in \mathbb{R}_+.$$

Then $H_d(A, B) \in \mathbb{R}_+$. □

Proposition 1.2.7. *Let X, Y are two normed spaces, and $F : \mathrm{X} \to \mathcal{P}_{cp}(\mathrm{Y})$. Then F is Lipschitzean if and only if there exist $L > 0$ such that for all $x, y \in \mathrm{X}$ we have*

$$H_d(F(x), F(y)) \leq Ld(x, y).$$

Proof. Let $x, y \in \mathrm{X}$ and F is Lipschitzean set valued map, then for each $a \in F(x)$ we have

$$d(a, F(y)) = \|a - b\| \leq L\|x - y\| \quad \textit{where } b \in F(y)$$

then

$$H_d(F(x), F(y)) \leq L\|x - y\|. \tag{1.2}$$

Conversely if F verify (1.2), then for all $a \in F(x)$ we have

$$d(a, F(y)) \leq H_d(F(x), F(y)) \leq L\|x - y\|.$$

After that there exist $b_0 \in F(y)$ such that

$$d(a, F(y)) = \|a - b_0\| \leq L\|x - y\|.$$

That means $a \in F(y) + B(0, L\|x - y\|)$, i.e. F is Lipschitzean. □

1.3 Fixed point theorem

Theorem 1.3.1. *[22] (Covitz-Nadler). Let* (X, d) *be a complete metric space. If* $T : \mathrm{X} \to \mathcal{P}_{cl}(\mathrm{X})$ *is a contraction, then* T *admits at least a fixed point, i.e., there exists* $u \in \mathrm{X}$ *such that* $u \in T(u)$*.*

Theorem 1.3.2. *[30] (nonlinear alternative of Leray-Schauder). Let* X *be a Banach space,* C *a closed convex subset of* X*,* U *an open subset of* C *with* $0 \in U$*. Suppose that* $T : \bar{U} \to \mathcal{P}_{cp,c}(C)$ *is an upper semicontinuous compact map. Then either* T *has a fixed point in* $\bar{U}$*, or there exist* $u \in \partial U$ *and* $\lambda \in (0, 1)$ *such that* $u \in \lambda T(u)$*.*

Theorem 1.3.3. *[44] Let* X *be a Banach space. Suppose that*

(i) the set valued map $T : K \subseteq \mathrm{X} \to \mathcal{P}(K)$*is upper semicontinuous,*

(ii) K *is a nonempty, compact, convex set in* X*,*

(iii) the set $T(u)$ *is closed, and convex for all* $u \in K$*.*

Then T *has a fixed point in* K*.*

1.4 Measurable multifunctions and measurable selection

Definition 1.4.1. *[23]. Let* $(\Omega, \mathfrak{A})$ *be a measurable space and* X *be a Banach space. A set valued map* $F : \Omega \to \mathcal{P}(\mathrm{X})$ *is called measurable if* $\{x \in \Omega; F(x) \cap B \neq \emptyset\} \in \mathfrak{A}$ *for each open set* B *in* X*.*

Definition 1.4.2. *[27]. Let* $f : [a, b] \to \mathbb{R}$*. We say that* f *is measurable if for each open set* U *in* $\mathbb{R}$*, we have*

$$f^{-1}(U) \in \mathcal{M}$$

where $\mathcal{M}$ *be a* $\sigma-$*algebra.*

Proposition 1.4.1. *[20]. Let* $(\Omega, \mathfrak{A})$ *be a measurable space and* X *be a separable Banach space. Let* $f : \Omega \to \mathrm{X}$ *be a measurable map and* $\iota : \Omega \to \mathbb{R}_+$ *be a measurable function. Then the set valued map* $\omega \in \Omega \mapsto B(f(\omega), \iota(\omega))$ *(the closed ball with center* $f(\omega)$ *and radius* $\iota(\omega)$*) is a measurable.*

Proposition 1.4.2. *[20]. Let $F_1, F_2 : \Omega \to \mathrm{Y}$ are set valued map with compact valued . Then the set valued map $\omega \in \Omega \mapsto F_1(\omega) \cap F_2(\omega)$ is a measurable.*

Proposition 1.4.3. *[20]. Let $F_1, F_2 : \Omega \to \mathrm{Y}$ are set valued map with compact valued. Then the set valued map $\omega \in \Omega \mapsto F_1(\omega) \cup F_2(\omega)$ is a measurable.*

Definition 1.4.3. *[35] Let $a, b \in \mathbb{R}$ and X, Y be two separable Banach spaces.*
A set valued map $F : [a, b] \times \mathrm{X} \to \mathcal{P}_{cp}(\mathrm{Y})$ is said to be Carathéodory if

(i) for every $x \in \mathrm{X}$ the set valued map $F(., x) : [a, b] \to \mathcal{P}(\mathrm{Y})$ is measurable,

(ii) for almost $\omega \in [a, b]$ the set valued map $F(\omega, .) : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ is upper semicontinuous.

Definition 1.4.4. *[35] A Carathéodory set valued map $F : [a, b] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ is said* L^1*-Carathéodory if*

(i) F is Carathéodory set valued map.

(ii) for each $l > 0$ there exist $\chi_l \in \mathrm{L}^1([a, b]; \mathbb{R})$ such that

$$\|F(\omega, x)\| \underset{def}{=} \sup\{v; v \in F(\omega, x)\} \leq \chi_l(\omega)$$

for all $|x_i| \leq l$ $(1 \leq i \leq n)$ and almost $\omega \in [0, 1]$.

Definition 1.4.5. *[35] Let $a, b \in \mathbb{R}$ and Y be a Banach space.*
A function $f : [a, b] \to \mathrm{Y}$ is said to be a measurable selection of a set valued map $F : [a, b] \to \mathcal{P}_{cp}(\mathrm{Y})$ if f is measurable and

$$f(\omega) \in F(\omega) \quad a.e.\ \omega \in [a, b].$$

Lemma 1.4.1. *[33] Let J be a subset in $\mathbb{R}^n$, Y is a complet separable space and $F : J \to \mathcal{P}_{cl}(\mathrm{Y})$ be a measurable set valued map. Then F admits a measurable selection.*

Proposition 1.4.4. *[35]. Let $F : [0, 1] \times \mathrm{X}_0 \to \mathcal{P}_{cp}(\mathrm{X})$ be a Carathéodory set valued map and*
$f : [0, 1] \to \mathrm{X}_0$ be a measurable function. Then the set valued map $\omega \in [0, 1] \mapsto F(\omega, f(\omega))$ is measurable.

Lemma 1.4.2. *Let $G : [0,1] \times \mathbb{R}^n \to \mathcal{P}_{c,cp}(\mathbb{R})$ be an L^1- Carathéodory set valued map and let a linear continuous θ from $\mathrm{L}^1([0,1];\mathbb{R})$ to $\mathcal{C}([0,1];\mathbb{R})$. Then the operator $\theta \circ S_G$ defined by $\theta \circ S_G(u) = \theta(S_{G,u})$ has a closed graph, where*

$$S_{G,u} = \big\{v \in \mathrm{L}^1([0,1];\mathbb{R} : v(\omega) \in G(\omega, u(\omega)) \ \ \textit{for almost } \omega \in [0,1])\big\}.$$

Definition 1.4.6. *Let $G : [0,1] \times \mathbb{R}^n \to \mathcal{P}_{c,cp}(\mathbb{R})$, we say that G is integrably bounded if there exist $\phi \in \mathrm{L}^1([0,1];\mathbb{R})$ such that for any $v \in S_{G,u}$ we have*

$$|v(\omega)| \le \phi(\omega) \ \ \textit{for almost } \omega \in [0,1]$$

Lemma 1.4.3. *[46] Let $F : [0,1] \to \mathcal{P}(\mathbb{R})$ be a measurable set valued map and $x : [0,1] \to \mathbb{R}$ a measurable function. Then for every measurable function $y : [0,1] \to \mathbb{R}^+$ there exist a function v such that*

$$\begin{aligned} v(\omega) \in F(\omega) \ \ &\textit{for all} \ \ \omega \in [0,1], \\ |v(\omega) - x(\omega)| \le d(x(\omega), F(\omega)) + y(\omega) \ \ &\textit{for almost} \ \ \omega \in [0,1]. \end{aligned}$$

1.5 Riemann-Liouville integral and derivative

Definition 1.5.1. *The Euler gamma function noted by $\Gamma(.)$ is defined by*

$$\Gamma(t) = \int_0^{+\infty} \omega^{t-1} \exp(-\omega) d\omega, \qquad t > 0$$

Remark 1.5.1. *For all $t > 0$ we have $\ \ t\Gamma(t) = \Gamma(t+1)$.*

Definition 1.5.2. *[40, 42] Let $\varsigma > 0$ and $\varphi \in \mathrm{L}^1([0,1];\mathbb{R})$. The following integral*

$$I_{0+}^{\varsigma}\varphi(t) = \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1}\varphi(s) ds,$$

is named the Riemann–Liouville integral of order ς.

Lemma 1.5.1. *The fractional integration operators I_{0+}^{ς} with $\varsigma > 0$ are bounded from $\mathcal{C}([0,1];\mathbb{R})$ into $\mathcal{C}([0,1];\mathbb{R})$, and for each $f \in \mathcal{C}([0,1];\mathbb{R})$ we have*

$$\|I_{0+}^{\varsigma} f\| \le \frac{1}{\Gamma(\varsigma+1)} \|f\|.$$

Proof. For any $f \in \mathcal{C}([0,1];\mathbb{R})$ we have

$$\begin{aligned} |I_{0+}^{\varsigma} f(\omega)| &= \frac{1}{\Gamma(\varsigma)} \Big| \int_0^{\omega} (\omega - s)^{\varsigma-1} f(s) ds \Big| \\ &\leq \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - s)^{\varsigma-1} |f(s)| ds \\ \leq \frac{\|f\|}{\Gamma(\varsigma)} \times \int_0^{\omega} (\omega - s)^{\varsigma-1} ds &= \frac{\|f\|}{\Gamma(\varsigma)} \times \frac{\omega^{\varsigma}}{\varsigma}, \end{aligned}$$

then

$$\|I_{0+}^{\varsigma} f\| \leq \frac{\|f\|}{\Gamma(\varsigma + 1)},$$

□

Definition 1.5.3. *[40, 42] Let φ a function given on the interval $[0,1]$. The fractional derivative of Riemann-Liouville of order $\varsigma > 0$ is defined by*

$$\begin{aligned} \mathcal{D}_{0+}^{\varsigma} \varphi(\omega) &= \frac{1}{\Gamma(n - \varsigma)} \Big(\frac{d}{d\omega}\Big)^n \int_0^{\omega} (\omega - s)^{n-\varsigma-1} \varphi(s) ds, \\ &= \Big(\frac{d}{d\omega}\Big)^n I_{0+}^{n-\varsigma} \varphi(\omega), \end{aligned}$$

where $n = [\varsigma] + 1$ and $[\varsigma]$ denotes the greatest integer number less than ς.

Corollary 1. *[40] Let $\varsigma > 0$ and $n = [\varsigma] + 1$.*
The equality $\mathcal{D}_{0+}^{\varsigma} \varphi(\omega) = 0$ is valid if and only if

$$\varphi(\omega) = \sum_{j=1}^{n} c_j \omega^{\varsigma-j}$$

Lemma 1.5.2. *[36] Let $u \in \mathrm{L}^1(0,1)$ and $\delta_1 > \delta_2 > 0$. Then,*

(i) $I_{0+}^{\delta_1} I_{0+}^{\delta_2} u(\omega) = I_{0+}^{\delta_1+\delta_2} u(\omega)$,

(ii) $\mathcal{D}_{0+}^{\delta_2} I_{0+}^{\delta_1} u(\omega) = I_{0+}^{\delta_1-\delta_2} u(\omega)$,

(iii) $\mathcal{D}_{0+}^{\delta_1} I_{0+}^{\delta_1} u(\omega) = u(\omega)$.

Lemma 1.5.3. *For a given $f \in \mathrm{L}^1([0,1];\mathbb{R})$ the solution of*

$$\mathcal{D}_{0+}^{\varsigma} \varphi(\omega) = f(\omega),$$

is given by

$$\varphi(\omega) = I^{\varsigma}_{0+} f(\omega) + \sum_{j=1}^{n} c_j \omega^{\varsigma - j},$$

where $n = [\varsigma] + 1$.

Proof. From precedent lemma, we have $f(\omega) = \mathcal{D}^{\varsigma}_{0+} I^{\varsigma}_{0+} f(\omega)$, then writing

$$\mathcal{D}^{\varsigma}_{0+} \varphi(\omega) = f(\omega)$$

is equivalent to

$$\mathcal{D}^{\varsigma}_{0+} (\varphi(\omega) - I^{\varsigma}_{0+} f(\omega)) = 0.$$

Then

$$\varphi(\omega) = I^{\varsigma}_{0+} f(\omega) + \sum_{j=1}^{n} c_j \omega^{\varsigma - j}.$$

$\square$

Lemma 1.5.4. *[36] If* $\varsigma > 0$ *and* $\mu > 0$*, then*

(i) $\mathcal{D}^{\varsigma}_{0+} \omega^{\mu-1} = \begin{cases} \frac{\Gamma(\mu)}{\Gamma(\mu-\varsigma)} \omega^{\mu-\varsigma-1}, & if \quad \varsigma - \mu > 0 \\ 0, & if \quad \mu - \varsigma \in \mathbb{Z}^- \end{cases}$

(ii) $I^{\varsigma}_{0+} \omega^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\varsigma+1)} \omega^{\mu+\varsigma}$.

1.6 Some properties of functional analysis

Lemma 1.6.1. *Let* K *be a compact set in a Banach space* $(\mathrm{X}, \|.\|)$*. Then for each* $x \in \mathrm{X}$ *there exist* $y_0 \in K$ *such that*

$$d(x, K) = \inf_{y \in K} \|x - y\| = \|x - y_0\|.$$

Proof. Let $x \in \mathrm{X}$, by definition of inf, for each $n \geq 1$ there exist $y_n \in K$ such that

$$\inf_{y \in K} \|x - y\| \leq \|x - y_n\| < \inf_{y \in K} \|x - y\| + \frac{1}{n}$$

and by definition of the compact set, there exist a subsequence $(y_{n_k}) \in K$ such that (y_{n_k}) converge to $y_0 \in K$ and

$$\inf_{y \in K} \|x - y\| \leq \|x - y_{n_k}\| < \inf_{y \in K} \|x - y\| + \frac{1}{n_k},$$

that means

$$\inf_{y\in K}\|x-y\| = \|x-y_0\|.$$

$\square$

Theorem 1.6.1. *(Dunford - Pettis)[18] Let $\mathcal{A}$ be a bounded set in $\mathrm{L}^1([0,\infty[,\mathbb{R})$. Then $\mathcal{A}$ is weakly compact in $\mathrm{L}^1([0,\infty[,\mathbb{R})$ if and only if*

(i) for all $\varepsilon>0$ there exist $\delta>0$ such that for all $f\in\mathcal{A}$

$$\int_A |f| < \varepsilon \quad \textit{for } |A|<\delta,$$

(ii) for all $\varepsilon>0$ there exist $A\subset[0,\infty[$ measurable such that for all $f\in\mathcal{A}$

$$|A|<\infty \quad \textit{and} \int_{[0,\infty[-A} |f| < \varepsilon.$$

Definition 1.6.1. *(equicontinuous set)[18]. Let A be a subset of $\mathcal{C}([a,b],\mathbb{R})$, we say that A is equicontinuous set if for every $\varepsilon>0$ there exist $\delta>0$ such that*

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon \quad \textit{for all} \quad f\in A.$$

Theorem 1.6.2. *(Ascoli- Arzelà) [18]. Let K be a bounded subset of $\mathcal{C}([a,b],\mathbb{R})$ and equicontinuous set. Then the closure of K is compact in $\mathcal{C}([a,b],\mathbb{R})$.*

Theorem 1.6.3. *[18]. Let A be a bounded set in $\mathrm{L}^1(\mathbb{R})$. Assume that*

$$\lim_{h\to 0}\|\tau_h f - f\|_1 = 0 \quad \textit{for each} \quad f\in A.$$

Then the closure of $A_{|[0,1]}$ in $\mathrm{L}^1([0,1])$ is compact in $\mathrm{L}^1([0,1])$, where $\tau_h f(\omega)=f(\omega+h)$.

Theorem 1.6.4. *[18]. Let (f_n) be a sequence in $\mathrm{L}^1([0,1])$ and let $f\in\mathrm{L}^1([0,1])$ be such that*

$$\|f_n-f\|_1\to 0.$$

Then, there exist a subsequence (f_{n_k}) such that

$$f_{n_k}(\omega)\to f(\omega) \quad \textit{a.e. on } [0,1].$$

Definition 1.6.2. *[21] Let* X *be a Banach space, and let* $\mathcal{A}$ *be a subset in* X, *we called* $\mathcal{A}$ *is weakly compact if* $\mathcal{A}$ *compact set in weak topology* $\sigma(\mathrm{X}, \mathrm{X}')$.

Proposition 1.6.1. *[21] Let* X *be a Banach space. A set* $\mathcal{A}$ *is weakly compact if and only if for any sequence* (x_n) *in* $\mathcal{A}$ *contains a subsequence* (x_{n_k}) *that weakly converge in* X.

Proposition 1.6.2. *[35]. Let* $\mathcal{A} \subset \mathrm{L}^1([0,1];\mathbb{R})$ *such that*

(i) $\mathcal{A}(\omega)$ *are relatively compact for a.e* $\omega \in [0,1]$.

(ii) There exist $h \in \mathrm{L}^1([0,1];\mathbb{R})$ *such that*

$$v(\omega) \leq h(\omega) \quad \text{for } v \in \mathcal{A} \text{ and almost } \omega \in [0,1].$$

Then $\mathcal{A}$ *is weakly compact in* $\mathrm{L}^1([0,1];\mathbb{R})$.

Chapter 2

Existence results for semilinear fractional differential inclusions

2.1 Existence results for semilinear fractional differential inclusions 1

Let

$$\begin{cases} \mathcal{D}_{0^+}^{\varsigma} z(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{A}_1 z(\omega), \ldots, \mathcal{A}_n z(\omega)), & \text{for almost } \omega \in [0,1], \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{\mu_1} z(1) = p I_{0^+}^{\mu_2} \mathrm{g}(1, z(1)), \end{cases} \tag{2.1}$$

where $1 < \varsigma < 2$, p, $\mu_2 > 0$ and $0 \leq \mu_1 \leq \varsigma - 1$, $\mathrm{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}(\mathbb{R})$ is a multivalued, $I_{0^+}^{\mu_2}$ is Riemann-Liouville fractional integral of order μ_2, $\mathcal{A}_j$ is an operator (generally not necessary linear) defined on the space of continuous functions on $[0,1]$ into itself for all $1 \leq j \leq n$.

Lemma 2.1.1. *For a given $v \in \mathrm{L}^1([0,1], \mathbb{R})$, a function z is a solution to*

$$\begin{cases} \mathcal{D}_{0^+}^{\varsigma} z(\omega) = v(\omega), & \omega \in [0,1], 1 < \varsigma < 2, \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{\mu_1} z(1) = p I_{0^+}^{\mu_2} \mathrm{g}(1, z(1)), \end{cases} \tag{2.2}$$

if and only if

$$
\begin{aligned}
z(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-\mu_1-1}v(\Lambda)d\Lambda \\
&\quad -\frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\mu_2-1}\mathrm{g}(\Lambda,z(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}. \qquad (2.3)
\end{aligned}
$$

Proof. Assume that z satisfies (2.2), by lemma 1.5.3 we have

$$
z(\omega) = I_{0+}^{\varsigma}v(\omega) - c_1\omega^{\varsigma-1} - c_2\omega^{\varsigma-2}, \qquad (2.4)
$$

where $c_1, c_2 \in \mathbb{R}$.

By using first boundary condition, we get that $c_2 = 0$, furthermore we require an additional approach for handling the second boundary condition

$$
\mathcal{D}_{0+}^{\mu_1}z(1) = I_{0+}^{\varsigma-\mu_1}v(1) - \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\mu_1)}c_1 = pI_{0+}^{\mu_2}\mathrm{g}(1,z(1)), \qquad (2.5)
$$

this means that

$$
\begin{aligned}
c_1 &= \frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-\mu_1-1}v(\Lambda)d\Lambda \\
&\quad -\frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\mu_2-1}\mathrm{g}(\Lambda,z(\Lambda))d\Lambda.
\end{aligned}
$$

By replacing c_1 in (2.4) we get (2.3).

Conversely, if z satisfies (2.3) by lemma 1.5.2 and lemma 1.5.4, we have $\mathcal{D}_{0+}^{\varsigma}z(\omega) = v(\omega)$, and

$$
\mathcal{D}_{0+}^{\mu_1}z(1) = pI_{0+}^{\mu_2}\mathrm{g}(1,z(1)),
$$

and we can easily to see that $z(0) = 0$. □

Let $\mathbb{X} = \mathcal{C}([0,1],\mathbb{R})$ the space of continuous functions on $[0,1]$ equipped with the norm

$$
\|z\|_{\mathbb{X}} = \max_{\omega\in[0,1]}|z(\omega)|.
$$

It's evident $(\mathbb{X}, \|.\|_{\mathbb{X}})$ is a complete space.

Let

$$
S_{\mathcal{F},z} = \{v \in \mathrm{L}^1([0,1];\mathbb{R}) : v(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{A}_1z(\omega), \dots, \mathcal{A}_nz(\omega))), \quad \text{for almost } \omega \in [0,1]\}.
$$

2.1.1 The case where $\mathcal{F}$ is convex

Now we prove the existence of solutions for the problem with a convex valued $\mathcal{F}$, by applying a fixed point theorem for set valued maps.

Theorem 2.1.1. *Supposons that the following hypothesis* $(A1)-(A4)$ *are met.*

(A1) $\mathcal{F}:[0,1]\times\mathbb{R}^{n+1}\to\mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory set valued.*

(A2) There exist $f_1,f_2\in\mathrm{L}^\infty([0,1],\mathbb{R}^+)$ *and* $\varphi,\psi_1,\ldots,\psi_n:\mathbb{R}_+\to\mathbb{R}_+^*$ *continuous, nondecreasing functions with*

$$\begin{aligned}\|\mathcal{F}(\omega,x,x_1,\ldots,x_n)\| &= \sup\{|v|:v\in\mathcal{F}(\omega,x,x_1,\ldots,x_n)\}\\ &\le f_1(\omega)\varphi(|x|)+f_2(\omega)\sum_{j=1}^{n}\psi_j(|x_j|),\end{aligned}$$

for almost $\omega\in[0,1]$.

(A3) The operators $\mathcal{A}_j$, $j=1,\ldots,n$, *are bounded, i.e., transform all bounded set into bounded set.*

(A4) There are $\sigma,\bar{\sigma}>0$ *such that*

$$\|\mathcal{A}_j z\|_{\mathbb{X}}\le\bar{\sigma}\quad\forall z\in\bar{B}_{\mathbb{X}}(0,\sigma),\ 1\le j\le n,$$

and

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma+\left(\|f_1\|_\infty\varphi(\sigma)+\|f_2\|_\infty\sum_{j=1}^{n}\psi_j(\bar{\sigma})\right)\mathcal{Q}}>1,$$

where

$$\mathcal{Q}=\frac{1}{\Gamma(\varsigma+1)}+\frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)},$$

$$\tilde{\mathcal{Q}}_\sigma=\frac{\alpha_\sigma p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)},$$

with $\alpha_\sigma=\max\left\{|\mathrm{g}(\omega,x)|,(\omega,x)\in[0,1]\times[-\sigma,\sigma]\right\}$.

Then the problem (2.1) *has at least one solution.*

Lemma 2.1.2. *The first and the second conditions of this theorem implies that the set of selections of $\mathcal{F}$ defined by $S_{\mathcal{F},z}$ is non-empty.*

Proof. From $(A1)$ and by proposition 1.4.4 the set valued $\mathcal{F}$ has a selection measurable noted by v, and $v \in S_{\mathcal{F},z}$ for all $z \in \mathbb{X}$. $\square$

Proof. **of Theorem 2.1.1** Define an operator $\mathcal{N} : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ by

$$\begin{aligned}\mathcal{N}(z) = \Big\{ h \in \mathbb{X} : h(\omega) = &\frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda \\ &- [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma - \mu_1 - 1} v(\Lambda) d\Lambda \\ &- \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\mu_2 - 1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda] \omega^{\varsigma - 1}, v \in S_{\mathcal{F},z} \Big\}.\end{aligned} \tag{2.6}$$

We will show that $\mathcal{N}$ satisfies all hypotheses of the theorem 1.3.2. The proof will be given in five steps.

Step 1. ($\mathcal{N}(z)$ is convex valued). Let $z \in \mathbb{X}$, and $h_1, h_2 \in \mathcal{N}(z)$, then there exist $v_1, v_2 \in S_{\mathcal{F},z}$ such that for $i \in \{1, 2\}$ we have

$$\begin{aligned} h_i(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v_i(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma - \mu_1 - 1} v_i(\Lambda) d\Lambda \\ &\quad - \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\mu_2 - 1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda] \omega^{\varsigma - 1}. \end{aligned}$$

For $\lambda \in [0, 1]$ we have

$$\begin{aligned} \lambda h_1(\omega) + (1 - \lambda) h_2(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} \big(\lambda v_1(\Lambda) + (1 - \lambda) v_2(\Lambda)\big) d\Lambda \\ &\quad - \Big[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma - \mu_1 - 1} \big(\lambda v_1(\Lambda) + (1 - \lambda) v_2(\Lambda)\big) d\Lambda \\ &\quad - \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\mu_2 - 1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda \Big] \omega^{\varsigma - 1}. \end{aligned}$$

Since $\mathcal{F}$ is convex set valued map, then $\lambda h_1 + (1 - \lambda) h_2 \in \mathcal{N}(z)$.

Step 2. ($\mathcal{N}$ maps bounded sets into bounded sets in $\mathbb{X}$). Let $B_{\mathrm{r}} = \{z \in \mathbb{X}; \|z\|_{\mathbb{X}} < \mathrm{r}\}$. We necessitate an alternative approach to establish the existence of a constant $l > 0$ such that for every $z \in B_{\mathrm{r}}$, one has $\|h\|_{\mathbb{X}} \leq l$ for $h \in \mathcal{N}(z)$.

Let $z \in B_r$ and $h \in \mathcal{N}(z)$, then there exists $v \in S_{\mathcal{F},z}$ with

$$\begin{aligned} h(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v(\Lambda) d\Lambda \right. \\ &\quad \left. - \frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda \right] \omega^{\varsigma-1}, \end{aligned}$$

by $(A3)$ there exist $R > 0$ such that

$$\|\mathcal{A}_j z\|_{\mathbb{X}} \leq R.$$

By simple calculations, we have

$$\begin{aligned} |h(\omega)| &\leq \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} |v(\Lambda)| d\Lambda + \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} |v(\Lambda)| d\Lambda \\ &\quad + \frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2-1} |\mathrm{g}(\Lambda, z(\Lambda))| d\Lambda. \end{aligned} \tag{2.7}$$

Then

$$\begin{aligned} \|h\|_{\mathbb{X}} &\leq l = \left[\frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)} \right] \times \left[\|f_1\|_\infty \varphi(r) + \|f_2\|_\infty \sum_{j=1}^n \psi_j(R) \right] \\ &\quad + \frac{\alpha_{\mathrm{r}} p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}. \end{aligned} \tag{2.8}$$

Step 3. ($\mathcal{N}$ maps bounded sets into equicontinuous sets in $\mathbb{X}$). Let B_{r} be a bounded set in $\mathbb{X}$ as in Step 2. Let $0 \leq t_1 \leq t_2 \leq 1$ and $z \in B_{\mathrm{r}}$. For each $h \in \mathcal{N}(z)$, then there exists $v \in S_{\mathcal{F},z}$ such that

$$\begin{aligned} & |h(t_2) - h(t_1)| \\ \leq & \left| \int_{t_1}^{t_2} \frac{(t_2-\Lambda)^{\varsigma-1} v(\Lambda)}{\Gamma(\varsigma)} d\Lambda + \int_0^{t_1} \frac{[(\omega_2-\Lambda)^{\varsigma-1} - (\omega_1-\Lambda)^{\varsigma-1}] v(\Lambda)}{\Gamma(\varsigma)} d\Lambda \right. \\ & + \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v(\Lambda) d\Lambda - \frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda \right] \\ & \left. \times \left(t_2^{\varsigma-1} - t_1^{\varsigma-1} \right) \right| \\ \leq & \left[\|f_1\|_\infty \varphi(\mathrm{r}) + \|f_2\|_\infty \sum_{j=1}^n \psi_j(R) \right] \left[\frac{\left| t_2^\varsigma - t_1^\varsigma \right|}{\Gamma(\varsigma+1)} + \frac{\left| t_2^{\varsigma-1} - t_1^{\varsigma-1} \right|}{(\varsigma-\mu_1)\Gamma(\varsigma)} \right] \\ & + \frac{\alpha_{\mathrm{r}} p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)} \left(\left| t_2^{\varsigma-1} - t_1^{\varsigma-1} \right| \right). \end{aligned}$$

Consequently

$$|h(t_2) - h(t_1)| \to 0 \quad as \quad t_2 \to t_1.$$

Step 4. ($\mathcal{N}$ upper semicontinuous). Let (z_m) be set of element in $\mathbb{X}$ with $z_m \to z_*$ and (h_m) be a sequence converges to h_* with $h_m \in \mathcal{N}(z_m)$ for any $m \geq 1$, we need to show that $h_* \in \mathcal{N}(z_*)$. Now $h_m \in \mathcal{N}(z_m)$ implies that there exists $v_m \in S_{\mathcal{F},z_m}$ such that

$$\begin{aligned} h_m(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1} v_m(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v_m(\Lambda)d\Lambda \\ &\quad -\frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu_2-1}\mathrm{g}(\Lambda, z_m(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}. \end{aligned}$$

We prove that there exists $v_* \in S_{\mathcal{F},z_*}$ such that for every $\omega \in [0,1]$,

$$\begin{aligned} h_*(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1} v_*(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v_*(\Lambda)d\Lambda \\ &\quad -\frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu_2-1}\mathrm{g}(\Lambda, z_*(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}. \end{aligned}$$

Let us consider the continuous linear operator $\theta : \mathrm{L}^1([0,1],\mathbb{R}) \to \mathcal{C}([0,1],\mathbb{R})$ given by

$$(\theta v)(\omega) = \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1} v(\Lambda)d\Lambda - \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v(\Lambda)d\Lambda.$$

Thus, by lemma 1.4.2 it follows that $\theta \circ S_{\mathcal{F}}$ is a closed graph operator.

Further, we have

$$h_m(\omega) - \left(\frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu_2-1}\mathrm{g}(\Lambda, z_m(\Lambda))d\Lambda\right)\omega^{\varsigma-1} \in \theta(S_{\mathcal{F},z_m}).$$

Since $z_m \to z_*$, therefore, we have

$$\begin{aligned} h_*(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1} v_*(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v_*(\Lambda)d\Lambda \\ &\quad -\frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu_2-1}\mathrm{g}(\Lambda, z_*(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}, \quad v_* \in S_{\mathcal{F},z_*}. \end{aligned}$$

Since $\mathcal{N}$ closed graph and completely continuous, then $\mathcal{N}$ upper semicontinuous

Step 5. (a priori bounds on solutions). Let $z \in \mathbb{X}$ such that $z \in \lambda\mathcal{N}(z)$ for $\lambda \in (0,1)$. Then for every $\omega \in [0,1]$ we have

$$\begin{aligned} |z(\omega)| &\leq \left[\frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right] \times \left[\|f_1\|_\infty \varphi(\|z\|_{\mathbb{X}}) + \|f_2\|_\infty \sum_{j=1}^{n} \psi_j(\|\mathcal{A}_j z\|_{\mathbb{X}})\right] \\ &\quad + \frac{\alpha_{\|z\|} p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}. \end{aligned} \tag{2.9}$$

Let $\mathcal{V} = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < \sigma\}$.

It is distinctly that $\mathcal{V}$ is an open set of $\mathbb{X}$ and $0 \in \mathcal{V}$. From step 2, step 3 and step 4 we can conclude that the restriction of $\mathcal{N}$ in $\bar{\mathcal{V}}$ is upper semicontinuous, completely continuous and has a compact convex values. From the choice of $\mathcal{V}$, for all $z \in \partial\mathcal{V}$ and $\lambda \in (0,1)$ and by fourth condition in theorem we get $z \notin \lambda\mathcal{N}(z)$. Therefore, by theorem 1.3.2, we deduce that $\mathcal{N}$ has a fixed point $z \in \bar{\mathcal{V}}$, which represents a solution of the problem 2.1. This finishes the proof. □

Remark 2.1.1. *If we remplace the hypothesis* $(A3)$ *and* $(A4)$ *by*

$(A3')$ *The operator* $\mathcal{A}_j$ *is linear continuous, for every* $1 \leq j \leq n$.

$(A4')$ *There are* $\sigma > 0$ *such that*

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma + \left(\|f_1\|_\infty \varphi(\sigma) + \sum_{j=1}^{n} \psi_j(\sigma \times \max_{1\leq j\leq n} \|\mathcal{A}_j\|)\|f_2\|_\infty\right)\mathcal{Q}} > 1,$$

the resultat of theorem (2.1.1) *still valid.*

Corollary 2. *Let us assume that the following statement be true.*

(A1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory set valued map.*

(A2) There exist $f_1, f_2 \in \mathrm{L}^\infty([0,1], \mathbb{R}^+)$ *and* $\varphi, \psi_1, \ldots, \psi_n : \mathbb{R}_+ \to \mathbb{R}_+^*$ *continuous, nondecreasing functions with*

$$\begin{aligned} \|\mathcal{F}(\omega, x, x_1, \ldots, x_n)\| &= \sup\{|v| : v \in \mathcal{F}(\omega, x, x_1, \ldots, x_n)\} \\ &\leq f_1(\omega)\varphi(|x|) + f_2(\omega)\sum_{j=1}^{n} \psi_j(|x_j|), \end{aligned}$$

for almost $\omega \in [0,1]$.

$(A3')$ *The operators* $\mathcal{A}_j$, $j = 1, \ldots, n$ *are linear continuous.*

$(A4')$ *There are* $\sigma > 0$ *with*

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma + \left(\|f_1\|_\infty \varphi(\sigma) + \sum_{j=1}^{n} \psi_j(\sigma \times \max_{1\leq j\leq n} \|\mathcal{A}_j\|)\|f_2\|_\infty\right)\mathcal{Q}} > 1,$$

where

$$\mathcal{Q} = \frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)},$$

$$\tilde{\mathcal{Q}}_\sigma = \frac{\alpha_\sigma p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)},$$

with $\alpha_\sigma = \max\left\{|\mathrm{g}(\omega,x)|, (\omega,x) \in [0,1]\times[-\sigma,\sigma]\right\}$.

Then there is at least one solution to problem (2.1).

Proof. For all $1 \leq j \leq n$, the operator $\mathcal{A}_j$ convert all bounded sets to bounded set, and we put: $\bar{\sigma}_j = \|\mathcal{A}_j\| \times \sigma$, and $\bar{\sigma} = \max\limits_{1\leq j\leq n} \bar{\sigma}_j$. Then by theorem 2.1.1 the problem (2.1) has at least one solution. □

Example 2. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}^{1.7}_{0^+} z(\omega) \in \mathcal{F}(\omega, z(\omega), I^{0.1}_{0^+} z(\omega), I^{0.3}_{0^+} z(\omega)), \quad \omega \in [0,1], \\ z(0) = 0, \\ \mathcal{D}^{0.2}_{0^+} z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 (1-\Lambda)^3 (\Lambda + \frac{z(\Lambda)}{4}) d\Lambda, \end{cases} \tag{2.10}$$

with

$$\mathcal{F}(\omega,x,x_1,x_2) = \left[\frac{\omega}{101}\sin(|x|) + \omega\frac{\ln(6|x_1|+1)}{4} + \frac{1+x_2^2}{212}; |x|\frac{\exp(\omega)}{101} + \exp(\omega)\frac{\ln(6|x_1|+1)}{4} + \frac{1+x_2^2}{212}\right].$$

In this problem, we get

$$\|\mathcal{F}(\omega,x,x_1,x_2)\| = \sup\{|v|, v \in \mathcal{F}(\omega,x,x_1,x_2)\} \leq \frac{\exp(\omega)}{101}|x| + \exp(\omega)\left(\frac{\ln(6|x_1|+1)}{4} + \frac{1+x_2^2}{212}\right).$$

It is evident that $\mathcal{F}$ *has convex compact valued and of* L^1*-Carathéodory type.*
Let $f_1(\omega) = f_2(\omega) = \exp(\omega)$, $\varphi(|x|) = \frac{|x|}{101}$, $\psi_1(|x_1|) = \frac{\ln(6|x_1|+1)}{4}$, $\psi_2(|x_2|) = \frac{1+x_2^2}{212}$.
It is obvious that $\|\mathcal{A}_1\| = \frac{1}{\Gamma(1.1)}$ *and* $\|\mathcal{A}_2\| = \frac{1}{\Gamma(1.3)}$.
Taking $\sigma = 11$, *then* $\tilde{\mathcal{Q}}_\sigma \approx 0.6094$, $\mathcal{Q} \approx 1.3811$, *and* $\|f_1\|_\infty = \|f_2\|_\infty \approx 2.7183$.
Then we obtain

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma + \mathcal{Q}\left(\|f_1\|_\infty \varphi(\sigma) + \|f_2\|_\infty \sum_{j=1}^{2} \psi_j(\sigma \times \max\limits_{1\leq j\leq 2}\|\mathcal{A}_j\|)\right)} \approx 1.4231.$$

Consequently, by corollary 2 the considered problem (2.1) *has at least one solution.*

In the theorem above, the boundedness condition of operators $\mathcal{A}_j$, $j = 1, \ldots, n$ is replaced by a boundedness condition on the functions ψ_j, $j = 1, \ldots, n$.

Theorem 2.1.2. *Let us assume that the following statement $(B1) - (B4)$ be true.*

(B1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory set valued map.*

(B2) There exist $f_1, f_2 \in \mathrm{L}^\infty([0,1],[0;\infty))$ *and* $\varphi, \psi_1, \ldots, \psi_n : [0,+\infty) \to (0,+\infty)$ *continuous functions with* φ *is nondecreasing function such that*

$$\begin{aligned} \|\mathcal{F}(\omega, x, x_1, \ldots, x_n)\| &= \sup\{|v| : v \in \mathcal{F}(\omega, x, x_1, \ldots, x_n)\} \\ &\leq f_1(\omega)\varphi(|x|) + f_2(\omega)\sum_{j=1}^{n} \psi_j(|x_j|), \end{aligned}$$

for almost $\omega \in [0,1]$.

(B3) The functions ψ_j, $j = 1, ..., n$ *are bounded i.e., there are* $c_1, \ldots, c_n > 0$ *such that for every* $x \in \mathbb{R}$ *we have* $|\psi_j(x)| \leq c_j$ *for* $1 \leq j \leq n$.

(B4) There exists $\sigma > 0$ *such that*

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma + \left(\|f_1\|_\infty \varphi(\sigma) + \sum_{j=1}^{n} c_j \|f_2\|_\infty\right)\mathcal{Q}} > 1,$$

where

$$\mathcal{Q} = \frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)},$$
$$\tilde{\mathcal{Q}}_\sigma = \frac{\alpha_\sigma p \Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)},$$

with $\alpha_\sigma = \max\left\{|\mathrm{g}(\omega,x)|, (\omega,x) \in [0,1] \times [-\sigma,\sigma]\right\}$.

Then there is at least one solution to problem (2.1).

Proof. The proof is slightly different from the proof of theorem 2.1.1. Namely, in step 2, instead of (2.8), we use the estimation

$$\begin{aligned} |h(\omega)| \leq \; & \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}|v(\Lambda)|d\Lambda + \frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1}|v(\Lambda)|d\Lambda \\ & + \frac{p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu_2-1}|\mathrm{g}(\Lambda, z(\Lambda))|d\Lambda, \end{aligned}$$

then

$$\|h\|_{\mathbb{X}} \leq \left[\frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right] \times \left[\|f_1\|_\infty \varphi(r) + \sum_{j=1}^{n} c_j \|f_2\|_\infty\right] + \frac{\alpha_r p \Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)} = l,$$

and in step 5, instead of (2.9) we use the estimation

$$|z(\omega)| \leq \left[\frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right] \times \left[\|f_1\|_\infty \varphi(\|z\|) + \sum_{j=1}^{n} c_j \|f_2\|_\infty\right] + \frac{\alpha_{\|z\|} p \Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}.$$

□

Example 3. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.7} z(\omega) \in \mathcal{F}(\omega, z(\omega), I_{0^+}^{0.1} z(\omega), I_{0^+}^{0.3} z(\omega)), \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{0.2} z(1) = \frac{4}{\Gamma(4)} \int_0^1 (1-\Lambda)^3 (\Lambda + \frac{z(\Lambda)}{4}) d\Lambda, \end{cases} \tag{2.11}$$

where

$$\mathcal{F}(\omega, x, x_1, x_2) = \left[\frac{\omega}{10} \sin(|x|) + \omega \exp(-6x_1^2) + \frac{1}{2+x_2^2}; |x| \frac{\exp(\omega)}{10} + \exp(\omega - 6x_1^2) + \frac{1}{1+x_2^2}\right].$$

In this problem, we have

$$\|\mathcal{F}(\omega, x, x_1, x_2)\| = \sup\{|v|, v \in \mathcal{F}(\omega, x, x_1, x_2)\} \leq \frac{\exp(\omega)}{10}|x| + \exp(\omega)\left(\exp(-6x_1^2) + \frac{1}{1+x_2^2}\right).$$

The set valued map $\mathcal{F}$ is convex compact valued and of L^1*-Carathéodory.*
Let $f_1(\omega) = \frac{\exp(\omega)}{10}$, $f_2(\omega) = \exp(\omega)$, $\varphi(|x|) = |x|$, $\psi_1(|x_1|) = \exp(-6x_1^2)$, $\psi_2(|x_2|) = \frac{1}{1+x_2^2}$ and $c_1 = c_2 = 1$.
Taking $\sigma = 17$, so $\tilde{\mathcal{Q}}_\sigma \approx 0.8534$, $\mathcal{Q} \approx 1.3811$, $\|f_1\|_\infty \approx 0.2718$ and $\|f_2\|_\infty \approx 2.7183$. Then we obtain

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma + \mathcal{Q}\left(\|f_1\|_\infty \varphi(\sigma) + \sum_{j=1}^{2} c_j \|f_2\|_\infty\right)} \approx 1.1531.$$

Consequently, by theorem 2.1.2 the considered problem (2.11) *has at least one solution.*

Remark 2.1.2. *If we replace the hypothesis* $(B4)$ *by*

(B4′) *There exists* $\alpha, \beta > 0$ *with*

$$|\mathrm{g}(\omega, x)| \leq \alpha, \text{ for } (\omega, x) \in [0,1] \times \mathbb{R},$$

and

$$|\varphi(x)| \leq \beta, \text{ for } x \in \mathbb{R}^{+},$$

the result of theorem 2.1.2 still true. Indeed, it is clear that the boundedness of the function g *on* $[0,1] \times \mathbb{R}$ *suffices to realize the hypothesis* $(B4)$*. Then we state the following corollary.*

Corollary 3. *Assume that*

(B1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory set valued map.*

(B2) *There exist* $f_1, f_2 \in \mathrm{L}^{\infty}([0,1],[0;+\infty))$ *and* $\varphi, \psi_1, \ldots, \psi_n : \mathbb{R}_+ \to \mathbb{R}_+^*$ *continuous functions with* φ *is nondecreasing such that*

$$\begin{aligned} \|\mathcal{F}(\omega, x, x_1, \ldots, x_n)\| &= \sup\{|v| : v \in \mathcal{F}(\omega, x, x_1, \ldots, x_n)\} \\ &\leq f_1(\omega)\varphi(|x|) + f_2(\omega)\sum_{j=1}^{n} \psi_j(|x_j|). \end{aligned}$$

(B3) *The functions* φ_j*,* $j = 1, \ldots, n$ *are bounded i.e., there exist* $c_1, \ldots, c_n > 0$ *such that for all* $x \in \mathbb{R}$ *we have* $|\psi_j(x)| \leq c_j$ *for* $x_i \in \mathbb{R}$ *and for almost* $\omega \in [0,1]$*.*

(B4′) *There exists* $\alpha, \beta > 0$ *with*

$$|\mathrm{g}(\omega, x)| \leq \alpha, \text{ for } (\omega, x) \in [0,1] \times \mathbb{R},$$

and

$$|\varphi(x)| \leq \beta, \text{ for } x \in [0;+\infty).$$

Then there is at least one solution to problem (2.1)*.*

Example 4. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.7} z(\omega) \in \mathcal{F}(\omega, z(\omega), I_{0^+}^{0.1} z(\omega), I_{0^+}^{0.3} z(\omega)), \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{0.2} z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 (1-\Lambda)^3 \left(\frac{5\cos(z(\Lambda))}{(7+\Lambda)^2}\right) d\Lambda, \end{cases} \tag{2.12}$$

where

$$\mathcal{F}(\omega, x, x_1, x_2) = \left[\omega \arctan(|x|) + \omega \exp(-6x_1^2) + \frac{1}{2+x_2^2}; \frac{\pi}{2}\exp(\omega) + \exp(\omega - 6x_1^2) + \frac{1}{1+x_2^2}\right].$$

In this problem, we have

$$\|\mathcal{F}(\omega, x, x_1, x_2)\| = \sup\{|v|, v \in \mathcal{F}(\omega, x, x_1, x_2)\} \leq \exp(\omega)(\frac{\pi}{2} + \exp(-6x_1^2) + \frac{1}{1+x_2^2}).$$

It is evident that $\mathcal{F}$ is convex compact valued and of L^1*-Carathéodory type.*
Let $f_1(\omega) = f_2(\omega) = \exp(\omega)$, $\varphi(|x|) = \frac{\pi}{2}$, $\psi_1(|x_1|) = \exp(-6x_1^2)$, $\psi_2(|x_2|) = \frac{1}{1+x_2^2}$.
Then the hypotheses of corollary 3 are fulfilment. Therefore, (3.9) *have a solution.*

2.1.2 The case where $\mathcal{F}$ is nonconvex

Now we study the existence solution of the problem (2.1) when the set valued map $\mathcal{F}$ is not necessarily convex valued.

Theorem 2.1.3. *Let us assume that the following statement valid.*

(C1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp}(\mathbb{R})$ *is a multivalued with*

1. *$\mathcal{F}$ is integrably bounded and the set valued map $\omega \in [0,1] \mapsto \mathcal{F}(\omega, x, x_1, \ldots, x_n)$ is measurable for $x, x_i \in \mathbb{R}$ with $1 \leq i \leq n$;*
2. *there exist $f \in \mathrm{L}^\infty([0,1], \mathbb{R}^+)$ such that for almost $\omega \in [0,1]$ and $x, y, x_l, y_l \in \mathbb{R}$ with $1 \leq l \leq n$,*
   $H_d(\mathcal{F}(\omega, x, x_1, \ldots, x_n), \mathcal{F}(\omega, y, y_1, \ldots, y_n)) \leq f(\omega)\left[|x-y| + \sum_{j=1}^{n} |x_j - y_j|\right].$

(C2) There are $\theta > 0$ with for $\omega \in [0,1]$ and $x, y \in \mathbb{R}$, we have

$$|\mathrm{g}(\omega, x) - \mathrm{g}(\omega, y)| \leq \theta|x-y|.$$

(C3) The operator $\mathcal{A}_j$ is σ_j-Lipschitz for all $1 \leq j \leq n$.

(C4) $\left[\left(\frac{\|f\|_\infty}{\Gamma(\varsigma+1)} + \frac{\|f\|_\infty}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right) \times \left(1 + \sum_{j=1}^{n} \sigma_j\right) + \frac{\theta p \Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}\right] < 1.$

Then there exist a solution for the problem (2.1).

Proof. From $(C1)$ for almost $\omega \in [0,1]$ the set valued map $\mathcal{F}(\omega, .)$ is upper semicontinuous, then from proposition 7.9 page 229 in [33] the set valued map $\omega \in [0,1] \mapsto \mathcal{F}(\omega, z(\omega), \mathcal{A}_1 z(\omega), \dots, \mathcal{A}_n z(\omega))$ is measurable and from lemma 1.4.1 there exist a measurable function h such that

$h(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{A}_1 z(\omega), \dots, \mathcal{A}_n z(\omega))$ for $z \in \mathbb{X}$, then $h \in S_{\mathcal{F},z}$. Define the set valued map $\mathcal{N}$ by

$$\mathcal{N}(z) = \Bigg\{ h \in \mathbb{X} : h(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma - \mu_1 - 1} v(\Lambda) d\Lambda - \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2 - 1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda] \omega^{\varsigma - 1}, v \in S_{\mathcal{F},z} \Bigg\}, \tag{2.13}$$

we will show the set valued map investigated theorem 1.3.1.

Step 1. $(\mathcal{N}(y) \in \mathcal{P}_{cl}(\mathbb{X}))$. Let (y_n) be set of element in $\mathcal{N}(y)$ with $y_n \to \tilde{y}$ in $\mathbb{X}$. Then $\tilde{y} \in \mathbb{X}$ and there exists $v_n \in S_{\mathcal{F},y}$ with for $\omega \in [0,1]$,

$$y_n(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_n(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma - \mu_1 - 1} v_n(\Lambda) d\Lambda - \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2 - 1} \mathrm{g}(\Lambda, y(\Lambda)) d\Lambda \right] \omega^{\varsigma - 1}.$$

Let $\mathcal{V} = \{v_n \in S_{\mathcal{F},z}; n \geq 1\}$, since $\mathcal{F}$ has a compact values, and by proposition 1.6.2 there exist a subsequence to obtain that v_{n_k} converges weakly to v in $\mathrm{L}^1([0,1];\mathbb{R})$. Therefore $v \in S_{\mathcal{F},z}$ and for $\omega \in [0,1]$

$$\frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_{n_k}(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v_{n_k}(\Lambda) d\Lambda \right] \omega^{\varsigma-1} \longrightarrow \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v(\Lambda) d\Lambda \right] \omega^{\varsigma-1},$$

then for every $\omega \in [0,1]$, we have

$$\tilde{y}(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma - \mu_1 - 1} v(\Lambda) d\Lambda - \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2 - 1} \mathrm{g}(\Lambda, y(\Lambda)) d\Lambda \right] \omega^{\varsigma - 1}.$$

Hence $\tilde{y} \in \mathcal{N}(y)$.

Step 2. ($\mathcal{N}$ is contraction.) We prove that there exist $0 < \gamma < 1$ with

$$H_d(\mathcal{N}(y), \mathcal{N}(\tilde{y})) \leq \gamma \|y - \tilde{y}\|_{\mathbb{X}} \qquad y, \tilde{y} \in \mathbb{X}.$$

Let $y, \tilde{y} \in \mathbb{X}$ and $h_1 \in \mathcal{N}(y)$, there exists a function measurable v_1 such that

$$\begin{aligned}
v_1(\omega) &\in \mathcal{F}(\omega, y(\omega), \mathcal{A}_1 y(\omega), \ldots, \mathcal{A}_n y(\omega)) \quad \text{for almost } \omega \in [0,1], \qquad (2.14)\\
h_1(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v_1(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v_1(\Lambda) d\Lambda \right.\\
&\left. - \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2-1} \mathrm{g}(\Lambda, y(\Lambda)) d\Lambda \right] \omega^{\varsigma-1} \quad \text{for all } \omega \in [0,1].
\end{aligned}$$

By $(C1)$, we have

$$\begin{aligned}
& H_d\Big(\mathcal{F}(\omega, y(\omega), \mathcal{A}_1 y(\omega), \ldots, \mathcal{A}_n y(\omega)); \mathcal{F}(\omega, \tilde{y}(\omega), \mathcal{A}_1 \tilde{y}(\omega), \ldots, \mathcal{A}_n \tilde{y}(\omega))\Big)\\
\leq\ & f(\omega)\left(|y(\omega) - \tilde{y}(\omega)| + \sum_{j=1}^{n} |\mathcal{A}_j y(\omega) - \mathcal{A}_j \tilde{y}(\omega)|\right).
\end{aligned}$$

Then by proposition 1.4.4 the set valued map $\omega \mapsto \mathcal{F}(\omega, \mathcal{A}_1 z(\omega), \ldots, \mathcal{A}_n z(\omega))$ is measurable, from lemma 1.4.3 has a measurable selection noted by v_2 such that

$$|v_1(\omega) - v_2(\omega)| \leq f(\omega)\left(|y(\omega) - \tilde{y}(\omega)| + \sum_{j=1}^{n} |\mathcal{A}_j y(\omega) - \mathcal{A}_j \tilde{y}(\omega)|\right), \quad \omega \in [0,1].$$

For $\omega \in [0,1]$, define the element

$$\begin{aligned}
h_2(\omega) = &\frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v_2(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} v_2(\Lambda) d\Lambda\\
&- \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2-1} \mathrm{g}(\Lambda, \tilde{y}(\Lambda)) d\Lambda] \omega^{\varsigma-1} \text{ for a.e. } \omega \in [0,1].
\end{aligned}$$

Thus,

$$\begin{aligned}
|h_1(\omega) - h_2(\omega)| \leq &\frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} |v_1(\omega) - v_2(\omega)| d\Lambda\\
&+ \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\mu_1-1} |v_1(\omega) - v_2(\omega)| d\Lambda\\
&+ \frac{p\Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu_2-1} |\mathrm{g}(\Lambda, y(\Lambda)) - \mathrm{g}(\Lambda, \tilde{y}(\Lambda))| d\Lambda\\
&\leq \left[\left(\frac{\|f\|_\infty}{\Gamma(\varsigma+1)} + \frac{\|f\|_\infty}{(\varsigma - \mu_1)\Gamma(\varsigma)}\right) \times \left(1 + \sum_{j=1}^{n} \sigma_j\right) + \frac{\theta p \Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}\right] \|y - \tilde{y}\|_{\mathbb{X}}.
\end{aligned}$$

Then

$$\|h_1 - h_2\|_{\mathbb{X}} \leq \left[\left(\frac{\|f\|_\infty}{\Gamma(\varsigma+1)} + \frac{\|f\|_\infty}{(\varsigma - \mu_1)\Gamma(\varsigma)}\right) \times \left(1 + \sum_{j=1}^{n} \sigma_j\right) + \frac{\theta p \Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}\right] \|y - \tilde{y}\|_{\mathbb{X}}.$$

Analogously, interchanging the roles of y and $\tilde{y}$, we get

$$H_d(\mathcal{N}(y), \mathcal{N}(\tilde{y})) \leq \gamma \|y - \tilde{y}\|_{\mathbb{X}}, \quad y, \tilde{y} \in \mathbb{X},$$

with

$$\gamma = \left[\left(\frac{\|f\|_\infty}{\Gamma(\varsigma+1)} + \frac{\|f\|_\infty}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right) \times \left(1 + \sum_{j=1}^{n} \sigma_j\right) + \frac{\theta p \Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}\right].$$

That means $\mathcal{N}$ is a contraction.

From step 1 and step 2 and theorem 1.3.1 the set valued map $\mathcal{N}$ has a fixed point, this point represent a solution of (2.1). The proof is fulfilled. □

Corollary 4. *Assume that the conditions (C1) and (C2) of the precedent theorem are satisfied, and we assume that*

$(C3')$ *$\mathcal{A}_j$ is linear and bounded operator for all $1 \leq j \leq n$.*

$(C4')$ $\left[\left(\frac{\|f\|_\infty}{\Gamma(\varsigma+1)} + \frac{\|f\|_\infty}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right) \times \left(1 + \sum_{j=1}^{n} \|\mathcal{A}_j\|\right) + \frac{\theta p \Gamma(\varsigma - \mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}\right] < 1.$

Then there is at least one solution to problem 2.1.

Proof. The operator $\mathcal{A}_j$ is $\|A_j\|$-Lipschitz, by precedent theorem the problem (2.1) has solution in $\mathbb{X}$. □

2.1.3 Application

Let

$$\begin{cases} \mathcal{D}_{0^+}^{\varsigma} z(\omega) \in \mathcal{F}(\omega, z(\omega), I_{0^+}^{\delta_1} z(\omega), \ldots, I_{0^+}^{\delta_n} z(\omega)), & \text{for almost } \omega \in [0,1], \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{\mu_1} z(1) = p I_{0^+}^{\mu_2} \mathrm{g}(1, z(1)), \end{cases} \tag{2.15}$$

where $1 < \varsigma < 2, p, \mu_2 > 0$ and $0 \leq \mu_1 \leq \varsigma - 1$, $\mathrm{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}(\mathbb{R})$ is a multivalued, $I_{0^+}^{\mu_2}, I_{0^+}^{\delta_1}, \ldots, I_{0^+}^{\delta_n}$ is Riemann-Liouville fractional integral of order $\mu_2, \delta_1, \ldots, \delta_n$.

Theorem 2.1.4. *Assume that the following assertions $(A1), (A2), (A3)$ and $(A4)$ hold.*

(A1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a* L^1*-Carathéodory set valued map.*

(A2) There exist $f_1, f_2 \in \mathrm{L}^\infty([0,1], \mathbb{R}^+)$ *and* $\varphi, \psi_1, \ldots, \psi_n : \mathbb{R}_+ \to \mathbb{R}_+^*$ *continuous, nondecreasing functions such that*

$$\begin{aligned} \|\mathcal{F}(\omega, x, x_1, \ldots, x_n)\| &= \sup\{|v| : v \in \mathcal{F}(\omega, x, x_1, \ldots, x_n)\} \\ &\leq f_1(\omega)\varphi(|x|) + f_2(\omega) \sum_{j=1}^{n} \psi_j(|x_j|), \end{aligned}$$

$(A3)$ *There exists* $L > 0$ *such that*

$$\frac{L}{\tilde{\mathcal{Q}}_L + \left(\|f_1\|_\infty \varphi(L) + \sum_{j=1}^{n} \varphi_j\left(\frac{L}{\min\limits_{1\leq j\leq n} \Gamma(\delta_j+1)}\right)\|f_2\|_\infty \right)\mathcal{Q}} > 1,$$

where

$$\mathcal{Q} = \frac{1}{\Gamma(\varsigma+1)} + \frac{1}{(\varsigma-\mu_1)\Gamma(\varsigma)},$$

$$\tilde{\mathcal{Q}}_L = \frac{\alpha_L p \Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)},$$

with $\alpha_L = \max\left\{|\mathrm{g}(\omega, x)|, (\omega, x) \in [0,1] \times [-L, L]\right\}$.

Then there is at least one solution to problem (2.15).

Proof. From corollary 2 the problem (2.15) has a solution when we take $\mathcal{A}_j = I_{0^+}^{\delta_j}$. □

Theorem 2.1.5. *Let us assume that the following statement are true.*

(C1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp}(\mathbb{R})$ *is a set valued map such that*

1. $\mathcal{F}$ *is integrably bounded and the set valued map* $\omega \to \mathcal{F}(\omega, x, x_1, \ldots, x_n)$ *is measurable for* $x, x_l \in \mathbb{R}$ *with* $1 \leq l \leq n$ *;*
2. *there exist* $f \in \mathrm{L}^\infty([0,1], \mathbb{R}^+)$ *with for almost* $\omega \in [0,1]$
   and $x, y, x_l, y_l \in \mathbb{R}$ *with* $1 \leq l \leq n$,
   $H_d(\mathcal{F}(\omega, x, x_1, \ldots, x_n), \mathcal{F}(\omega, y, y_1, \ldots, y_n)) \leq f(\omega)\left[|x-y| + \sum_{j=1}^{n} |x_j - y_j|\right].$

(C2) There exist $\theta > 0$ *with for* $\omega \in [0,1]$ *and* $a, b \in \mathbb{R}$, *we have*

$$|\mathrm{g}(\omega, a) - \mathrm{g}(\omega, b)| \leq \theta |a - b|.$$

(C4) $\left[\left(\frac{\|f\|_\infty}{\Gamma(\varsigma+1)}+\frac{\|f\|_\infty}{(\varsigma-\mu_1)\Gamma(\varsigma)}\right)\times\left(1+\sum_{j=1}^{n}\frac{1}{\Gamma(\delta_j+1)}\right)+\frac{\theta p\Gamma(\varsigma-\mu_1)}{\Gamma(\mu_2+1)\Gamma(\varsigma)}\right]<1.$

Then there is at least one solution to problem (2.15).

2.2 Existence results for semilinear fractional differential inclusions 2

Now we study the existence of the semilinear fractional differential inclusions for multi-term boundary value problem (2.16) with the following data:

$$\begin{cases}\mathcal{D}^{\varsigma}_{0^+}z(\omega)\in\mathcal{F}(\omega,z(\omega),\mathcal{D}^{\gamma_1}_{0^+}z(\omega),\ldots,\mathcal{D}^{\gamma_n}_{0^+}z(\omega)), & \text{for almost } \omega\in[0,1],\\ z(0)=0,\\ \mathcal{D}^{\nu}_{0^+}z(1)=pI^{\mu}_{0^+}\mathrm{g}(1,z(1)),\end{cases}\tag{2.16}$$

with $1<\varsigma<2, p,\mu\geq 0$, $\mathrm{g}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $\mathcal{F}:[0,1]\times\mathbb{R}^{n+1}\to\mathcal{P}(\mathbb{R})$ is a set valued map, $0=\gamma_0<\gamma_1<\cdots<\gamma_n<1$, $\varsigma>\gamma_n+1$ and $0\leq\nu<\gamma_n$.

Consider $\mathcal{C}([0,1],\mathbb{R})$ the space of all continuous functions defined on $[0,1]$ and define the space $\mathbb{X}=\{z: z,\mathcal{D}^{\gamma_1}_{0^+}z,\ldots,\mathcal{D}^{\gamma_n}_{0^+}z\in\mathcal{C}([0,1],\mathbb{R})\}$,
$(0<\gamma_1<\cdots<\gamma_n<1)$ equipped with the norm

$$\|z\|_{\mathbb{X}}=\max_{\omega\in[0,1]}|z(\omega)|+\sum_{j=1}^{n}\max_{\omega\in[0,1]}|\mathcal{D}^{\gamma_j}_{0^+}z(\omega)|.$$

Lemma 2.2.1. $(\mathbb{X},\|.\|_{\mathbb{X}})$ *is a Banach space.*

Proof. Let (z_m) be a sequence in $\mathbb{X}$ such that for every $\varepsilon>0$ there exist $m_0\in\mathbb{N}$, for any $m\geq m_0$ we have for every $p\in\mathbb{N}$.

$$\|z_{m+p}-z_m\|_{\mathbb{X}}<\varepsilon.\tag{2.17}$$

For $m\geq m_0$, and $p\in\mathbb{N}$

$$\|z_{m+p}-z_m\|_\infty<\varepsilon$$

the space $\mathcal{C}([0,1],\mathbb{R})$ is a Banach space, then there exist $z\in\mathcal{C}([0,1],\mathbb{R})$ such that

$$\|z_m-z\|_\infty\to 0,$$

and

$$\begin{aligned}\left|I_{0^+}^{1-\gamma_j} z_m(\omega) - I_{0^+}^{1-\gamma_j} z(\omega)\right| &= \left|I_{0^+}^{1-\gamma_j}(z_m - z)(\omega)\right| = \left|\frac{1}{\Gamma(1-\gamma_j)}\int_0^\omega (\omega-\Lambda)^{-\gamma_j}(z_m - z)(\Lambda)d\Lambda\right| \\ &\leq \frac{1}{\Gamma(1-\gamma_j)}\int_0^\omega (\omega-\Lambda)^{-\gamma_j}\|z_m - z\|_\infty d\Lambda = \frac{\omega^{1-\gamma_j}\|z_m - z\|_\infty}{(1-\gamma_j)\Gamma(1-\gamma_j)} \\ &\leq \frac{\|z_m - z\|_\infty}{\Gamma(2-\gamma_j)},\end{aligned}$$

that means

$$\|I_{0^+}^{1-\gamma_j} z_m - I_{0^+}^{1-\gamma_j} z\|_\infty \leq \frac{\|z_m - z\|_\infty}{\Gamma(2-\gamma_j)},$$

then

$$\|I_{0^+}^{1-\gamma_j} z_m - I_{0^+}^{1-\gamma_j} z\|_\infty \to 0 \quad \text{for } 1 \leq j \leq n,$$

from (2.17), there exist $v_j \in \mathcal{C}([0,1],\mathbb{R})$ such that for every $m \geq m_0$

$$\|\mathcal{D}_{0^+}^{\gamma_j} z_m - v_j\|_\infty < \varepsilon$$

Since $\mathcal{D}_{0^+}^{\gamma_j} z_m = \frac{d}{d\omega} I_{0^+}^{1-\gamma_j} z_m$ and by theorem 2.5 page 92 in [12] we get $v_j = \mathcal{D}_{0^+}^{\gamma_j} z$.
That means $\mathbb{X}$ is Banach space. □

For all $z \in \mathbb{X}$, define the set of selections of the set valued map $\mathcal{F}$ by

$$S_{\mathcal{F},z} = \{v \in \mathrm{L}^1([0,1],\mathbb{R}) : v(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{D}_{0^+}^{\gamma_1} z(\omega), \ldots, \mathcal{D}_{0^+}^{\gamma_n} z(\omega)) \ \text{ for almost } \omega \in [0,1]\}. \tag{2.18}$$

Theorem 2.2.1. *Let A be a subset of $\mathbb{X}$ such that*

(i) A is bounded,i.e, there exist $M > 0$ such that for all $f \in A$

$$\|f\|_{\mathbb{X}} \leq M.$$

(ii) A equicontinuous,i.e, given any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that for all $t, s \in [0,1]$ we have

$$|t-s| < \delta(\varepsilon) \Rightarrow |f(t) - f(s)| + \sum_{j=1}^n |\mathcal{D}_{0^+}^{\gamma_j} f(t) - \mathcal{D}_{0^+}^{\gamma_j} f(s)| < \varepsilon$$

for all $f \in A$.

Then the set A is relatively compact set in $\mathbb{X}$.

Proof. Let $\varepsilon > 0$, by property (ii) there exist $\delta(\varepsilon) > 0$ such that for all $t, s \in [0,1]$ we have

$$|t-s| < \delta(\varepsilon) \Rightarrow |f(t) - f(s)| + \sum_{j=1}^{n} |\mathcal{D}_{0^+}^{\gamma_j} f(t) - \mathcal{D}_{0^+}^{\delta_j} f(s)| < \frac{\varepsilon}{4}.$$

Since $[0,1] \subset \underset{t\in[0,1]}{\cup}]t-\delta(\varepsilon), t+\delta(\varepsilon)[$ and $[0,1]$ is a compact set in $\mathbb{R}$, then there exist a finite number of points $t_l \in [0,1] (1 \leq l \leq p)$ such that

$$[0,1] \subset \overset{p}{\underset{l=1}{\cup}}]t_l - \delta(\varepsilon), t_l + \delta(\varepsilon)[.$$

Second there exist a finite number of points $y_m \in \mathbb{R}$ $(1 \leq m \leq q)$ such that
$-M = y_1 < y_2 < \cdots < y_q = M$ and $y_{m+1} - y_m < \frac{\varepsilon}{2}$, $1 \leq m \leq q-1$,
where M is the constant appearing in property (i).
Let then $\{\sigma_j; 1 \leq j \leq n\}$ denote the fini te set formed by all the applications from the set $\{1, \ldots, p\}$ into the set $\{1, \ldots, q\}$ and let the subsets $A_j (1 \leq j \leq n)$ of A be defined by

$$A_j = \{f \in A; |f(t_l) - y_{\sigma_j(l)}| + \sum_{j=1}^{n} |\mathcal{D}_{0^+}^{\gamma_j} f(t_l) - y_{\sigma_j(l)}| < \frac{\varepsilon}{4}, 1 \leq l \leq p\}.$$

Then we claim that

$$A \subset \overset{n}{\underset{j=1}{\cup}} A_j \quad \text{and} \quad diam A_j \leq \varepsilon \quad \text{(see [21] page 24)}.$$

To prove our assertion let $f \in A$ be any function in A. Since

$$|f(t_l)| + \sum_{j=1}^{n} |\mathcal{D}_{0^+}^{\gamma_j} f(t_l)| \leq M,$$

there exist for all $l \in \{1, \ldots, p\}$ an integer $k = k(f,l) \in \{1, \ldots, q\}$ such that

$$|f(t_l) - y_{k(f,l)}| + \sum_{j=1}^{n} |\mathcal{D}_{0^+}^{\gamma_j} f(t_l) - y_{k(f,l)}| \leq \frac{\varepsilon}{4}.$$

Then by definition of the application σ_j, there exist an integer $j(f) \in \{1, \ldots, n\}$ such that the application $\sigma_{j(f)}(l) = k(f,l)$, or equivalently such that $f \in A_{\sigma_j(f)}$. Hence $A \subset \overset{n}{\underset{j=1}{\cup}} A_j$.
Given any integer $j \in \{1, \ldots, n\}$, and let $f, g \in A_j$ and $t \in [0,1]$.

Since $[0,1] \subset \bigcup_{l=1}^{p}]t_l - \delta(\varepsilon), t_l + \delta(\varepsilon)[$, there exist $l = l(t)$ such that $t \in]t_l - \delta(\varepsilon), t_l + \delta(\varepsilon)[$. Then

$$
\begin{aligned}
|f(t) - g(t)| + \sum_{j=1}^{n} |\mathcal{D}_{0+}^{\gamma_j} f(t) - \mathcal{D}_{0+}^{\gamma_j} g(t)| &\le |f(t) - f(t_l)| + \sum_{j=1}^{n} |\mathcal{D}_{0+}^{\gamma_j} f(t) - \mathcal{D}_{0+}^{\gamma_j} f(t_l)| \\
+|f(t_l) - y_{k(f,l)}| + \sum_{j=1}^{n} |\mathcal{D}_{0+}^{\gamma_j} f(t_l) - y_{k(f,l)}| &+ |g(t_l) - y_{k(f,l)}| + \sum_{j=1}^{n} |\mathcal{D}_{0+}^{\gamma_j} g(t_l) - y_{k(f,l)}| \\
&+|g(t) - g(t_l)| + \sum_{j=1}^{n} |\mathcal{D}_{0+}^{\gamma_j} g(t) - \mathcal{D}_{0+}^{\gamma_j} g(t_l)| < \varepsilon.
\end{aligned}
$$

Hence $diam A_j < \varepsilon$, and the proof is complete. □

2.2.1 The case where $\mathcal{F}$ is convex

Theorem 2.2.2. *Supposons that the following assertions* $(B1), (B2)$ *and* $(B3)$ *hold*

(B1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is a Carathéodory set valued map.*

(B2) There exist $f \in \mathrm{L}^{\infty}([0,1],[0;+\infty))$ *and* $\psi_0, \dots, \psi_n : \mathbb{R}_+ \to \mathbb{R}_+^*$ *continuous, nondecreasing with*

$$
\|\mathcal{F}(\omega, \xi_0, \dots, \xi_n)\| = \sup\{|v| : v \in \mathcal{F}(\omega, \xi_0, \dots, \xi_n)\} \le f(\omega) \sum_{i=0}^{n} \psi_i(|\xi_i|),
$$

for $\xi_i \in \mathbb{R}$ *and almost* $\omega \in [0,1]$.

(B3) There are $\sigma > 0$ *with*

$$
\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma + (\sum_{j=0}^{n} \psi_j(\sigma)) \|f\|_\infty \mathcal{Q}} > 1,
$$

with

$$
\mathcal{Q} = \sum_{i=0}^{n} \left(\frac{1}{\Gamma(\varsigma - \gamma_i + 1)} + \frac{1}{(\varsigma - \nu)\Gamma(\varsigma - \gamma_i)} \right),
$$

$$
\tilde{\mathcal{Q}}_r = \sum_{j=0}^{n} \frac{\alpha_r p \Gamma(\varsigma - \nu)}{\Gamma(\mu + 1)\Gamma(\varsigma - \gamma_j)},
$$

for $r > 0$, *such that* $\alpha_r = \max\left\{|\mathrm{g}(\omega, \xi)|, (\omega, \xi) \in [0,1] \times [-r, r]\right\}$.

Then the problem (2.16) *has at least one solution.*

Lemma 2.2.2. *The first and the second conditions of this theorem implies that the set of selections of $\mathcal{F}$ defined by $S_{\mathcal{F},z}$ is non-empty.*

Proof. From (B1) and proposition 1.4.4 the set valued map $\omega \in [0,1] \mapsto \mathcal{F}(\omega, x_0, \dots, x_n)$ is measurable, by lemma 1.4.1 has a selection measurable noted by h.
By second condition $(B2)$, we have

$$|h(\omega)| \leq f(\omega) \sum_{j=0}^{n} \psi_j(\|z\|),$$

that means $h \in \mathrm{L}^1([0,1];\mathbb{R})$, then $S_{\mathcal{F},z}$ is non-empty. $\square$

Proof. of Theorem 2.2.2. Define an operator $\mathcal{N} : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ by

$$\begin{aligned} \mathcal{N}(z) = \Big\{ h \in \mathbb{X} : h(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\nu-1} v(\Lambda) d\Lambda \\ &- \frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda] \omega^{\varsigma-1}, v \in S_{\mathcal{F},z} \Big\}. \end{aligned} \tag{2.19}$$

We will show that $\mathcal{N}$ satisfies all hypotheses of theorem 1.3.2. The proof will be given in five steps.

Step 1. ($\mathcal{N}(z)$ is convex valued).

Let $z \in \mathbb{X}$ and $h_1, h_2 \in \mathcal{N}(z)$, then there exist $v_1, v_2 \in S_{\mathcal{F},z}$ such that

$$\begin{aligned} h_i(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_i(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\nu-1} v_i(\Lambda) d\Lambda. \\ &\quad -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda] \omega^{\varsigma-1} \end{aligned}$$

for $i \in \{1, 2\}$. Then for $\lambda \in [0,1]$

$$\begin{aligned} \big(\lambda h_1 + (1-\lambda) h_2\big)(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} \big(\lambda v_1 + (1-\lambda) v_2\big)(\Lambda) d\Lambda \\ &- [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\nu-1} \big(\lambda v_1 + (1-\lambda) v_2\big)(\Lambda) d\Lambda \\ &- \frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda\big] \omega^{\varsigma-1}. \end{aligned}$$

Since $\mathcal{F}$ is convex set valued map, then $\lambda h_1 + (1-\lambda) h_2 \in S_{\mathcal{F},z}$, then, it is clear that $\mathcal{N}(z)$ is convex for $z \in \mathbb{X}$.

Step 2. ($\mathcal{N}$ set valued map convert any bounded sets to bounded sets).
Let $B_r = \{z \in \mathbb{X}; \|z\|_{\mathbb{X}} < r\}$. We prove that there are a constant $l > 0$ such that for any $z \in B_r$, one has $\|h\|_{\mathbb{X}} \leq l$ for all $h \in \mathcal{N}(z)$. Let $z \in B_r$ and $h \in \mathcal{N}(z)$, then there are $v \in S_{\mathcal{F},z}$ with

$$\begin{aligned} h(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma-\nu-1} v(\Lambda) d\Lambda \right. \\ & \left. - \frac{p\Gamma(\varsigma - \nu)}{\Gamma(\mu)\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\mu-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda \right] \omega^{\varsigma-1}. \end{aligned}$$

By simple calculation, we find

$$\begin{aligned} |\mathcal{D}_{0^+}^{\gamma_j} h(\omega)| &\leq \frac{1}{\Gamma(\varsigma - \gamma_j)} \int_0^\omega (\omega - \Lambda)^{\varsigma-\gamma_j-1} |v(\Lambda)| d\Lambda + \frac{1}{\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\varsigma-\nu-1} |v(\Lambda)| d\Lambda \\ & + \frac{p\Gamma(\varsigma - \nu)}{\Gamma(\mu)\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\mu-1} |\mathrm{g}(\Lambda, z(\Lambda))| d\Lambda \\ &\leq \left[\frac{1}{\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\varsigma-\gamma_j-1} d\Lambda + \frac{1}{\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\varsigma-\nu-1} d\Lambda \right] \|f\|_\infty \sum_{j=1}^n \psi_j(r) \\ & + \frac{\alpha_r p\Gamma(\varsigma - \nu)}{\Gamma(\mu)\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\mu-1} d\Lambda, \end{aligned}$$

where $0 \leq j \leq n$ and $\gamma_0 = 0$. Hence we get

$$\|h\|_{\mathbb{X}} \leq \tilde{\mathcal{Q}}_r + \|f\|_\infty \sum_{i=0}^n \psi_i(r) \mathcal{Q} = l \quad (\text{ a constant }).$$

Step 3. ($\mathcal{N}$ set valued map convert any bounded set to equicontinuous sets). Let B_r be a bounded set of $\mathbb{X}$ as in step 2, and let $0 \leq t_1 \leq t_2 \leq 1$ and $z \in B_r$. For each $h \in \mathcal{N}(z)$, then there exists $v \in S_{\mathcal{F},z}$ such that

$$\begin{aligned} & |\mathcal{D}_{0^+}^{\gamma_j} h(t_2) - \mathcal{D}_{0^+}^{\gamma_j} h(t_1)| \\ & \leq \int_{t_1}^{t_2} \frac{(t_2 - \Lambda)^{\varsigma-\gamma_j-1} v(\Lambda)}{\Gamma(\varsigma - \gamma_j)} d\Lambda + \int_0^{t_1} \frac{[(t_2 - \Lambda)^{\varsigma-\gamma_j-1} - (t_1 - \Lambda)^{\varsigma-\gamma_j-1}] v(\Lambda)}{\Gamma(\varsigma - \gamma_j)} d\Lambda \\ & + \left[\frac{1}{\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\varsigma-\nu-1} v(\Lambda) d\Lambda - \frac{p\Gamma(\varsigma - \nu)}{\Gamma(\mu)\Gamma(\varsigma - \gamma_j)} \int_0^1 (1 - \Lambda)^{\mu-1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda \right] \times \\ & \left(t_2^{\varsigma-\gamma_j-1} - t_1^{\varsigma-\gamma_j-1} \right) \\ & \leq \|f\|_\infty \sum_{j=0}^n \psi_j(r) \left[\frac{\left| t_2^{\varsigma-\gamma_j} - t_1^{\varsigma-\gamma_j} \right|}{\Gamma(\varsigma - \gamma_j + 1)} + \frac{\left| t_2^{\varsigma-\gamma_j-1} - t_1^{\varsigma-\gamma_j-1} \right|}{(\varsigma - \nu)\Gamma(\varsigma - \gamma_j)} \right] \\ & + \frac{\alpha_{\|z\|} p\Gamma(\varsigma - \nu)}{\Gamma(\mu + 1)\Gamma(\varsigma - \gamma_j)} \left(\left| t_2^{\varsigma-\gamma_j-1} - t_1^{\varsigma-\gamma_j-1} \right| \right). \end{aligned}$$

where $0 \le j \le n$ and $\gamma_0 = 0$. Therefore

$$|\mathcal{D}^{\gamma_j}_{0+}h(t_2) - \mathcal{D}^{\gamma_j}_{0+}h(t_1)| \to 0 \ \ as \ \ t_2 \to t_1.$$

From theorem 2.2.1 the closure of $\mathcal{N}(B_r)$ is compact set in $\mathbb{X}$. Consequently $\mathcal{N}$ is completely continuous.

Step 4. ($\mathcal{N}$ has a closed graph). Let $z_m \to z_*, h_m \in \mathcal{N}(z_m)$ converge to h_*, we need to show that $h_* \in \mathcal{N}(z_*)$. Now there exist v_m in $S_{\mathcal{F},z_m}$ such that

$$\begin{aligned} h_m(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_m(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v_m(\Lambda)d\Lambda \\ &\quad -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\mathrm{g}(\Lambda, z_m(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}, \end{aligned}$$

we show that there exist $v_* \in S_{\mathcal{F},z_*}$ such that for every $\omega \in [0,1]$,

$$\begin{aligned} h_*(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v_*(\Lambda)d\Lambda \\ &\quad -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\mathrm{g}(\Lambda, z_*(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}. \end{aligned}$$

While $\mathcal{F}$ is upper semicontinuous, then for $\varepsilon > 0$ there are m_0 with for $m \ge m_0$

$$\mathcal{F}(\omega, z_m(\omega), \mathcal{D}^{\gamma_1}_{0+}z_m(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0+}z_m(\omega)) \subseteq \mathcal{F}(\omega, z_*(\omega), \mathcal{D}^{\gamma_1}_{0+}z_*(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0+}z_*(\omega)) + B_\varepsilon(0).$$

Then there exist $v_*(\omega) \in \mathcal{F}(\omega, z_*(\omega), \mathcal{D}^{\gamma_1}_{0+}z_*(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0+}z_*(\omega))$ for almost $\omega \in [0,1]$ such that

$$v_m(\omega) \to v_*(\omega).$$

Then

$$\begin{aligned} &\Bigg| h_*(\omega) - \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda + \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v_*(\Lambda)d\Lambda \\ &\qquad -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\mathrm{g}(\Lambda, z_*(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}\Bigg| \\ &\qquad \le |h_m(\omega) - h_*(\omega)| + \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}|v_m(\Lambda) - v_*(\Lambda)|d\Lambda \\ &\qquad + \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}|v_m(\Lambda) - v_*(\Lambda)|d\Lambda \\ &\qquad -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\big|\mathrm{g}(\Lambda, z_m(\Lambda)) - \mathrm{g}(\Lambda, z_*(\Lambda))\big|d\Lambda\Bigg]\omega^{\varsigma-1}, \end{aligned}$$

according to Lebesgue DCT, we get $v_* \in S_{\mathcal{F},z_*}$ and

$$\begin{aligned} h_*(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1} v_*(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1} v_*(\Lambda)d\Lambda \right.\\ &\left. -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\mathrm{g}(\Lambda, z_*(\Lambda))d\Lambda\right]\omega^{\varsigma-1}. \end{aligned}$$

This means that $h_* \in \mathcal{N}(z_*)$.

Step 5. (a priori bounds on solutions). Let $z \in \mathbb{X}$ such that $z \in \lambda\mathcal{N}(z)$ for $\lambda \in (0,1)$. Then for every $\omega \in [0,1]$ we have

$$\begin{aligned} \sum_{j=0}^n |\mathcal{D}_{0^+}^{\gamma_j} z(\omega)| &\leq \sum_{j=0}^n \left[\frac{1}{\Gamma(\varsigma-\gamma_j)}\int_0^\omega (\omega-\Lambda)^{\varsigma-\gamma_j-1}d\Lambda + \frac{1}{\Gamma(\varsigma-\gamma_j)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}d\Lambda\right] \times \\ &\quad \|f\|_\infty \sum_{i=0}^n \psi_i \|z\|_{\mathbb{X}} + \tilde{\mathcal{Q}}_{\|z\|_{\mathbb{X}}} \\ &= \sum_{i=0}^n \left(\frac{1}{\Gamma(\varsigma-\gamma_i+1)} + \frac{1}{(\varsigma-\nu)\Gamma(\varsigma-\gamma_i)}\right)\|f\|_\infty \sum_{i=0}^n \psi_i \|z\|_{\mathbb{X}} + \tilde{\mathcal{Q}}_{\|z\|_{\mathbb{X}}}. \end{aligned}$$

where $0 \leq j \leq n$ and $\gamma_0 = 0$. Finally, we find

$$\|z\|_{\mathbb{X}} \leq \mathcal{Q}.\|f\|_\infty (\sum_{i=0}^n \psi_i(\|z\|_{\mathbb{X}})) + \tilde{\mathcal{Q}}_{\|z\|_{\mathbb{X}}}.$$

Let $\mathcal{V} = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < \sigma\}$.

It is distinctly that $\mathcal{V}$ is an open subset of $\mathbb{X}$ and $0 \in \mathcal{V}$. From steps 1, step 2, step 3 and step 4 we can conclude that the restriction of $\mathcal{N}$ in $\bar{\mathcal{V}}$ is upper semicontinuous, completely continuous and has a compact convex values. From the choice of $\mathcal{V}$ for all $z \in \partial\mathcal{V}$ and $\lambda \in (0,1)$ and by third condition (B3) in theorem we get $z \notin \lambda\mathcal{N}(z)$. Therefore, by theorem 1.3.2, we conclude that $\mathcal{N}$ has a fixed point $z \in \bar{\mathcal{V}}$, then, the problem 2.1 have a solution. □

Example 5. *Consider the following fractional differential inclusion problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.9} z(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{D}_{0^+}^{0.1} z(\omega), \mathcal{D}_{0^+}^{0.3} z(\omega)), \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{0.2} z(1) = \dfrac{4}{\Gamma(4)}\displaystyle\int_0^1 (1-\Lambda)^3(\Lambda + \frac{z(\Lambda)}{4})d\Lambda, \end{cases} \tag{2.20}$$

where

$$\mathcal{F}(\omega, x_0, x_1, x_2) = \left[\omega(\exp(-6|x_2|) + \arctan(|x_0|) + \frac{1}{1+|x_1|}; \exp(\omega) + \frac{\pi}{2} + 1\right].$$

In this problem, we have

$$\|\mathcal{F}(\omega,x_0,x_1,x_2)\| = \sup\{|v|, v\in\mathcal{F}(\omega,x_0,x_1,x_2)\} \le \exp(\omega)+\frac{\pi}{2}+1.$$

It is clear that

(i) $\mathcal{F}(\omega,x_0,x_1,x_2)\in\mathcal{P}_{c,cp}(\mathbb{R})$.

(ii) From example 1 the set valued $\mathcal{F}$ *is upper semicontinuous.*

(iii) From 1.4.1 the set valued $\mathcal{F}$ *is measurable.*

From definition of Carathéodory set valued map 1.4.4, the set valued map $\mathcal{F}$ *is Carathéodory set valued map.*
Let $f(\omega)=\exp(\omega)$, $\psi_0(|x_0|)=\frac{\pi}{2}$, $\psi_1(|x_1|)=1$ *and* $\psi_2(|x_2|)=1$.
Taking $\sigma=43$, *we find* $\tilde{\mathcal{Q}}_\sigma\approx 5.7506$, $\mathcal{Q}\approx 3.7447$ *and* $\|m\|_\infty\approx 2.7183$. *Then we get*

$$\frac{\sigma}{\tilde{\mathcal{Q}}_\sigma+\|m\|_\infty\mathcal{Q}\displaystyle\sum_{j=0}^{2}\psi_j(\sigma)}\approx 1.0214.$$

Consequently, there is at least one solution to problem (2.20).

Remark 2.2.1. *If we replace the hypothesis* $(B3)$ *by*

(B3)* *There are* $\alpha>0$ *and* $\alpha_1,\ldots,\alpha_n$ *such that*

$$|\mathrm{g}(\omega,x)|\le\alpha \quad \text{for all} \quad (\omega,x)\in[0,1]\times\mathbb{R},$$

and

$$|\psi_i(x)|\le\alpha_i \quad \text{for all} \quad x\in\mathbb{R} \text{ and } 0\le i\le n$$

the result of theorem 2.2.2 still true. Indeed, it is clear that the boundedness of the function g *on* $[0,1]\times\mathbb{R}^{n+1}$ *suffices to realize the hypothesis* $(B3)$. *Then we state the following corollary.*

Corollary 5. *Assume that*

(B1) $\mathcal{F}:[0,1]\times\mathbb{R}^{n+1}\to\mathcal{P}_{cp,c}(\mathbb{R})$ *Carathéodory set valued map.*

(B2) *There exist* $f \in \mathrm{L}^{\infty}([0,1],[0;+\infty])$ *and* $\psi_0,\ldots,\psi_n : \mathbb{R}_+ \to \mathbb{R}_+^*$ *continuous, nondecreasing such that*

$$\|\mathcal{F}(\omega,\xi_0,\ldots,\xi_n)\| = \sup\{|v| : v \in \mathcal{F}(\omega,\xi_0,\ldots,\xi_n)\} \leq f(\omega)\sum_{i=0}^{n}\psi_i(|\xi_i|),$$

with $\xi_i \in \mathbb{R}$ *and almost* $\omega \in [0,1]$.

(B3′) *There are* $\alpha > 0$ *and* $\alpha_1,\ldots,\alpha_n$ *with*

$$|\mathrm{g}(\omega,\xi)| \leq \alpha, \ \textit{for } (\omega,\xi) \in [0,1]\times\mathbb{R},$$

and

$$|\psi_i(\xi)| \leq \alpha_i, \ \textit{for } \xi \in \mathbb{R} \textit{ and } 0 \leq i \leq n.$$

Then there exist a solution for the problem (2.16).

The following example illustrates the previous situation.

Example 6. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.9} z(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{D}_{0^+}^{0.1} z(\omega), \mathcal{D}_{0^+}^{0.3} z(\omega)), \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{0.2} z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 (1-\Lambda)^3 . \dfrac{5\cos(z(\Lambda))}{(7+\Lambda)^2} d\Lambda, \end{cases} \tag{2.21}$$

with

$$\mathcal{F}(\omega,x_0,x_1,x_2) = \left[\omega(\exp(-6|x_2|) + \arctan(|x_0|) + \frac{1}{1+|x_1|}; \exp(\omega) + \frac{\pi}{2} + 1\right]$$

In this problem, we have

$$\|\mathcal{F}(\omega,x_0,x_1,x_2)\| = \sup\{|v|, v \in \mathcal{F}(\omega,x_0,x_1,x_2)\} \leq \exp(\omega) + \frac{\pi}{2} + 1.$$

It is obvious that $\mathcal{F}$ *is convex compact valued and of Carathéodory set valued map.*
Let $f(\omega) = \exp(\omega)$, $\psi_0(|x_0|) = \frac{\pi}{2}$, $\psi_1(|x_1|) = 1$ *and* $\psi_2(|x_2|) = 1$.
Then, from corollary 5 the problem (2.21) has at least one solution.

2.2.2 The case where $\mathcal{F}$ is nonconvex

Now we study the case when $\mathcal{F}$ is not necessarily convex set valued map.

Theorem 2.2.3. *Assume that the following assertions are met.*

(C1) $\mathcal{F} : [0,1] \times \mathbb{R}^{n+1} \to \mathcal{P}_{cp}(\mathbb{R})$ *is a set valued map with*

1. $\mathcal{F}$ *is integrably bounded and the set valued map* $\omega \mapsto \mathcal{F}(\omega, \xi_0, \ldots, \xi_n)$ *is measurable for all* $\xi_l \in \mathbb{R}$ *with* $0 \le l \le n$ *,*
2. *there exist* $f \in \mathrm{L}^\infty([0,1], \mathbb{R}^+)$ *such that for almost* $[0,1]$ *and all* $x_i, y_i \in \mathbb{R}$ *with* $0 \le i \le n$*,*
   $H_d(\mathcal{F}(\omega, x_0, \ldots, x_n), \mathcal{F}(\omega, y_0, \ldots, y_n)) \le f(\omega) \sum_{j=0}^{n} |x_j - y_j|.$

(C2) There exist $\theta > 0$ *such that, for all* $\omega \in [0,1]$ *and* $x, y \in \mathbb{R}$*, we have*

$$|\mathrm{g}(\omega, x) - \mathrm{g}(\omega, y)| \le \theta |x - y|.$$

(C3) $\displaystyle\sum_{j=0}^{n} \left(\frac{\|f\|_\infty}{\Gamma(\varsigma - \gamma_j + 1)} + \frac{\|f\|_\infty}{(\varsigma - \nu)\Gamma(\varsigma - \gamma_j)} + \frac{\theta p \Gamma(\varsigma - \nu)}{\Gamma(\mu + 1)\Gamma(\varsigma - \gamma_j)} \right) < 1.$

Then, the problem (2.16) *has at least one solution.*

Proof. From $(C1)(2)$ for almost $\omega \in [0,1]$ the set valued map $\mathcal{F}(\omega, .)$ is upper semicontinuous, then from proposition 7.9 page 229 in [33] the set valued $\omega \mapsto \mathcal{F}(\omega, z(\omega), \mathcal{D}_{0+}^{\gamma_1} z(\omega), \ldots, \mathcal{D}_{0+}^{\gamma_n} z(\omega))$ is measurable and from lemma 1.4.1 there exist a measurable function h such that
$h(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{D}_{0+}^{\gamma_1} z(\omega), \ldots, \mathcal{D}_{0+}^{\gamma_n} z(\omega))$ for $z \in \mathbb{X}$, then $h \in S_{\mathcal{F},z}$.
For prove the existence the solution for problem (2.16) we must show that the set valued map $\mathcal{N}$ that defines in (2.2.1) has a fixed point.
Now we prove that $\mathcal{N}$ satisfies all hypotheses of theorem 1.3.1. The proof is done in two steps.
Step 1. ($\mathcal{N}(z) \in \mathcal{P}_{cl}(\mathbb{X})$). Indeed, let (z_n) be a sequence in $\mathcal{N}(z)$ such that $z_n \to \tilde{z}$ in $\mathbb{X}$. Then $\tilde{z} \in \mathbb{X}$ and there exists $v_n \in S_{\mathcal{F},z}$ such that, for every $\omega \in [0,1]$,

$$z_n(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma - \nu - 1} v_n(\Lambda) d\Lambda \right.$$
$$\left. - \frac{p\Gamma(\varsigma - \nu)}{\Gamma(\mu)\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\mu - 1} \mathrm{g}(\Lambda, z(\Lambda)) d\Lambda \right] \omega^{\varsigma - 1}.$$

As the set valued map $\mathcal{F}$ has compact values, then the set $\mathcal{A} = \{v_n \in S_{\mathcal{F},z}; n \geq 1\}$ satisfied all hypothesis of proposition 1.6.2 we pass onto a subsequence to obtain that v_{n_k} converges weakly to v in $\mathrm{L}^1([0,1];\mathbb{R})$. Therefore, $v \in S_{\mathcal{F},z}$ and for each $\omega \in [0,1]$

$$\frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_n(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v_n(\Lambda)d\Lambda\right]\omega^{\varsigma-1}$$
$$\longrightarrow \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v(\Lambda)d\Lambda\right]\omega^{\varsigma-1},$$

we establish that, for every $\omega \in [0,1]$, we have

$$\begin{aligned}\tilde{z}(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v(\Lambda)d\Lambda\right. \\ &\left. -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\mathrm{g}(\Lambda, z(\Lambda))d\Lambda\right]\omega^{\varsigma-1}.\end{aligned}$$

Hence $\tilde{z} \in \mathcal{N}(z)$.

Step 2. There exists $\gamma < 1$ such that

$$H_d(\mathcal{N}(z), \mathcal{N}(\tilde{z})) \leq \gamma\|z-\tilde{z}\|_{\mathbb{X}} \text{for each } \ z, \tilde{z} \in \mathbb{X}.$$

Let $z, \tilde{z} \in \mathbb{X}$ and $h_1 \in \mathcal{N}(z)$. Then by lemma 1.4.3 there exists a function measurable v_1 such that, for each $\omega \in [0,1]$,

$$\begin{aligned}v_1(\omega) &\in \mathcal{F}(\omega, z(\omega), \ldots, \mathcal{D}^{\delta_n}z(\omega)), \\ h_1(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_1(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-\nu-1}v_1(\Lambda)d\Lambda\right. \\ &\left. -\frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\mu-1}\mathrm{g}(\Lambda, z(\Lambda))d\Lambda\right]\omega^{\varsigma-1}.\end{aligned} \tag{2.22}$$

From $(C1)$, we have

$$\begin{aligned}&H_d\Big(\mathcal{F}(\omega, z(\omega), \mathcal{D}^{\gamma_1}_{0^+}z(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0^+}z(\omega)); \mathcal{F}(\omega, \tilde{z}(\omega), \mathcal{D}^{\gamma_1}_{0^+}\tilde{z}(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0^+}\tilde{z}(\omega))\Big) \\ \leq\ & f(\omega)(\sum_{j=0}^n |\mathcal{D}^{\gamma_j}_{0^+}z(\omega) - \mathcal{D}^{\gamma_j}_{0^+}\tilde{z}(\omega)|).\end{aligned}$$

Then there are $w \in \mathcal{F}(\omega, \tilde{z}(\omega), \mathcal{D}^{\gamma_1}_{0^+}\tilde{z}(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0^+}\tilde{z}(\omega))$ with

$$|v_1(\omega) - w| \leq f(\omega)(\sum_{j=0}^n |\mathcal{D}^{\gamma_j}_{0^+}z(\omega) - \mathcal{D}^{\gamma_j}_{0^+}\tilde{z}(\omega)|), \quad \omega \in [0,1].$$

Then by proposition 1.4.4 implies that the set valued map $\omega \mapsto \mathcal{F}(\omega, \tilde{z}(\omega), \mathcal{D}^{\gamma_1}\tilde{z}(\omega), \ldots, \mathcal{D}^{\gamma_n}_{0^+}\tilde{z}(\omega))$ is measurable, from lemma 1.4.3 has a measurable selection noted by v_2 such that

$$|v_1(\omega) - v_2(\omega)| \leq f(\omega)(\sum_{j=0}^{n} |\mathcal{D}^{\gamma_j}_{0^+} z(\omega) - \mathcal{D}^{\gamma_j}_{0^+}\tilde{z}(\omega)|).$$

For each $\omega \in [0,1]$, let us define

$$\begin{aligned} h_2(\omega) = &\frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v_2(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-\nu-1} v_2(\Lambda) d\Lambda \\ &- \frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\mu-1} \mathrm{g}(\Lambda, \tilde{z}(\Lambda)) d\Lambda] \omega^{\varsigma-1} \text{ for almost } \omega \in [0,1]. \end{aligned}$$

Hence

$$\begin{aligned} |\mathcal{D}^{\gamma_j}_{0^+} h_1(\omega) - \mathcal{D}^{\gamma_j}_{0^+} h_2(\omega)| \leq &\frac{1}{\Gamma(\varsigma-\gamma_j)} \int_0^{\omega} (\omega-\Lambda)^{\varsigma-\gamma_j-1} |v_1(\omega) - v_2(\omega)| d\Lambda \\ &+ \frac{1}{\Gamma(\varsigma-\gamma_j)} \int_0^1 (1-\Lambda)^{\varsigma-\nu-1} |v_1(\omega) - v_2(\omega)| d\Lambda \\ &+ \frac{p\Gamma(\varsigma-\nu)}{\Gamma(\mu)\Gamma(\varsigma-\gamma_j)} \int_0^1 (1-\Lambda)^{\mu-1} |\mathrm{g}(\Lambda, z(\Lambda)) - \mathrm{g}(\Lambda, \tilde{z}(\Lambda))| d\Lambda \\ &\leq \left[\frac{\|f\|_\infty}{\Gamma(\varsigma-\gamma_j+1)} + \frac{\|f\|_\infty}{(\varsigma-\nu)\Gamma(\varsigma-\gamma_j)} + \frac{\theta p\Gamma(\varsigma-\nu)}{\Gamma(\mu+1)\Gamma(\varsigma-\gamma_j)}\right] \|z-\tilde{z}\|_{\mathbb{X}}, \end{aligned}$$

where $0 \leq j \leq n$ with $\gamma_0 = 0$. Therefore

$$\|h_1 - h_2\|_{\mathbb{X}} \leq \left[\sum_{j=0}^{n} (\frac{\|f\|_\infty}{\Gamma(\varsigma-\gamma_j+1)} + \frac{\|f\|_\infty}{(\varsigma-\nu)\Gamma(\varsigma-\gamma_j)} + \frac{\theta p\Gamma(\varsigma-\nu)}{\Gamma(\mu+1)\Gamma(\varsigma-\gamma_j)})\right] \|z-\tilde{z}\|_{\mathbb{X}}.$$

Analogously, interchanging the roles of z and $\tilde{z}$, we get

$$H_d(\mathcal{N}(z), \mathcal{N}(\tilde{z})) \leq \gamma \|z-\tilde{z}\|_{\mathbb{X}} \quad \text{for each} \quad z, \tilde{z} \in \mathbb{X},$$

with

$$\gamma = \sum_{j=0}^{n} \left(\frac{\|f\|_\infty}{\Gamma(\varsigma-\gamma_j+1)} + \frac{\|f\|_\infty}{(\varsigma-\nu)\Gamma(\varsigma-\gamma_j)} + \frac{\theta p\Gamma(\varsigma-\nu)}{\Gamma(\mu+1)\Gamma(\varsigma-\gamma_j)}\right).$$

While $\mathcal{N}$ is a contraction set valued map, and from theorem 1.3.1 we get $\mathcal{N}$ has a fixed point z which is represent a solution of (2.16). The proof is fulfilled. □

Example 7. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}^{1.9}_{0^+} z(\omega) \in \mathcal{F}(\omega, z(\omega), \mathcal{D}^{0.1}_{0^+} z(\omega), \mathcal{D}^{0.3}_{0^+} z(\omega)), \\ z(0) = 0, \\ \mathcal{D}^{0.2}_{0^+} z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 (1-\Lambda)^3 . \frac{5\cos(z(\Lambda))}{(7+\Lambda)^2} d\Lambda, \end{cases} \tag{2.23}$$

where

$$\mathcal{F}(\omega, x_0, x_1, x_2) = \left[\frac{\sin x_0}{9+\omega^2} + \frac{1}{31}\arctan x_1 - \pi; -\frac{1}{2}\right] \cup \left[1; \frac{\exp\omega}{51}|x_2| + 1\right],$$

and

$$\mathrm{g}(\omega, x) = \frac{5}{(7+\omega)^2}\cos x.$$

We have

(i) $\mathcal{F}$ *is closed valued but is not convex valued.*

(ii) From lemma 1.4.3 $\mathcal{F}$ *is measurable.*

(iii) $n = 2, \varsigma = 1.9, \gamma_1 = 0.1, \gamma_2 = 0.3, \nu = 0.2,$ *and* $\mu = p = 4$

In view of the data given above, we have for $\omega \in [0,1]$ *and* $x, y \in \mathbb{R}$,

$$H_d(\mathcal{F}(\omega, x_0, x_1, x_2); \mathcal{F}(\omega, y_0, y_1, y_2)) \leq \frac{1}{9+\omega^2}|x_0 - y_0| + \frac{1}{31}|x_1 - y_1| + \frac{\exp\omega}{51}|x_2 - y_2|,$$

$$|\mathrm{g}(\omega, x) - \mathrm{g}(\omega, y)| \leq \frac{5}{49}|x - y|,$$

$$f(\omega) = \frac{1}{9+\omega^2} + \frac{1}{31} + \frac{\exp\omega}{51},$$

and also

$$\gamma = \sum_{j=0}^{n}\left(\frac{\|f\|_\infty}{\Gamma(\varsigma - \gamma_j + 1)} + \frac{\|f\|_\infty}{(\varsigma - \delta_j)\Gamma(\varsigma - \gamma_j)} + \frac{\theta p \Gamma(\varsigma - \nu)}{\Gamma(\mu+1)\Gamma(\varsigma - \gamma_j)}\right) \approx 0.7864 < 1.$$

Hence, all the hypotheses of theorem 2.2.3 are satisfied. Then the problem (2.23) *has at least one solution.*

Chapter 3

Study the existence and uniqueness for fractional differential inclusions in Lebesgue and Sobolev spaces

In this chapter we study the existence and uniqueness for BVP of fractional differential inclusion

$$\begin{cases} \mathcal{D}_{0^+}^{\varsigma} z(\omega) \in \mathcal{H}(\omega, z(\omega)), \quad \text{for almost } \omega \in [0,1], \\ z(0) = 0, \\ z(1) = pI_{0^+}^{\mu_1} k_1(\xi_1, z(\xi_1)) + qI_{0^+}^{\mu_2} k_2(\xi_2, z(\xi_2)), \end{cases} \tag{3.1}$$

where $1 < \varsigma < 2$, $p, q \geq 0$, $\mu_1, \mu_2 \geq 1$, $0 < \xi_1, \xi_2 \leq 1$, $k_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $1 \leq j \leq 2$, and $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set valued map.

Definition 3.0.1. *A function z is said to be a solution of the problem (3.1), if there exists a function $v \in \mathrm{L}^1([0,1], \mathbb{R})$ with $v(\omega) \in \mathcal{H}(\omega, z(\omega))$ for almost $\omega \in [0,1]$ such that*

$$\begin{cases} \mathcal{D}_{0^+}^{\varsigma} z(\omega) = v(\omega), \quad 1 < \varsigma < 2, \\ z(0) = 0, \\ z(1) = pI_{0^+}^{\mu_1} k_1(\xi_1, z(\xi_1)) + qI_{0^+}^{\mu_2} k_2(\xi_2, z(\xi_2)). \end{cases}$$

Lemma 3.0.1. *For a given* $y \in \mathrm{L}^1([0,1],\mathbb{R})$*, a function* z *is a solution to*

$$\begin{cases} \mathcal{D}^{\varsigma}_{0^+} z(\omega) = y(\omega), \quad 1 < \varsigma < 2, \quad \textit{for almost} \;\; \omega \in [0,1], \\ z(0) = 0, \\ z(1) = pI^{\mu_1}_{0^+} k_1(\xi_1, z(\xi_1)) + qI^{\mu_2}_{0^+} k_2(\xi_2, z(\xi_2)), \end{cases} \tag{3.2}$$

if and only if

$$\begin{aligned} z(\omega) \;\; &= \;\; \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} y(\Lambda) d\Lambda - \left[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} y(\Lambda) d\Lambda \right. \\ &- \left. \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} k_1(\Lambda, z(\Lambda)) d\Lambda - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2(\Lambda, z(\Lambda)) d\Lambda \right] \omega^{\varsigma - 1} \end{aligned} \tag{3.3}$$

Proof. Assume that z satisfies (3.2), from lemma 1.5.3 we get

$$z(\omega) = I^{\varsigma}_{0^+} y(\omega) - c_1 \omega^{\varsigma-1} - c_2 \omega^{\varsigma-2},$$

where $c_1, c_2 \in \mathbb{R}$.

The first boundary condition, we obtain $I^{\varsigma}_{0^+} y(\omega) = 0$, $c_1 \omega^{\varsigma-1} = 0$ and $\lim_{\omega \to 0} = +\infty$, then $z(0) = 0$ that means $c_2 = 0$. From the second boundary condition, we get

$$z(1) \;\; = \;\; I^{\varsigma}_{0^+} y(1) - c_1 = pI^{\mu_1}_{0^+} k_1(\xi_1, z(\xi_1)) + qI^{\mu_2}_{0^+} k_2(\xi_2, z(\xi_2)),$$

this means that

$$\begin{aligned} c_1 = \frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} y(\Lambda) d\Lambda &- \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} k_1(\Lambda, z(\Lambda)) d\Lambda \\ &- \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2(\Lambda, z(\Lambda)) d\Lambda, \end{aligned} \tag{3.4}$$

we replace the value of c_1 with the value obtained in (3.4) we get the integral equation (3.3).
Conversely, if z satisfies (3.3) by lemma 1.5.2 and 1.5.4, we have $\mathcal{D}^{\varsigma}_{0^+} z(\omega) = y(\omega)$.
By simple calculation, we get

$$z(1) = pI^{\mu_1}_{0^+} k_1(\xi_1, z(\xi_1)) + qI^{\mu_2}_{0^+} k_2(\xi_2, z(\xi_2)),$$

and we can easily to see that $z(0) = 0$. □

3.1 Existence and uniqueness result in L^1 space

Consider $\mathbb{X}_1 = \mathrm{L}^1([0,1],\mathbb{R})$ the space of all integrable functions defined on $[0,1]$ provided by the norm

$$\|z\|_{\mathbb{X}_1} = \int_0^1 |z(\omega)|d\omega.$$

It's clearly that $(\mathbb{X}_1, \|.\|_{\mathbb{X}_1})$ is a Banach space.

For each $z \in \mathbb{X}_1$, the set of selections of $\mathcal{H}$ is defined by

$$S_{\mathcal{H},z} = \{v \in \mathrm{L}^1([0,1],\mathbb{R}) : v(\omega) \in \mathcal{H}(\omega, z(\omega)) \text{ for almost } \omega \in [0,1]\}. \tag{3.5}$$

3.1.1 Existence result

In this work we have studied the existence and uniqueness for (3.1) in $\mathbb{X}_1$ when $\mathcal{H}$ has a convex values.

Theorem 3.1.1. *Assume that the following assertions $(B1)-(B4)$ hold*

(B1) The set valued map $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ is a Carathéodory map.

(B2) Existence two functions $\varphi \in \mathrm{L}^1([0,1],\mathbb{R}^+)$ and $\psi \in \mathrm{L}^\infty([0,1],\mathbb{R}^+)$ with

$$\|\mathcal{H}(\omega, x)\| = \sup\{|v| : v \in \mathcal{H}(\omega, x)\} \leq \varphi(\omega) + \psi(\omega)|x|,$$

for $x \in \mathbb{R}$ and almost $\omega \in [0,1]$.

(B3) There exists $C_1, C_2 > 0$ such that

$$|k_i(\omega, x) - k_i(\omega, y)| \leq C_i|x - y|,$$

and

$$k_i(\omega, 0) = 0,$$

for $\omega \in [0,1]$ and $i \in \{1,2\}$.

(B4) $\dfrac{2\|\psi\|_\infty}{\Gamma(\varsigma+1)} + \dfrac{C_1 p \xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)} + \dfrac{C_2 q \xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)} < 1.$

Then there exists a solution for the problem (3.1).

Lemma 3.1.1. *The first and the second conditions of this theorem implies that $S_{\mathcal{H},z}$ is non-empty set.*

Proof. The set valued map $\mathcal{H}$ satifies all hypothesis of proposition 1.4.4, then for every $z \in \mathbb{X}_1$ there exist a measurable function f such that $f(\omega) \in \mathcal{H}(\omega, z(\omega))$ for almost $\omega \in [0,1]$. From $(B2)$ we have

$$|f(\omega)| \leq \varphi(\omega) + \psi(\omega)|z(\omega)|, \quad \text{for almost} \quad \omega \in [0,1],$$

after that

$$\int_0^1 |f(\Lambda)| d\Lambda \leq \|\varphi\|_1 + \|\psi\|_\infty \|z\|_{\mathbb{X}_1} < +\infty,$$

that means $f \in \mathrm{L}^1([0,1];\mathbb{R})$, then $S_{\mathcal{H},z}$ is non-empty. □

Proof. **of theorem 3.1.1** Define an operator $\mathcal{N} : \mathbb{X}_1 \to \mathcal{P}(\mathbb{X}_1)$ by

$$\begin{aligned}\mathcal{N}(z) = \Bigg\{ h \in \mathbb{X}_1 : \ h(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda \\ - \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda \\ - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} k_1(\Lambda, z(\Lambda)) d\Lambda \\ - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2(\Lambda, z(\Lambda)) d\Lambda \Bigg] \omega^{\varsigma - 1}, v \in S_{\mathcal{H},z} \Bigg\}. \end{aligned} \tag{3.6}$$

We will show that $\mathcal{N}$ satisfies all hypotheses of the theorem 1.3.3.

Step 1. ($\mathcal{N}(z)$ is convex valued). Let $z \in \mathbb{X}_1$ and $h_1, h_2 \in \mathcal{N}(z)$, then there exist $v_1, v_2 \in S_{\mathcal{H},z}$ such that

$$\begin{aligned} h_i(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v_i(\Lambda) d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1 - \Lambda)^{\varsigma - 1} v_i(\Lambda) d\Lambda \\ &- \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} k_1(\Lambda, z(\Lambda)) d\Lambda - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2(\Lambda, z(\Lambda)) d\Lambda \Bigg] \omega^{\varsigma - 1}, \end{aligned}$$

for $i \in \{1,2\}$. Then for $\lambda \in [0,1]$

$$\begin{aligned}\big(\lambda h_1 + (1-\lambda)h_2\big)(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}\big(\lambda v_1 + (1-\lambda)v_2\big)(\Lambda)d\Lambda \\ &\quad -\Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^{1}(1-\Lambda)^{\varsigma-1}\big(\lambda v_1 + (1-\lambda)v_2\big)(\Lambda)d\Lambda \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda, z(\Lambda))d\Lambda \\ &\quad -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda, z(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}.\end{aligned}$$

Since $\mathcal{H}$ is convex set valued map, then $\lambda h_1 + (1-\lambda)h_2 \in S_{\mathcal{H},z}$, therefore we can easily to see that $\mathcal{N}(z)$ is convex for each $z \in \mathbb{X}_1$.

Step 2. ($\mathcal{N}$ maps bounded sets into bounded sets in $\mathbb{X}_1$). Let $B_r = \{z \in \mathbb{X}_{\Bbbk}; \|z\| \leq r\}$, we show that there exists a constant $l > 0$ such that for each $z \in B_r$ and $h \in \mathcal{N}(z)$ one has $\|h\|_{\mathbb{X}_{\Bbbk}} \leq l$.

$$\begin{aligned} h(\omega) = \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^{1}(1-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda \\ -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda, z(\Lambda))d\Lambda - \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda, z(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}. \quad (3.7)\end{aligned}$$

From the conditions $(B2)$ and $(B3)$ we have

$$\begin{aligned}|h(\omega)| &\leq \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}|v(\Lambda)|d\Lambda + \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^{1}(1-\Lambda)^{\varsigma-1}|v(\Lambda)|d\Lambda \\ &\quad +\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}|k_1(\Lambda, z(\Lambda))|d\Lambda + \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}|k_2(\Lambda, z(\Lambda))|d\Lambda\Bigg]\omega^{\varsigma-1} \\ &\leq \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}\big(\varphi(\Lambda) + \psi(\Lambda)|z(\Lambda)|\big)d\Lambda \\ &\quad +\Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^{1}(1-\Lambda)^{\varsigma-1}\big(\varphi(\Lambda) + \psi(\Lambda)|z(\Lambda)|\big)d\Lambda \\ &\quad +\frac{C_1 p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}|z(\Lambda)|d\Lambda + \frac{C_2 q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}|z(\Lambda)|d\Lambda\Bigg]\omega^{\varsigma-1} \\ &\leq \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)}\int_0^{\omega}\big(\varphi(\Lambda) + \psi(\Lambda)|z(\Lambda)|\big)d\Lambda + \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^{1}\big(\varphi(\Lambda) + \psi(\Lambda)|z(\Lambda)|\big)d\Lambda \\ &\quad +\frac{C_1 p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}\int_0^{\xi_1}|z(\Lambda)|d\Lambda + \frac{C_2 q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\int_0^{\xi_2}|z(\Lambda)|d\Lambda\Bigg]\omega^{\varsigma-1}.\end{aligned}$$

By simple calculation, we get

$$\|f\|_{\mathbb{X}_1} = \int_0^1 |f(\Lambda)|d\Lambda \;\leq\; \frac{2\|\varphi\|_1}{\Gamma(\varsigma+1)} + \left(\frac{2\|\psi\|_\infty}{\Gamma(\varsigma+1)} + \frac{C_1 p\xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)} + \frac{C_2 q\xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)}\right)\|z\|_{\mathbb{X}_{\not{p}}}$$
$$\leq l = \frac{2\|\varphi\|_1}{\Gamma(\varsigma+1)} + \left(\frac{2\|\psi\|_\infty}{\Gamma(\varsigma+1)} + \frac{C_1 p\xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)} + \frac{C_2 q\xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)}\right)r.$$

Step 3. ($\mathcal{N}$ set valued map convert any bounded sets into relatively compact sets in $\mathbb{X}_1$). Let B_r the ball in $\mathbb{X}_1$ as previous step . From precedent step the set $\mathcal{N}(B_r)$ is bounded in $\mathbb{X}_1$, now we prove that the closure of $\mathcal{N}(B_r)$ is compact in $\mathbb{X}_1$.

For $f \in \mathcal{N}(B_r)$, there are an element $z \in B_r$ with $f \in \mathcal{N}(z)$, let $\widehat{f}$ is a function defined by

$$\widehat{f}(\omega) = \begin{cases} f(\omega) & if \;\; \omega \in [0,1], \\ 0 & if \;\; \omega \in \mathbb{R} - [0,1]. \end{cases}$$

Let A be a subset in $\mathrm{L}^1(\mathbb{R})$ defined by

$$A = \left\{\widehat{f}; f \in \mathcal{N}(B_r)\right\}.$$

Let be a function $\widehat{f} \in A$, we have

$$\|\widehat{f}\|_{\mathrm{L}^1(\mathbb{R})} = \|f\|_{\mathbb{X}_1} \leq l,$$

where l is a constant as a step 2, that means A is bounded in $\mathrm{L}^1(\mathbb{R})$.

Now we prove that

$$\|\Lambda_h \widehat{f} - \widehat{f}\|_{\mathrm{L}^1(\mathbb{R})} \to 0 \quad \text{as} \quad h \to 0.$$

where $\Lambda_h \widehat{f}(\omega) = \widehat{f}(\omega + h)$.

Let $h \in (-1, 1)$.

Case $h > 0$

$\int_{-\infty}^{+\infty} |\widehat{f}(\omega+h) - \widehat{f}(\omega)|d\omega = \int_{-h}^{0} |f(\omega+h)|d\omega + \int_0^{1-h} |f(\omega+h) - f(\omega)|d\omega + \int_{1-h}^{1} |f(\omega)|d\omega.$

- $|f(\omega+h)| \leq \frac{2(\omega+h)^{\varsigma-1}}{\Gamma(\varsigma)} \times \left(\|\varphi\|_1 + \|\psi\|_\infty \|z\|_{\mathbb{X}_{\not{p}}}\right) + (\omega+h)^{\varsigma-1} \times \left(\frac{C_1 p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)\|z\|_{\mathbb{X}_{\not{p}}}.$

-

$$\begin{aligned}
|f(\omega+h)-f(\omega)| &\leq \Big|\int_{\omega}^{\omega+h}(\omega+h-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda \\
&+\int_{0}^{\omega}\big((\omega+h-\Lambda)^{\varsigma-1}-(\omega-\Lambda)^{\varsigma-1}\big)v(\Lambda)d\Lambda\Big| \\
&+\Big|\frac{1}{\Gamma(\varsigma)}\int_{0}^{1}(1-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda-\frac{p}{\Gamma(\mu_1)}\int_{0}^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda \\
&-\frac{q}{\Gamma(\mu_2)}\int_{0}^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\Big|\times\Big((\omega+h)^{\varsigma-1}-\omega^{\varsigma-1}\Big) \\
&\leq \frac{h^{\varsigma-1}}{\Gamma(\varsigma)}\int_{\omega}^{\omega+h}|v(\Lambda)|d\Lambda+\frac{h^{\varsigma-1}}{\Gamma(\varsigma)}\int_{0}^{\omega}|v(\Lambda)|d\Lambda \\
&+\Big|\frac{1}{\Gamma(\varsigma)}\int_{0}^{1}(1-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda-\frac{p}{\Gamma(\mu_1)}\int_{0}^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda \\
&-\frac{q}{\Gamma(\mu_2)}\int_{0}^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\Big|\Big((\omega+h)^{\varsigma-1}-\omega^{\varsigma-1}\Big) \\
&\leq \frac{2h^{\varsigma-1}}{\Gamma(\varsigma)}\Big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_{\not k}}\Big)+\Big(\frac{\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}}{\Gamma(\varsigma)} \\
&+\frac{C_1p\xi_1^{\mu_1-1}\|z\|_{\mathbb{X}_1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}\|z\|_{\mathbb{X}_1}}{\Gamma(\mu_2)}\Big)\Big((\omega+h)^{\varsigma-1}-\omega^{\varsigma-1}\Big)
\end{aligned}$$

- $|f(\omega)|\leq\frac{2t^{\varsigma-1}}{\Gamma(\varsigma)}\times\big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}\big)+\omega^{\varsigma-1}\times\big(\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\big)\|z\|_{\mathbb{X}_1}$.

By simple calculation, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty}|\widehat{f}(\omega+h)-\widehat{f}(\omega)|d\omega \leq \big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}\big)\big(&\frac{h^\varsigma-3(1-h)^\varsigma+3}{\Gamma(\varsigma+1)}+\frac{2(1-h)h^{\varsigma-1}}{\Gamma(\varsigma)}\big) \\
&+\frac{2-2(1-h)^\varsigma}{\varsigma}\big(\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\big)\|z\|_{\mathbb{X}_1}
\end{aligned}$$

as $h\to 0$ we find that $\int_{-\infty}^{+\infty}|\widehat{f}(\omega+h)-\widehat{f}(\omega)|d\omega\to 0$.

Case $h<0$

$\int_{-\infty}^{+\infty}|\widehat{f}(\omega+h)-\widehat{f}(\omega)|d\omega=\int_{0}^{-h}|f(\omega)|d\omega+\int_{-h}^{1}|f(\omega+h)-f(\omega)|d\omega+\int_{1}^{1-h}|f(\omega+h)|d\omega$

- $|f(\omega)|\leq\frac{2t^{\varsigma-1}}{\Gamma(\varsigma)}\times\big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}\big)+\omega^{\varsigma-1}\times\big(\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\big)\|z\|_{\mathbb{X}_1}$.

- $$\begin{aligned}
|f(\omega+h)-f(\omega)| \leq & \Bigg|\int_0^{\omega+h}\big((\omega+h-\Lambda)^{\varsigma-1}-(\omega-\Lambda)^{\varsigma-1}\big)v(\Lambda)d\Lambda \\
& +\int_\omega^{\omega+h}(\omega-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda\Bigg| \\
& +\Bigg|\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda \\
& -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda \\
& -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\Bigg| \times \left|(\omega+h)^{\varsigma-1}-\omega^{\varsigma-1}\right| \\
& \leq \frac{(-h)^{\varsigma-1}}{\Gamma(\varsigma)}\int_\omega^{\omega+h}|v(\Lambda)|d\Lambda+\frac{(-h)^{\varsigma-1}}{\Gamma(\varsigma)}\int_0^\omega|v(\Lambda)|d\Lambda \\
& +\Bigg|\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda \\
& -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda \\
& -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\Bigg|\Big((\omega+h)^{\varsigma-1}-\omega^{\varsigma-1}\Big) \\
& \leq \frac{2(-h)^{\varsigma-1}}{\Gamma(\varsigma)}\Big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}\Big) \\
& +\left(\frac{\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}}{\Gamma(\varsigma)}+\frac{C_1p\xi_1^{\mu_1-1}\|z\|_{\mathbb{X}_1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}\|z\|_{\mathbb{X}_1}}{\Gamma(\mu_2)}\right) \\
& \times\Big|(\omega+h)^{\varsigma-1}-\omega^{\varsigma-1}\Big|
\end{aligned}$$

- $|f(\omega+h)| \leq \frac{2(\omega+h)^{\varsigma-1}}{\Gamma(\varsigma)}\times\big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}\big)+(\omega+h)^{\varsigma-1}\times\big(\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\big)\|z\|_{\mathbb{X}_1}$.

By simple calculation, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty}|\widehat{f}(\omega+h)-\widehat{f}(\omega)|d\omega \leq & \big(\|\varphi\|_1+\|\psi\|_\infty\|z\|_{\mathbb{X}_1}\big) \\
& \times\left(\frac{(-h)^\varsigma-3(1+h)^\varsigma+3}{\Gamma(\varsigma+1)}+\frac{2(1+h)(-h)^{\varsigma-1}}{\Gamma(\varsigma)}\right) \\
& +\frac{2-2(1+h)^\varsigma}{\varsigma}\big(\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\big)\|z\|_{\mathbb{X}_1},
\end{aligned}$$

as $h\to 0$ we get $\int_{-\infty}^{+\infty}|\widehat{f}(\omega+h)-\widehat{f}(\omega)|d\omega\to 0$, i.e., $\|\Lambda_h\widehat{f}-\widehat{f}\|_{L^1(\mathbb{R})}\to 0 \quad \text{as} \quad h\to 0$.

By theorem 1.6.3 we have the closure of $A_{|[0,1]}=\mathcal{N}(B_r)$ is compact set in $L^1([0,1])$.

The set valued map $\mathcal{N}$ converts each bounded set into relatively compact set, then it's a completely continuous map.

Step 4. ($\mathcal{N}$ has a closed graph). Let $z_n \to z_*, f_n \in \mathcal{N}(z_n)$ and $f_n \to f_*$, we need to show that $f_* \in \mathcal{N}(z_*)$. Now $f_n \in \mathcal{N}(z_n)$ implies that there exists $v_n \in S_{\mathcal{H},z_n}$ such that

$$\begin{aligned} f_n(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_n(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-1}v_n(\Lambda)d\Lambda \right.\\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z_n(\Lambda))d\Lambda \\ &\quad \left. -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z_n(\Lambda))d\Lambda\right]\omega^{\varsigma-1}, \end{aligned}$$

we will prove the existence $v_* \in S_{\mathcal{H},z_*}$ such that for all $\omega \in [0,1]$,

$$\begin{aligned} f_*(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda \right.\\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z_*(\Lambda))d\Lambda \\ &\quad \left. -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z_*(\Lambda))d\Lambda\right]\omega^{\varsigma-1}, \end{aligned}$$

from $\|z_n - z_*\|_{\mathbb{X}_1} \to 0$, then by theorem 1.6.4 there exist a subsequence $z_{n_k} \in \mathbb{X}_1$ such that

$$z_{n_k}(\omega) \to z_*(\omega) \quad \text{for almost } \omega \in [0,1]. \tag{3.8}$$

From proposition 1.2.2 the upper continuity of $\mathcal{H}(\omega,.)$ dictates that there exists $\delta > 0$ such that for every $x \in \mathbb{R}$, we have

$$|x - z_*(\omega)| < \delta \ \text{ implies that } \ \mathcal{H}(\omega,x) \subset \mathcal{H}(\omega,z_*(\omega)) + B_\varepsilon(0),$$

and from (3.8) there exist $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have

$$|z_{n_k}(\omega) - z_*(\omega)| < \delta \quad \text{a.e. on } [0,1],$$

that means $\mathcal{H}(\omega, z_{n_k}(\omega)) \subset \mathcal{H}(\omega, z_*(\omega)) + B_\varepsilon(0)$, for almost $\omega \in [0,1]$.
Since $v_{n_k}(\omega) \in \mathcal{H}(\omega, z_{n_k}(\omega))$ it follows that there exist $v_*(\omega) \in \mathcal{H}(\omega, z_*(\omega))$ for almost $\omega \in [0,1]$ such that

$$|v_{n_k}(\omega) - v_*(\omega)| < \varepsilon \text{ a.e. on } [0,1], \text{ for } k \geq k_0$$

i.e., $$v_{n_k}(\omega) \to v_*(\omega) \text{ a.e. on } [0,1].$$

Therefore

$$\begin{aligned}\int_0^1 \Bigg| f_*(\omega) \quad - \quad & \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda \\ & -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda, z_*(\Lambda))d\Lambda \\ & -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda, z_*(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}\Bigg|d\omega \\ & \leq \int_0^1 |f_{n_k}(\omega) - f_*(\omega)|d\omega + \int_0^1 |\frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}|v_{n_k} - v_*(\Lambda)|d\Lambda d\omega \\ & + \int_0^1 \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-1}|v_{n_k} - v_*(\Lambda)|d\Lambda d\omega \\ & + \int_0^1 \frac{p\omega^{\varsigma-1}}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}|k_1(\Lambda, z_{n_k}(\Lambda))d\Lambda - k_1(\Lambda, z_*(\Lambda))|d\Lambda d\omega \\ & + \int_0^1 \frac{q\omega^{\varsigma-1}}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}|k_2(\Lambda, z_{n_k}(\Lambda))d\Lambda - k_2(\Lambda, z_*(\Lambda))|d\Lambda d\omega.\end{aligned}$$

Returning to Lebesgue DCT we get $v_* \in S_{\mathcal{H},z_*}$ and

$$\begin{aligned} f_*(\omega) \quad = \quad & \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda \\ & -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda, z_*(\Lambda))d\Lambda \\ & -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda, z_*(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}.\end{aligned}$$

This means that $f_* \in \mathcal{N}(z_*)$.

From step 2, step 3 and step 4 the set valued map $\mathcal{N}$ is upper semicontinuous.

Step 5. (Construction the compact set in $\mathrm{L}^1([0,1])$). Let ε_1 and ε_2 two functions defined on $(-1,1)$ by

$$\varepsilon_1(h) = \begin{cases} \frac{(-h)^\varsigma - 3(1+h)^\varsigma + 3}{\Gamma(\varsigma+1)} + \frac{2(1+h)(-h)^{\varsigma-1}}{\Gamma(\varsigma)} & if\ h < 0, \\ \frac{h^\varsigma - 3(1-h)^\varsigma + 3}{\Gamma(\varsigma+1)} + \frac{2(1-h)h^{\varsigma-1}}{\Gamma(\varsigma)} & if\ h > 0, \end{cases}$$

and

$$\varepsilon_2(h) = \begin{cases} \|\psi\|_\infty\left(\frac{(-h)^\varsigma - 3(1+h)^\varsigma + 3}{\Gamma(\varsigma+1)} + \frac{2(1+h)(-h)^{\varsigma-1}}{\Gamma(\varsigma)}\right) + \frac{2-2(1+h)^\varsigma}{\varsigma}\left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right) & if\ h < 0, \\ \|\psi\|_\infty\left(\frac{h^\varsigma - 3(1-h)^\varsigma + 3}{\Gamma(\varsigma+1)} + \frac{2(1-h)h^{\varsigma-1}}{\Gamma(\varsigma)}\right) + \frac{2-2(1-h)^\varsigma}{\varsigma}\left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right) & if\ h > 0. \end{cases}$$

Let

$$K = \bar{B}(0, \frac{A}{1-B}) \cap \left\{ f \in \mathbb{X}_1; \int_{-\infty}^{+\infty} |\widehat{f}(\omega + h) - \widehat{f}(\omega)| d\omega \leq \|\varphi\|_1 \varepsilon_1(h) + \frac{A\varepsilon_2(h)}{1-B} \right\},$$

where

$A = \frac{2\|\varphi\|_1}{\Gamma(\varsigma+1)}$, $B = \frac{2\|\psi\|_\infty}{\Gamma(\varsigma+1)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)}$, and $\widehat{f}$ the function that defined in step 3.

As step 2 and step 3, for any $z \in K$ and for any $f \in \mathcal{N}(z)$, we have

$$\|f\|_{\mathbb{X}_1} \leq A + B\|z\|_{\mathbb{X}_1} \leq \frac{A}{1-B},$$

and

$$\int_{-\infty}^{+\infty} |\widehat{f}(\omega + h) - \widehat{f}(\omega)| d\omega \leq \|\varphi\|_1 \varepsilon_1(h) + \frac{A\varepsilon_2(h)}{1-B},$$

that means that $f \in K$.

The element $0 \in K$ that means the set K is nonempty.

Let $\widehat{A} = \{\widehat{f}; f \in K\}$, from step 2 we have

$$\|\widehat{f}\|_{L^1(\mathbb{R})} = \|f\|_{\mathbb{X}_1} \leq \tfrac{A}{1-B},$$

and

$$\|\Lambda_h \widehat{f} - \widehat{f}\|_{L^1(\mathbb{R})} \leq \|\varphi\|_1 \varepsilon_1(h) + \tfrac{A\varepsilon_2(h)}{1-B} \quad \to 0 \quad \text{as } h \to 0.$$

Then by theorem 1.6.3 the closure of $K = A_{|[0,1]}$ is compact set in $\mathbb{X}_1$, and it's clearly that K is closed and convex set in $\mathbb{X}_1$.

We define $\mathcal{M} : z \in K \mapsto \mathcal{M}(z) = \mathcal{N}(z) \in \mathcal{P}_{c,cl}(K)$

Let O_K be an open set in K i.e., $O_K = K \cap O$, where O is a open set in $\mathbb{X}_1$.

$$\mathcal{M}^{-1}(O_K) = \{z \in K; \mathcal{M}(z) \subseteq O\} = \{z \in K; \mathcal{N}(z) \subseteq O\} = K \cap \mathcal{N}^{-1}(O),$$

since the map $\mathcal{N}$ is upper semicontinuous, we have $\mathcal{N}^{-1}(O)$ is open set in $\mathbb{X}_1$ we conclude that $\mathcal{M}^{-1}(O_K) = K \cap \mathcal{N}^{-1}(O)$ is open set in K that means the set valued map $\mathcal{M}$ is upper semicontinuous.

From step 1 and step 4 the set $\mathcal{M}(z)$ is convex and closed in $\mathbb{X}_1$.

Using theorem 1.3.3, we conclude that $\mathcal{M}$ has a fixed point $z \in K$, which is a solution of the problem (3.1). □

Remark 3.1.1. *If the function $\varphi = 0$, then the function $z = 0$ is a solution of* (3.1).

Corollary 6. *Assume that the following assertions $(B1) - (B4)$ in theorem 3.1.1 hold, and $\varphi = 0$. Then the function $z = 0$ is a solution for the problem* (3.1).

Proof. Let $z = 0$, then

$z(1) = pI_{0^+}^{\mu_1} k_1(\omega, 0) + qI_{0^+}^{\mu_2} k_2(\omega, 0) = 0$.

From $(B2)$, we have $\mathcal{H}(\omega, 0) = \{0\}$ and $\mathcal{D}_{0^+}^{\varsigma} 0 = 0 \in \{0\}$, which $z = 0$ is a solution of problem (3.1).

□

Theorem 3.1.2. *Suppose that the hypothesis $(B1^*), (B2), (B3)$ and $(B4^*)$ are hold, where*

$(B1^*)$ *$\mathcal{H} : [0, 1] \times \mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ satisfies*

1. *The set valued map $\mathcal{H}(\omega, .)$ is Lipschitzean,*
2. *$\mathcal{H}(., x)$ is measurable for each $x \in \mathbb{R}$.*

$(B4^*)$ $\frac{2\max\{L, \|\psi\|_\infty\}}{\Gamma(\varsigma+1)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\varsigma\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\varsigma\Gamma(\mu_2)} < 1$.

Then the problem (3.1) *has a unique solution in $\mathbb{X}_1$.*

Proof. The set valued map $\mathcal{H}(\omega, .)$ is Lipschitzean, then from proposition 1.2.3 the set valued map $F(\omega, .)$ is upper semicontinuous, that means the assumption $(B1)$ is valid, by theorem 3.1.1 the problem (3.1) has a solution, we prove now this solution is unique.

Let z_1 and z_2 are two solutions of the problem (3.1), then there exist $v_1 \in S_{\mathcal{H},z_1}$ and $v_2 \in S_{\mathcal{H},z_2}$ such that for each $i \in \{1,2\}$

$$\begin{aligned} z_i(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v_i(\Lambda)d\Lambda - \Big[\frac{1}{\Gamma(\varsigma)}\int_0^{1}(1-\Lambda)^{\varsigma-1}v_i(\Lambda)d\Lambda \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z_i(\Lambda))d\Lambda \\ &\quad -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z_i(\Lambda))d\Lambda\Big]\omega^{\varsigma-1}, \quad \omega\in[0,1]. \end{aligned}$$

For every $\omega \in [0,1]$ we have

$$\begin{aligned} |z_1(\omega)-z_2(\omega)| &\le \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}\big|v_1(\Lambda)-v_2(\Lambda)\big|d\Lambda \\ &\quad +\frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)}\int_0^{1}(1-\Lambda)^{\varsigma-1}\big|v_1(\Lambda)-v_2(\Lambda)\big|d\Lambda \\ &\quad +\frac{pt^{\varsigma-1}}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}\big|k_1(\Lambda,z_1(\Lambda))-k_1(\Lambda,z_2(\Lambda))\big|d\Lambda \\ &\quad +\frac{qt^{\varsigma-1}}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}\big|k_2(\Lambda,z_1(\Lambda))-k_2(\Lambda,z_2(\Lambda))\big|d\Lambda, \end{aligned}$$

that means

$$\begin{aligned} \|z_1-z_2\|_{\mathbb{X}_1} &\le \frac{1}{\Gamma(\varsigma)}\int_0^{1}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}\big|v_1(\Lambda)-v_2(\Lambda)\big|d\Lambda d\omega \\ &\quad +\frac{1}{\Gamma(\varsigma+1)}\int_0^{1}(1-\Lambda)^{\varsigma-1}\big|v_1(\Lambda)-v_2(\Lambda)\big|d\Lambda \\ &\quad +\frac{p}{\varsigma\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}\big|k_1(\Lambda,z_1(\Lambda))-k_1(\Lambda,z_2(\Lambda))\big|d\Lambda \\ &\quad +\frac{q}{\varsigma\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}\big|k_2(\Lambda,z_1(\Lambda))-k_2(\Lambda,z_2(\Lambda))\big|d\Lambda, \end{aligned}$$

using Fubini theorem, we get

$$\begin{aligned} \|z_1-z_2\|_{\mathbb{X}_1} &\le \frac{1}{\Gamma(\varsigma)}\int_0^{1}\int_{\Lambda}^{1}(\omega-\Lambda)^{\varsigma-1}\big|v_1(\Lambda)-v_2(\Lambda)\big|d\omega d\Lambda + \frac{1}{\Gamma(\varsigma+1)}\int_0^{1}\big|v_1(\Lambda)-v_2(\Lambda)\big|d\Lambda \\ &\quad +\frac{C_1p\xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)}\int_0^{\xi_1}\big|z_1(\Lambda)-z_2(\Lambda)\big|d\Lambda + \frac{C_2q\xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)}\int_0^{\xi_2}\big|z_1(\Lambda)-z_2(\Lambda)\big|d\Lambda \\ &\le \big(\frac{2L}{\Gamma(\varsigma+1)}+\frac{C_1p\xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)}\big)\|z_1-z_2\|_{\mathbb{X}_1}. \end{aligned}$$

Since $\frac{2L}{\Gamma(\varsigma+1)}+\frac{C_1p\xi_1^{\mu_1-1}}{\varsigma\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\varsigma\Gamma(\mu_2)}<1$, then $\|z_1-z_2\|_{\mathbb{X}_1}=0$, i.e., $z_1=z_2$, which means the solution is unique. □

Corollary 7. *Assume that assumptions* $(B1^*), (B2), (B3)$ *and* $(B4^*)$ *are hold, and* $\varphi = 0$. *Then the function* $z = 0$ *is the unique solution for the problem* (3.1).

3.1.2 Topological Structure

We present a result on the topological structure of the set of solutions of problem (3.1). Let

$$S = \{y \in \mathbb{X}_1 : y \text{ is a solution of problem } (3.1)\}.$$

It follows that from theorem 3.1.1 that S is nonempty set.

Theorem 3.1.3. *If the following* $(B1), (B2), (B3)$ *and* $(B4)$ *hold. Then the set* S *its a compact in* $\mathbb{X}_1$.

Proof. The proof this theorem is done in three steps.

Step 1. (S is bounded in $\mathbb{X}_1$). Let z be an element in S, then from step 2 in theorem 3.1.1, we have

$$\|z\|_{\mathbb{X}_1} \leq A + B\|z\|_{\mathbb{X}_1},$$

where A and B are defined in step 5 and $B < 1$, then

$$(1 - B)\|z\|_{\mathbb{X}_1} \leq A,$$

that mean S is bounded set in $\mathbb{X}_1$.

Step 2. (S is closed in $\mathbb{X}_1$). Let (z_n) be a sequence in S converges to z. the element (z_n, z_n) belongs in graph of multivalued $\mathcal{N}$ and from step 4 in theorem 3.1.1 the multivalued $\mathcal{N}$ has a closed graph, then S is closed in $\mathbb{X}_1$.

Step 3. (S is compact in $\mathbb{X}_1$). Let $\widehat{S} = \{\widehat{z} \in \mathrm{L}^1(\mathbb{R}); z \in S\}$, where

$$\widehat{z}(\omega) = \begin{cases} z(\omega) & if \quad \omega \in [0, 1], \\ 0 & if \quad \omega \in \mathbb{R} - [0, 1]. \end{cases}$$

From step 1 the set S is bounded in $\mathbb{X}_1$ and

$$\|\Lambda_h \widehat{z} - \widehat{z}\|_{\mathrm{L}^1(\mathbb{R})} \leq \|\varphi\|_1 \varepsilon_1(h) + \frac{A\varepsilon_2(h)}{1 - B},$$

where $\varepsilon_1, \varepsilon_2$ are defined in step 5 in theorem 3.1.1. Then from theorem 1.6.3 the closure of S is compact set in $\mathbb{X}_1$.

From step 1, step 2 and step 3, we conclude that the set S is compact in $\mathbb{X}_1$. $\square$

Theorem 3.1.4. *If the following* $(B1), (B2), (B3)$ *and* $(B4)$ *hold, and supposons that*

$(B5)$

$$\mathcal{H}(\omega, \lambda x + (1-\lambda)y) = \lambda \mathcal{H}(\omega, x) + (1-\lambda)\mathcal{H}(\omega, y),$$
$$k_i(\omega, \lambda x + (1-\lambda)y) = \lambda k_i(\omega, x) + (1-\lambda)k_i(\omega, y),$$

for $i \in \{1,2\}, (\omega, \lambda) \in [0,1] \times [0,1]$ *and* $x, y \in \mathbb{R}$.

Then the set of solutions of problem (3.1) *is nonempty, compact and convex set in* $\mathbb{X}_1$.

Proof. From theorem 3.1.3 the set of solution is nonempty and compact set in $\mathbb{X}_1$. We prove now the set S is convex.

Let z_1, z_2 are two solutions for problem (3.1), and let $\lambda \in [0,1]$

$$\begin{aligned}
\lambda z_1(\omega) + (1-\lambda) z_2(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} \big(\lambda v_1(\Lambda) + (1-\lambda) v_2(\Lambda)\big) d\Lambda \\
&\quad - \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} \big(\lambda v_1(\Lambda) + (1-\lambda) v_2(\Lambda)\big) d\Lambda. \\
&\quad - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} \big(\lambda k_1(\Lambda, z_1(\Lambda)) + (1-\lambda) k_1(\Lambda, z_2(\Lambda))\big) d\Lambda \\
&\quad - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} \big(\lambda k_2(\Lambda, z_1(\Lambda)) + (1-\lambda) k_2(\Lambda, z_2(\Lambda))\big) d\Lambda \Bigg] \omega^{\varsigma-1} \\
&= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} \big(\lambda v_1(\Lambda) + (1-\lambda) v_2(\Lambda)\big) d\Lambda \\
&\quad - \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} \big(\lambda v_1(\Lambda) + (1-\lambda) v_2(\Lambda)\big) d\Lambda. \\
&\quad - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} \big(k_1(\Lambda, \lambda z_1(\Lambda) + (1-\lambda) k_1\big(\Lambda, z_2(\Lambda)\big)\big) d\Lambda \\
&\quad - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2\big(\Lambda, \big(\lambda z_1(\Lambda) + (1-\lambda) z_2(\Lambda)\big)\big) d\Lambda \Bigg] \omega^{\varsigma-1}.
\end{aligned}$$

The set valued map $\mathcal{H}$ is convex, we conclude

$\lambda v_1 + (1-\lambda) v_2 \in S_{\mathcal{H}, \lambda z_1 + (1-\lambda) z_2}$ and $\lambda z_1 + (1-\lambda) z_2$ be an element in S. $\square$

3.1.3 Examples

Example 8. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.9} z(\omega) \in \mathcal{H}(\omega, z(\omega)), \\ z(0) = 0, \\ z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 \Lambda(1-\Lambda)^3 z(\Lambda) d\Lambda, \end{cases} \tag{3.9}$$

where

$$\mathcal{H} : (\omega, x) \in [0,1] \times \mathbb{R} \mapsto \left[\frac{|x \sin \omega|}{2}; \frac{\omega |x|}{2} + \arctan \omega \right] \in \mathcal{P}(\mathbb{R}).$$

In this problem, we have

(i) the set valued map $\mathcal{H}$ is carathéodory set valued, and $\mathcal{H}(\omega, x)$ is nonempty, convex and compact set in $\mathbb{R}$.

(ii)

$$\|\mathcal{H}(\omega, x)\| = \sup\{|v|, v \in \mathcal{H}(\omega, x)\} \leq \varphi(\omega) + \psi(\omega)|x|.$$

where $\varphi(\omega) = \arctan \omega$ and $\psi(\omega) = \frac{\omega}{2}$.

(iii) $k_1(\omega, x) = tx$ and $k_2(\omega, x) = 0$.

(iv) $\varsigma = 1.9, p = \mu_1 = 4, q = 0$ and $C_1 = 1$.

Since

$$\frac{2\|\psi\|_\infty}{\Gamma(\varsigma + 1)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\varsigma \Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\varsigma \Gamma(\mu_2)} \approx 0.8981 < 1.$$

Consequently, by theorem 3.1.1 the considered (3.9) admits a solution.

Example 9. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.9} z(\omega) = \omega + \frac{\arctan(z(\omega))}{2}, \\ z(0) = 0, \\ z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 \Lambda(1-\Lambda)^3 z(\Lambda) d\Lambda, \end{cases} \tag{3.10}$$

where $\mathcal{H}(\omega, x) = \{\omega + \frac{\arctan(x)}{2}\}$.

The set valued map $\mathcal{H}$ *is* $\frac{1}{2}$*-Lipschitzean.*

In this problem, we have

(i) $k_1(\omega, x) = tx$ *and* $k_2(\omega, x) = 0$,

(ii) $\varsigma = 1.9, p = \mu_1 = 4, q = 0$,

(iii) *we will take* $\psi(\omega) = \frac{1}{2}$ *and* $C_1 = 1$.

Since

$$\frac{2\max\{L, \|\psi\|_\infty\}}{\Gamma(\varsigma+1)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\varsigma\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\varsigma\Gamma(\mu_2)} \approx 0.8981 < 1.$$

Consequently, from theorem 3.1.2 the considered (3.10) *has a unique solution.*

3.2 Existence result in Sobolev spaces

Consider $\mathbb{X}_2 = \mathrm{W}^{1,1}([0,1];\mathbb{R})$ the space of all functions f such that $f \in \mathbb{X}_1$ and $f' \in \mathbb{X}_1$, provided by the norm

$$\|z\|_{\mathbb{X}_2} = \|z\|_{\mathbb{X}_1} + \|z'\|_{\mathbb{X}_1}$$

It's clearly that $(\mathbb{X}_2, \|.\|_{\mathbb{X}_2})$ is a complete space.

For $z \in \mathbb{X}_2$, the set of all selections of $\mathcal{H}$ is defined by

$$S_{\mathcal{H},z} = \{v \in \mathrm{L}^1([0,1],\mathbb{R}) : v(\omega) \in \mathcal{H}(\omega, z(\omega)) \text{ for almost } \omega \in [0,1]\}. \tag{3.11}$$

Theorem 3.2.1. *Suppose the following assertions are satisfied.*

(C1) $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is aset valued map with*

1. $\mathcal{H}$ *is integrably bounded and the set valued map* $\omega \in [0,1] \mapsto \mathcal{H}(\omega, x)$ *is measurable for all* $x \in \mathbb{R}$ *;*
2. *there exist* $m \in \mathrm{L}^\infty([0,1],\mathbb{R}^+)$ *such that for almost* $\omega \in [0,1]$ *and all* $x, y \in \mathbb{R}$,

$$H_d(\mathcal{H}(\omega, x), \mathcal{H}(\omega, y)) \leq m(\omega)|x - y|.$$

(C2) *There exists* $C_1, C_2 > 0$ *such that*

$$|k_i(\omega, x) - k_i(\omega, y)| \leq C_i|x - y|,$$

for $\omega \in [0, 1]$ *and* $i \in \{1, 2\}$.

(C3) $\gamma = \big(1 + \frac{1}{\varsigma}\big)\big(\frac{2\|m\|_\infty}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)}\big) < 1$.

Then, the problem (2.16) *has at least one solution in* $W^{1,1}([0, 1], \mathbb{R})$.

Proof. From $(C1)(2)$ for almost $\omega \in [0, 1]$ the multivalued $\mathcal{H}(\omega, .)$ is upper semicontinuous, then from proposition 7.9 page 229 in [33] the set valued $\omega \in [0, 1] \mapsto \mathcal{H}(\omega, z(\omega))$ is measurable and from lemma 1.4.1 there exist a measurable function f such that $f(\omega) \in \mathcal{H}(\omega, z(\omega))$ for $z \in \mathbb{X}$, then $f \in S_{\mathcal{H},z}$. Define the set valued map $\mathcal{N}$ by

$$\begin{aligned}\mathcal{N}(z) = \Bigg\{h \in \mathbb{X}_1 :\ h(\omega) = &\frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1 - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda \\ &- \frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} k_1(\Lambda, z(\Lambda)) d\Lambda \\ &- \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2(\Lambda, z(\Lambda)) d\Lambda\Bigg]\omega^{\varsigma - 1}, v \in S_{\mathcal{H},z}\Bigg\}.\end{aligned} \tag{3.12}$$

we will show satisfies conditions of theorem 1.3.1. The proof is done in two steps.

Step 1. ($\mathcal{N}(z) \in \mathcal{P}_{cl}(\mathbb{X}_2)$). let (z_n) be a sequence in $\mathcal{N}(z)$ converges to $\tilde{z}$ in $\mathbb{X}_2$. Then $\tilde{z} \in \mathbb{X}_2$ and there exist $v_n \in S_{\mathcal{H},z}$ such that, for every $\omega \in [0, 1]$,

$$\begin{aligned} z_n(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1 - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda \\ &\quad - \frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1} (\xi_1 - \Lambda)^{\mu_1 - 1} k_1(\Lambda, z(\Lambda)) d\Lambda - \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2} (\xi_2 - \Lambda)^{\mu_2 - 1} k_2(\Lambda, z(\Lambda)) d\Lambda\Bigg]\omega^{\varsigma - 1}.\end{aligned}$$

Let $\mathcal{A} = \{v_n \in S_{\mathcal{H},z}; n \geq 1\}$, since $\mathcal{H}$ has compact values, then by proposition 1.6.2, the set $\mathcal{A}$ is weakly compact in $\mathrm{L}^1([0, 1]; \mathbb{R})$, there exist a subsequence to obtain that v_{n_k} converges weakly to v in $\mathrm{L}^1([0, 1]; \mathbb{R})$. Therefore $v \in S_{\mathcal{H},z}$ and for each $\omega \in [0, 1]$

$$\begin{aligned}&\frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)}\int_0^1 (1 - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda]\omega^{\varsigma - 1} \\ &\to \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda - [\frac{1}{\Gamma(\varsigma)}\int_0^1 (1 - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda]\omega^{\varsigma - 1},\end{aligned}$$

then, for every $\omega \in [0,1]$, we have

$$\begin{aligned}\tilde{z}(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}v(\Lambda)d\Lambda\right.\\ &\left.-\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda - \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\right]\omega^{\varsigma-1}.\end{aligned}$$

Hence $\tilde{z} \in \mathcal{N}(z)$.

Step 2. ($\mathcal{N}$ is contraction.) We prove that there exist $0 < \gamma < 1$ such that

$$H_d(\mathcal{N}(z), \mathcal{N}(\tilde{z})) \leq \gamma\|z-\tilde{z}\|_{\mathbb{X}_2} \quad \text{for every} \quad z, \tilde{z} \in \mathbb{X}_2.$$

Let $z, \tilde{z} \in \mathbb{X}_2$ and $f_1 \in \mathcal{N}(z)$, there exists $v_1 \in S_{\mathcal{H},z}$ such that, for all $\omega \in [0,1]$,

$$\begin{aligned}f_1(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v_1(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}v_1(\Lambda)d\Lambda\right.\\ &-\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda\\ &\left.-\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\right]\omega^{\varsigma-1}.\end{aligned}$$

By $(C1)$, we have

$$H_d\Big(\mathcal{H}(\omega, z(\omega)); \mathcal{H}(\omega, \tilde{z}(\omega))\Big) \leq m(\omega)|z(\omega)-\tilde{z}(\omega)| \quad \omega \in [0,1].$$

Then by proposition 1.4.4 implies that the set-valued map $\omega \mapsto \mathcal{H}(\omega, \tilde{z}(\omega))$ is measurable, from lemma 1.4.3 has a measurable selection noted by v_2 such that

$$|v_1(\omega) - v_2(\omega)| \leq m(\omega)|z(\omega)-\tilde{z}(\omega)|.$$

For each $\omega \in [0,1]$, let us define

$$\begin{aligned}f_2(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v_1(\Lambda)d\Lambda - \left[\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}v_1(\Lambda)d\Lambda\right.\\ &-\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z(\Lambda))d\Lambda\\ &\left.-\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z(\Lambda))d\Lambda\right]\omega^{\varsigma-1}. \text{ for a.e. } \omega \in [0,1].\end{aligned}$$

Hence

$$|f_1(\omega)-f_2(\omega)|\leq \frac{1}{\Gamma(\varsigma)}\int_0^\omega(\omega-\Lambda)^{\varsigma-1}|v_1(\Lambda)-v_2(\Lambda)|d\Lambda$$
$$+\frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}|v_1(\Lambda)-v_2(\Lambda)|d\Lambda$$
$$+\frac{p\omega^{\varsigma-1}}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}|k_1(\Lambda,z(\Lambda))-k_1(\Lambda,\tilde{z}(\Lambda))|d\Lambda$$
$$+\frac{q\omega^{\varsigma-1}}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}|k_2(\Lambda,z(\Lambda))-k_2(\Lambda,\tilde{y}(\Lambda))|d\Lambda.$$

Then

$$\begin{aligned}\int_0^1|f_1(\omega)-f_2(\omega)|d\omega \ &\leq\ \frac{1}{\Gamma(\varsigma)}\int_0^1\int_0^\omega(\omega-\Lambda)^{\varsigma-1}|v_1(\Lambda)-v_2(\Lambda)|d\Lambda d\omega\\
&+\frac{1}{\Gamma(\varsigma+1)}\int_0^1(1-\Lambda)^{\varsigma-1}|v_1(\Lambda)-v_2(\Lambda)|d\Lambda\\
&+\frac{p}{\varsigma\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}|k_1(\Lambda,z(\Lambda))-k_1(\Lambda,\tilde{z}(\Lambda))|d\Lambda\\
&+\frac{q}{\varsigma\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}|k_2(\Lambda,z(\Lambda))-k_2(\Lambda,\tilde{z}(\Lambda))|d\Lambda,\end{aligned}$$

and

$$\begin{aligned}\int_0^1|f_1^{'}(\omega)-f_2^{'}(\omega)|d\omega \ &\leq\ \frac{1}{\Gamma(\varsigma-1)}\int_0^1\int_0^\omega(\omega-\Lambda)^{\varsigma-2}|v_1(\Lambda)-v_2(\Lambda)|d\Lambda d\omega\\
&+\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}|v_1(\Lambda)-v_2(\Lambda)|d\Lambda\\
&+\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}|k_1(\Lambda,z(\Lambda))-k_1(\Lambda,\tilde{z}(\Lambda))|d\Lambda\\
&+\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}|k_2(\Lambda,z(\Lambda))-k_2(\Lambda,\tilde{z}(\Lambda))|d\Lambda.\end{aligned}$$

By Fubini theorem, we get

$$\begin{aligned}\|f_1-f_2\|_{\mathbb{X}_2} \ &=\ \int_0^1|f_1(\omega)-f_2(\omega)|d\omega+\int_0^1|f_1^{'}(\omega)-f_2^{'}(\omega)|d\omega\\
&\leq\big(1+\frac{1}{\varsigma}\big)\big(\frac{2\|m\|_\infty}{\Gamma(\varsigma)}+\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\big)\|z-\tilde{z}\|_{\mathbb{X}_2}.\end{aligned}$$

Analogously, interchanging the roles of z and $\tilde{z}$, we get

$$H_d(\mathcal{N}(z),\mathcal{N}(\tilde{z}))\leq\gamma\|z-\tilde{z}\|_{\mathbb{X}_2}\ \text{ for every }\ z,\tilde{z}\in\mathbb{X}_2.$$

Since $\mathcal{N}$ is a contraction set valued map, it follows from theorem 1.3.1 that $\mathcal{N}$ has a fixed point $z \in \mathbb{X}_2$ which represent a solution of (3.1). The proof is fulfilled. $\square$

Example 10. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}_{0^+}^{1.7} z(\omega) \in \mathcal{H}(\omega, z(\omega)) \text{ for almost } t \in [0,1], \\ z(0) = 0, \\ z(1) = \dfrac{1}{\Gamma(4)} \displaystyle\int_0^1 (1-\Lambda)^3 \frac{\cos(z(\Lambda))}{(1+\Lambda)^2} d\Lambda, \end{cases} \tag{3.13}$$

where

$$\mathcal{H}(\omega, x) = \big[-\frac{1}{10}.\frac{|\sin x|}{1+\omega^2} - 1, -1\big] \cup \big[0, 1\big],$$

and

$$k_1(\omega, x) = \frac{\cos x}{(1+\omega)^2}, \;\; k_2(\omega, x) = 0,$$

let $\varsigma = 1.7, \mu_1 = 4, C_1 = p = \xi_1 = 1$ *and* $q = 0$.

For each $\omega \in [0,1]$ *and* $x, y \in \mathbb{R}$

$$|k_1(\omega, x) - k_1(\omega, y)| \le |x - y|,$$
$$m(\omega) = \frac{1}{10(1+\omega^2)},$$

and

$$\gamma = \big(1 + \frac{1}{\varsigma}\big)\big(\frac{2\|m\|_\infty}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)}\big) \approx 0.6143.$$

From theorem 3.2.1 the problem (3.13) *has a solution* $z \in \mathbb{X}_2$.

3.3 Existence and uniqueness results in sobolev fractional space

Let

$$\mathbb{X} = \{z \in \mathrm{L}^2([0,1];\mathbb{R}); D_{0^+}^{\beta} z \in \mathrm{L}^2([0,1];\mathbb{R})\} \;\; \text{where } 0 < \beta < \varsigma - 1.$$

The space $(\mathbb{X}, \|.\|_{\mathbb{X}})$ is complete reflexive space [34], with

$$\|z\|_{\mathbb{X}} = \|z\|_2 + \|D_{0^+}^{\beta} z\|_2.$$

In this section we study the existence solution for the problem (3.1).

Theorem 3.3.1. *Assume that the following assertions $(D1)-(D3)$ hold*

(D1) $\mathcal{H}:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ *is a Carathéodory set valued map and integrably bounded with* $\phi\in\mathcal{C}([0,1];\mathbb{R})$.

(D2) There are $C_1,C_2>0$ *such that*

$$|k_j(\omega,x)-k_j(\omega,y)|\leq C_j|x-y|,$$

and

$$k_j(\omega,0)=0,$$

for $\omega\in[0,1]$ *and* $j\in\{1,2\}$.

(D3) $\frac{C_1p\xi_1^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\xi_2^{\mu_2-1}}{\Gamma(\mu_2)}<1.$

Then there exists a solution for the problem (3.1).

Proof. For each measurable function z, the set $S_{\mathcal{H},z}$ is nonempty.

We use the iterative method, let (z_n) be a sequence of measurable function with $z_0\in\mathbb{X}$ such that

$$\begin{aligned} z_{n+1}(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega(\omega-\Lambda)^{\varsigma-1}v_n(\Lambda)d\Lambda-\Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1(1-\Lambda)^{\varsigma-1}v_n(\Lambda)d\Lambda \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-\Lambda)^{\mu_1-1}k_1(\Lambda,z_n(\Lambda))d\Lambda \\ &\quad -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-\Lambda)^{\mu_2-1}k_2(\Lambda,z_n(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}, \end{aligned} \tag{3.14}$$

with $v_n\in S_{\mathcal{H},z_n}$ for each $n\in\mathbb{N}$.

Step 1 ($z_n\in\mathbb{X}$).

We use the proof by recurrence, since $\mathcal{H}$ integrably bounded, then there exist a function $\phi\in\mathrm{L}^1([0,1];\mathbb{R})$ such that

$$v_n(\omega)\leq\phi(\omega)\ \text{ for almost }\omega\in[0,1],$$

then $z_0 \in \mathbb{X}$ and if $z_n \in \mathbb{X}$, we get

$$|z_{n+1}(\omega)| \leq \frac{2\|\phi\|_1}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} \int_0^{\xi_1} |z_n(\Lambda)| d\Lambda + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} \int_0^{\xi_2} |z_n(\Lambda)| d\Lambda,$$

hence $z_{n+1} \in \mathrm{L}^2([0,1];\mathbb{R})$ and

$$\begin{aligned} D_{0^+}^{\beta} z_{n+1}(\omega) &= \frac{1}{\Gamma(\varsigma-\beta)} \int_0^{\omega} (\omega-\Lambda)^{\varsigma-\beta-1} v_n(\Lambda) d\Lambda \\ &- \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\beta)} \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} v_n(\Lambda) d\Lambda \\ &- \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1-\Lambda)^{\mu_1-1} k_1(\Lambda, z_n(\Lambda)) d\Lambda \\ &- \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2-\Lambda)^{\mu_2-1} k_2(\Lambda, z_n(\Lambda)) d\Lambda \Bigg] \omega^{\varsigma-\beta-1}, \end{aligned}$$

that means

$$|D_{0^+}^{\beta} z_{n+1}(\omega)| \leq \frac{2\|v_n\|_1}{\Gamma(\varsigma-\beta)} + \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\beta)} \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right) \|z_n\|_2,$$

that means $D_{0^+}^{\beta} z_{n+1} \in \mathrm{L}^2([0,1];\mathbb{R})$, then the sequence (z_n) belongs to $\mathbb{X}$.

Step 2 (boundedness of (z_n) in $\mathbb{X}$).

Let $n \geq 1$ and $\omega \in [0,1]$, then

$$|z_{n+1}(\omega)| \leq \frac{2}{\Gamma(\varsigma+1)} \int_0^1 |v_n(\Lambda)| d\Lambda + \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} \right) \|z_n\|_2,$$

then by simple calculation, we have

$$|z_{n+1}(\omega)|^2 \leq \left(\frac{2\|\phi\|_1}{\Gamma(\varsigma+1)} \sum_{j=0}^{n} \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)^j + \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)^{n+1} \|z_0\|_2 \right)^2,$$

finally we get

$$\begin{aligned} \|z_{n+1}\|_2 &\leq \frac{2\|\phi\|_1}{\Gamma(\varsigma+1)} \sum_{j=0}^{n} \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)^j \\ &+ \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)^{n+1} \|z_0\|_2, \end{aligned}$$

i.e., the sequence (z_n) is bounded in $\mathrm{L}^2([0,1];\mathbb{R})$, and

$$\begin{aligned} D_{0^+}^{\beta} z_{n+1}(\omega) &= \frac{1}{\Gamma(\varsigma-\beta)} \int_0^{\omega} (\omega-\Lambda)^{\varsigma-\beta-1} v_n(\Lambda) d\Lambda - \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\beta)} \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} v_n(\Lambda) d\Lambda \\ &- \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1-\Lambda)^{\mu_1-1} k_1(\Lambda, z_n(\Lambda)) d\Lambda \\ &- \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2-\Lambda)^{\mu_2-1} k_2(\Lambda, z_n(\Lambda)) d\Lambda \Bigg] \omega^{\varsigma-\beta-1}, \end{aligned}$$

so

$$|D_{0^+}^{\beta} z_{n+1}(\omega)|^2 \leq \left(\frac{2\|v_n\|_1}{\Gamma(\varsigma-\beta)} + \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\beta)} \big(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} \big) \|z_n\|_2 \right)^2.$$

Since (z_n) is bounded in $\mathrm{L}^2([0,1];\mathbb{R})$ and $\mathcal{H}$ integrably bounded, we can easily to see that there exist $\gamma > 0$ such that

$$\|D_{0^+}^{\beta} z_n\|_2 \leq \gamma \text{ for every } n \geq 1. \tag{3.15}$$

From (3.15) and (3.15) we conclude that the sequence (z_n) is bounded in $\mathbb{X}$.

Step 3 (Passage to the limit).

Since (z_n) is bounded in $\mathbb{X}$ and $\mathbb{X}$ is reflexive Banach space, then has a subsequence (z_{n_k}) converges weakly to an element in $\mathbb{X}$ noted by $\bar{z}$. Now we show that $\bar{z}$ is solution to the problem (3.3).

Let (v_{n_k}) be a sequence in $\mathrm{L}^1([0,1];\mathbb{R})$ with

$$v_{n_k}(\omega) \in \mathcal{H}(\omega, z_{n_k}(\omega)) \text{ for almost } \omega \in [0,1].$$

By proposition 1.6.2 the sequence (v_{n_k}) has a subsequence converge weakly to $\bar{v}$ in $\mathrm{L}^1([0,1];\mathbb{R})$.

The sequence $z_{n_k}(\omega)$ be a bounded sequence in $\mathbb{R}$, then has a subsequence noted by $z_{n_k}(\omega)$ converge to $u(\omega)$ and $u(\omega) = \bar{z}(\omega)$ for each $\omega \in [0,1]$.

The sequence $(v_{n_k}(\omega))$ be a bounded sequence in $\mathbb{R}$ (because ϕ is bounded), then has a subsequence noted by $v_{n_k}(\omega)$ converge to $w(\omega)$ and $w(\omega) = \bar{v}(\omega)$.

The upper semicontinuous of $\mathcal{H}$ dictates that

$$\bar{v}(\omega) \in \mathcal{H}(\omega, \bar{z}(\omega)) \text{ for almost } \omega \in [0,1].$$

Passage to the limit of the equation (3.14), we get

$$\begin{aligned} \bar{z}(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega-\Lambda)^{\varsigma-1} \bar{v}(\Lambda) d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)} \int_0^1 (1-\Lambda)^{\varsigma-1} \bar{v}(\Lambda) d\Lambda \\ &\quad - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1-\Lambda)^{\mu_1-1} k_1(\Lambda, \bar{z}(\Lambda)) d\Lambda - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2-\Lambda)^{\mu_2-1} k_2(\Lambda, \bar{z}(\Lambda)) d\Lambda \Bigg] \omega^{\varsigma-1}, \end{aligned}$$

that means $\bar{z}$ is solution to problem (3.1), and $\bar{z} \in \mathbb{X}$. $\square$

Example 11. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}^{1.9}_{0^+} z(\omega) \in \mathcal{H}(\omega, z(\omega)) \text{ for almost } t \in [0,1], \\ z(0) = 0, \\ z(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 \Lambda(1-\Lambda)^3 z(\Lambda) d\Lambda, \end{cases} \tag{3.16}$$

where

$$\mathcal{H} : (\omega, x) \in [0,1] \times \mathbb{R} \mapsto \left[\frac{|x \sin \omega|}{2}; \frac{\omega |x|}{2} + \arctan \omega \right] \in \mathcal{P}(\mathbb{R}).$$

We have

- *(i) the set valued map $\mathcal{H}$ is carathéodory set valued, and $\mathcal{H}(\omega, x)$ is nonempty and compact set in $\mathbb{R}$.*
- *(iii) $k_1(\omega, x) = tx$ and $k_2(\omega, x) = 0$.*
- *(iv) $\varsigma = 1.9, p = \mu_1 = 4, q = 0$ and $C_1 = 1$.*

Since

$$\frac{C_1 p \xi_1^{\mu_1 - 1}}{\varsigma \Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\varsigma \Gamma(\mu_2)} < 1.$$

Consequently, by theorem 3.3.1 the considered (3.16) admits a solution.

Remark 3.3.1. *If the function ϕ is not continuous functions, but ϕ is measurable and bounded on $[0,1]$. Then the result of theorem 3.3.1 still valid.*

Theorem 3.3.2. *Assume that the following assertions $(D1^*), (D2)$ and $(D3)$ are satisfied*

$(D1^*)$ $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is integrably bounded with $\phi \in \mathcal{C}([0,1]; \mathbb{R})$ such that

- *The set-valued map $\mathcal{H}(\omega, .)$ is K-lipschitzean.*
- *The set-valued map $\mathcal{H}(., x)$ is measurable for $x \in \mathbb{R}$.*

(D2) There are $C_1, C_2 > 0$ such that

$$|k_j(\omega, x) - k_j(\omega, y)| \leq C_j |x - y|,$$

and

$$k_j(\omega, 0) = 0,$$

for $\omega \in [0,1]$ *and* $j \in \{1,2\}$.

(D3) $\frac{2K}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} < 1.$

Then the problem 3.1 has one solution in $\mathbb{X}$.

Proof. From theorem 3.3.1 the problem (3.1) has a solution, now we prove this solution is unique.

Let z_1 and z_2 be two solution for the problem (3.1), then there exist $v_1 \in S_{\mathcal{H},z_1}$ and $v_2 \in S_{\mathcal{H},z_2}$ such that

$$\begin{aligned} z_j(\omega) &= \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1} v_j(\Lambda)d\Lambda - \Bigg[\frac{1}{\Gamma(\varsigma)}\int_0^1 (1-\Lambda)^{\varsigma-1} v_j(\Lambda)d\Lambda \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1} (\xi_1-\Lambda)^{\mu_1-1} k_1(\Lambda, z_j(\Lambda))d\Lambda \\ &\quad -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2} (\xi_2-\Lambda)^{\mu_2-1} k_2(\Lambda, z_j(\Lambda))d\Lambda\Bigg]\omega^{\varsigma-1}, \end{aligned}$$

for $1 \le j \le 2$, then

$$\begin{aligned} |z_2(\omega) - z_1(\omega)| &\le \frac{2K}{\Gamma(\varsigma)}\int_0^1 |z_2(\Lambda) - z_1(\Lambda)|d\Lambda \\ &\quad + \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)\int_0^1 |z_2(\Lambda) - z_1(\Lambda)|d\Lambda, \end{aligned}$$

after that

$$\|z_2 - z_1\|_2 \le \left(\frac{2K}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)\|z_2 - z_1\|_2,$$

while

$$\frac{2K}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} < 1,$$

then $z_1 = z_2$. □

Example 12. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.9} z(\omega) = \omega + \frac{1.9\arctan(z(\omega))}{2} & \textit{for almost } \omega \in [0,1], \\ z(0) = 0, \\ z(1) = \frac{4}{\Gamma(4)}\int_0^1 \Lambda(1-\Lambda)^3 z(\Lambda)d\Lambda, \end{cases} \tag{3.17}$$

where $\mathcal{H}(\omega, x) = \{\omega + \frac{\arctan(x)}{2}\}$.

The set valued map $\mathcal{H}$ *is* $\frac{1.9}{2}$*-Lipschitzean.*

In this problem, we have

(i) $k_1(\omega, x) = 1.9\omega x$ *and* $k_2(\omega, x) = 0$,

(ii) $\varsigma = 1.9, p = \mu_1 = 4, q = 0$,

(iii) $C_1 = 1.9$.

Since

$$\frac{2L}{\Gamma(\varsigma)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)} \approx 0.8981 < 1.$$

Consequently, from theorem 3.3.2 the considered (3.17) *has a unique solution.*

Chapter 4

Existence and uniqueness results for IVP of nonlinear fractional inclusion on an unbounded domain

In this chapter we study the existence and uniqueness for IVP of fractional differential inclusion

$$\begin{cases} \mathcal{D}^{\varsigma}_{0+}z(\omega) \in \mathcal{H}(\omega, z(\omega)), \quad \text{for almost} \quad \omega \in J = [0, +\infty[\\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0+}z(0) = I^{\beta}_{0+}z(\xi), \end{cases} \tag{4.1}$$

where $1 < \varsigma < 2$, $0 < \xi < 1, \beta \geq 1$, and $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a set valued mapping.

Definition 4.0.1. *A function z is said to be a solution of problem (4.1), if there exists a function $v \in \mathrm{L}^1([0, 1], \mathbb{R})$ with $v(\omega) \in \mathcal{H}(\omega, z(\omega))$ for almost $\omega \in J$ such that*

$$\begin{cases} \mathcal{D}^{\varsigma}_{0+}z(\omega) = v(\omega), \quad \textit{for almost} \quad \omega \in J, 1 < \varsigma < 2, \\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0+}z(0) = I^{\beta}_{0+}z(\xi). \end{cases} \tag{4.2}$$

Lemma 4.0.1. *For a given $y \in \mathrm{L}^1(J, \mathbb{R})$, a function z is a solution to*

$$\begin{cases} \mathcal{D}^{\varsigma}_{0+} z(\omega) = y(\omega), \quad \textit{for almost} \;\; \omega \in J, 1 < \varsigma < 2, \\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0+} z(0) = I^{\beta}_{0+} z(\xi), \end{cases} \tag{4.3}$$

if and only if

$$z(\omega) \;\; = \;\; \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} y(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda. \tag{4.4}$$

Proof. Assume that z satisfies (4.15), from lemma 1.5.3 we have

$$z(\omega) = I^{\varsigma}_{0+} y(\omega) - c_1 \omega^{\varsigma-1} - c_2 \omega^{\varsigma-2}, \tag{4.5}$$

where $c_1, c_2 \in \mathbb{R}$.

By using first boundary condition, we get that $c_2 = 0$ and from the second boundary condition, we find

$$\mathcal{D}^{\varsigma-1}_{0+} z(\omega) = I^{\varsigma-1}_{0+} y(\omega) - c_1 \Gamma(\varsigma),$$

this means that

$$\mathcal{D}^{\varsigma-1}_{0+} z(0) = -c_1 \Gamma(\varsigma) = I^{\beta}_{0+} z(\xi),$$

that is

$$c_1 = \frac{-1}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda.$$

By replacing c_1 in (4.5) we get the equation (4.4).

Conversely, if z satisfies (4.4) by lemma 1.5.2 and lemma 1.5.4, we have $\mathcal{D}^{\varsigma}_{0+} z(\omega) = y(\omega)$, and

$$\mathcal{D}^{\varsigma-1}_{0+} z(\omega) = I^{\varsigma-1}_{0+} y(\omega) + \frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda,$$

by simple calculation, we get $\mathcal{D}^{\varsigma-1}_{0+} z(0) = I^{\beta}_{0+} z(\xi)$, and we can easily to see that $z(0) = 0$. $\square$

4.1 Study the existence and uniqueness in weight spaces

Let $\mathcal{C}(J;\mathbb{R})$ the space of all functions defined on J and continuous on J, and let

$$\mathbb{X} = \{z \in \mathcal{C}(J,\mathbb{R}); \sup_{\omega\in J} \frac{|z(\omega)|}{(1+\omega^{\varsigma-1})^2} < \infty\},$$

equipped the norm

$$\|z\|_{\mathbb{X}} = \sup_{\omega\in J} \frac{|z(\omega)|}{(1+\omega^{\varsigma-1})^2}.$$

Lemma 4.1.1. *The space* $(\mathbb{X}, \|.\|_{\mathbb{X}})$ *is Banach space.*

Proof. Let (z_n) be be a Cauchy sequence in $\mathbb{X}$. Then for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that

$$\|z_m - z_n\|_{\mathbb{X}} = \sup_{\omega\in J} \frac{|z_m(\omega) - z_n(\omega)|}{(1+\omega^{\varsigma-1})^2} < \frac{\varepsilon}{2} \text{ for } m \geq n \geq N.$$

We remark that

$$\frac{|z_m(\omega) - z_n(\omega)|}{(1+\omega^{\varsigma-1})^2} \leq \|z_m - z_n\|_{\mathbb{X}} < \frac{\varepsilon}{2} \text{ for all } \omega \in J.$$

The sequence $(\frac{z_n(\omega)}{(1+\omega^{\varsigma-1})^2})$ be a Cauchy sequence in $\mathbb{R}$, thus it's converge to $\frac{z(\omega)}{(1+\omega^{\varsigma-1})^2}$ for all $\omega \in J$, and

$$\begin{aligned} |z(\omega) - z(t_0) \quad | \quad &\leq |z_n(\omega) - z(\omega)| + |z_n(\omega) - z_n(t_0)| + |z_n(t_0) - z(t_0)| \\ &\leq (1+\omega^{\varsigma-1})^2 |\frac{z_n(\omega)}{(1+\omega^{\varsigma-1})^2} - \frac{z(\omega)}{(1+\omega^{\varsigma-1})^2}| + |z_n(\omega) - z_n(t_0)| \\ &+ (1+\omega^{\varsigma-1})^2 |\frac{z_n(t_0)}{(1+\omega^{\varsigma-1})^2} - \frac{z(t_0)}{(1+\omega^{\varsigma-1})^2}| \end{aligned}$$

as $\omega \to t_0$ and $n \to \infty$ we have $z(\omega) - z(t_0) \to 0$.

Then $z \in \mathcal{C}(J,\mathbb{R})$ and for each $\omega \in J$ there exist $N_1 = N_1(\varepsilon,\omega) \in \mathbb{N}$ such that

$$\frac{|z_n(\omega) - z(\omega)|}{(1+\omega^{\varsigma-1})^2} < \frac{\varepsilon}{2} \text{ for all } n \geq N_1.$$

Let $n \geq \max(N_1, N)$ then

$$\begin{aligned} |\frac{z(\omega)}{(1+\omega^{\varsigma-1})^2} \quad | \quad &\leq \frac{|z_N(\omega) - z(\omega)|}{(1+\omega^{\varsigma-1})^2} + \frac{|z_N(\omega)|}{(1+\omega^{\varsigma-1})^2} \\ &\leq \frac{|z_n(\omega) - z(\omega)|}{(1+\omega^{\varsigma-1})^2} + \frac{|z_N(\omega) - z_n(\omega)|}{(1+\omega^{\varsigma-1})^2} + \frac{|z_N(\omega)|}{(1+\omega^{\varsigma-1})^2} \\ &< \frac{\varepsilon}{2} + \|z_N - z_n\|_{\mathbb{X}} + \|z_N\|_{\mathbb{X}} < \varepsilon + \|z_N\|_{\mathbb{X}}. \end{aligned}$$

Then $z \in \mathbb{X}$, i.e., $\mathbb{X}$ is Banach space. $\square$

Lemma 4.1.2. *Let A be a bounded set in $\mathbb{X}$. Then A is relatively compact in $\mathbb{X}$ if the following conditions hold*

- *(i) A is equicontinuous in $[0, +\infty)$.*
- *(ii) for all $\varepsilon > 0$, there exist $x > 0$, for all $z \in A$ and $\omega \geq x$*

$$\frac{z(\omega)}{(1+\omega^{\varsigma-1})^2} < \varepsilon.$$

Proof. For each $\varepsilon > 0$, by (ii) there exist $x > 0$ such that for each $z \in A$ and $\omega \geq x$, we have

$$\frac{z(\omega)}{(1+\omega^{\varsigma-1})^2} < \varepsilon.$$

Let

$$A_{|[0,x]} = \{z_{|[0,x]}, z \in A\}.$$

In view of (i) and theorem 1.6.2 we get that the closure of $A_{|[0,x]}$ is compact set in $\mathcal{C}([0,x], \mathbb{R})$, that means there exist $z_1, \ldots, z_n \in A$ such that

$$\max_{\omega \in [0,x]} \frac{|z(\omega) - z_j(\omega)|}{(1+\omega^{\varsigma-1})^2} < \varepsilon \quad \text{for some } 1 \leq j \leq n.$$

Thence

$$\|z - z_j\|_{\mathbb{X}} = \max\Big\{\max_{\omega \in [0,x]} \frac{|z(\omega) - z_j(\omega)|}{(1+\omega^{\varsigma-1})^2}, \sup_{\omega > x} \frac{|z(\omega) - z_j(\omega)|}{(1+\omega^{\varsigma-1})^2}\Big\} < \varepsilon.$$

Hence A is precompact,i.e., the closure of A is compact set in $\mathcal{C}(J, \mathbb{R})$. $\square$

4.1.1 Existence result in convex case

Theorem 4.1.1. *Assume that the following assertions $(B1) - (B3)$ hold*

$(B1)$ *$\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ such that for every measurable function $z : J \to \mathbb{R}$ we have $\omega \in J \mapsto \mathcal{H}(\omega, z(\omega))$ is measurable.*

$(B2)$ *There are $\varphi, \psi \in \mathrm{L}^1(J, \mathbb{R}) \cap \mathrm{L}^\infty(J, \mathbb{R})$ such that*

(i) $\|\mathcal{H}(\omega, x)\| = \sup\{|v| : v \in \mathcal{H}(\omega, x)\} \leq \varphi(\omega) + \frac{\psi(\omega)}{(1+\omega^{\varsigma-1})^2}|x|$,

(ii) $\forall \varepsilon > 0, \exists b_\varepsilon > 0$ *such that* $\int_{b_\varepsilon}^{+\infty} |\varphi(\Lambda)| d\Lambda < \frac{\varepsilon}{2}$ *and* $\int_{b_\varepsilon}^{+\infty} |\psi(\Lambda)| d\Lambda < \frac{\varepsilon}{2}$.

$(B3)$ $\frac{\|\psi\|_1}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi + \frac{2\xi^\varsigma}{\varsigma} + \frac{\xi^{2\varsigma-1}}{2\varsigma-1}) < 1.$

Then the problem (4.1) *has a least one solution.*

Proof. Using the conditions $(B1), (B2)(i)$ and by applying lemma 1.4.1 we can easily to see that $S_{\mathcal{H},z} \neq \emptyset$ for all $z \in \mathbb{X}$. Define an operator $\mathcal{N} : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ by

$$\begin{aligned} \mathcal{N}(z) = \Bigg\{ h \in \mathbb{X} : h(\omega) = &\frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda \\ &+ \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda, v \in S_{\mathcal{H},z} \Bigg\}. \end{aligned} \tag{4.6}$$

We will show that $\mathcal{N}$ satisfies all hypotheses of the theorem 1.3.3.

Step 1. ($\mathcal{N}$ is convex valued.) Let $z \in \mathbb{X}$, and $h_1, h_2 \in \mathcal{N}(z)$, then there exist $v_1, v_2 \in S_{\mathcal{H},z}$ such that

$$h_i(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_i(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda \quad \text{for } i \in \{1, 2\}.$$

Then for $\lambda \in [0, 1]$ we have

$$\begin{aligned} \lambda h_1(\omega) + (1 - \lambda) h_2(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} \big(\lambda v_1(\Lambda) + (1 - \lambda) v_2(\Lambda)\big) d\Lambda \\ &\quad + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda \end{aligned}$$

Since $\mathcal{H}$ is convex set-valued, then $\lambda h_1 + (1 - \lambda) h_2 \in \mathcal{N}(z)$.

Step 2. ($\mathcal{N}$ bounded set into bounded set.)

Let $B_r = \{z \in \mathbb{X}, \|z\|_{\mathbb{X}} < r\}$ and let $h \in \mathcal{N}(z)$ with $z \in B_r$, then there exist $v \in S_{\mathcal{H},z}$ such that

$$h(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda.$$

For all $\omega \in J$ we have

$$\begin{aligned} \frac{|h(\omega)|}{1 + \omega^{\varsigma-1}} &\leq \frac{1}{\Gamma(\varsigma)} \int_0^\omega |v(\Lambda)| d\Lambda + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi |z(\Lambda)| d\Lambda \\ &\leq \frac{\|\varphi\|_1 + \|\psi\|_1 \|z\|_{\mathbb{X}}}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)} (\xi + \frac{\xi^\varsigma}{\varsigma}) \|z\|_{\mathbb{X}} \\ &= \frac{\|\varphi\|_1}{\Gamma(\varsigma)} + \|z\|_{\mathbb{X}} \bigg(\frac{\|\psi\|_1}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)} (\xi + \frac{2\xi^\varsigma}{\varsigma} + \frac{\xi^{2\varsigma-1}}{2\varsigma - 1}) \bigg), \end{aligned}$$

remark that

$$\int_0^\xi |z(\Lambda)|d\Lambda = \int_0^\xi (1+\Lambda^{\varsigma-1})^2 \times \frac{|z(\Lambda)|}{(1+\Lambda^{\varsigma-1})^2}d\Lambda \;\leq\; \|z\|_{\mathbb{X}} \int_0^\xi (1+\Lambda^{\varsigma-1})^2 d\Lambda$$
$$= (\xi + \frac{2\xi^\varsigma}{\varsigma} + \frac{\xi^{2\varsigma-1}}{2\varsigma-1})\|z\|_{\mathbb{X}}.$$

Then

$$\|h\|_{\mathbb{X}} \leq l = \frac{\|\varphi\|_1}{\Gamma(\varsigma)} + r\left(\frac{\|\psi\|_1}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi + \frac{2\xi^\varsigma}{\varsigma} + \frac{\xi^{2\varsigma-1}}{2\varsigma-1})\right),$$

and

$$\lim_{\omega\to+\infty} \frac{|h(\omega)|}{(1+\omega^{\varsigma-1})^2} \leq \lim_{\omega\to+\infty} \frac{l}{1+\omega^{\varsigma-1}} = 0.$$

Step 3. ($\mathcal{N}$ bounded set into equicontinuous set.)

Let B_r the set as defined in step 2, and $t_1, t_2 \in J$ where $t_1 \leq t_2$, then

$$\begin{aligned}
|\frac{h(t_1)}{(1+t_1^{\varsigma-1})^2} - \frac{h(t_2)}{(1+t_2^{\varsigma-1})^2}| \;=\; & \Bigg|\frac{1}{\Gamma(\varsigma)}\int_{t_1}^{t_2} \frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} v(\Lambda)d\Lambda \\
& + \frac{1}{\Gamma(\varsigma)}\int_0^{t_1}\left(\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}\right)v(\Lambda)d\Lambda \\
& + \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^\xi (\xi-\Lambda)^{\beta-1}z(\Lambda)d\Lambda\right] \\
& \times (\frac{t_1^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2} - \frac{t_2^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2})\Bigg| \\
& \leq \frac{1}{\Gamma(\varsigma)} \times \frac{(t_2-t_1)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}\int_{t_1}^{t_2}|v(\Lambda)|d\Lambda \\
& + \frac{1}{\Gamma(\varsigma)}\int_0^{t_1}|v(\Lambda)|d\Lambda \times \sup_{\Lambda\in[0,t_1]}\left|\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}\right| \\
& + \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^\xi (\xi-\Lambda)^{\beta-1}z(\Lambda)d\Lambda\right] \\
& \times |\frac{t_1^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2} - \frac{t_2^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}| \\
& \leq \frac{1}{\Gamma(\varsigma)} \times \frac{(t_2-t_1)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} \times \left(\|\varphi\|_1 + \|\psi\|_1\|z\|_{\mathbb{X}}\right) \\
& + \frac{1}{\Gamma(\varsigma)} \times \sup_{\Lambda\in[0,t_1]}\left|\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}\right| \\
& \times \left(\|\varphi\|_1 + \|\psi\|_1\|z\|_{\mathbb{X}}\right)
\end{aligned}$$

$$+\left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^{\xi}(\xi-\Lambda)^{\beta-1}z(\Lambda)d\Lambda\right]$$
$$\times\left|\frac{t_1^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}-\frac{t_2^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}\right|$$
$$\leq\frac{1}{\Gamma(\varsigma)}\times\frac{(t_2-t_1)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}\times\Big(\|\varphi\|_1+\|\psi\|_1\|z\|_{\mathbb{X}}\Big)$$
$$+\frac{1}{\Gamma(\varsigma)}\times\sup_{\Lambda\in[0,t_1]}\left|\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}-\frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}\right|$$
$$\times\Big(\|\varphi\|_1+\|\psi\|_1\|z\|_{\mathbb{X}}\Big)$$
$$+\frac{c_\xi\|z\|_{\mathbb{X}}}{\Gamma(\varsigma)\Gamma(\beta)}\times\left|\frac{t_1^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}-\frac{t_2^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}\right|,$$

where $c_\xi=\int_0^{\xi}(1+\Lambda^{\varsigma-1})^2(\xi-\Lambda)^{\beta-1}d\Lambda$. As $t_2\to t_1$ we have $\left|\frac{h(t_1)}{(1+t_1^{\varsigma-1})^2}-\frac{h(t_2)}{(1+t_2^{\varsigma-1})^2}\right|\to 0$.

Step 4. ($\mathcal{N}$ has a closed graph.)

Let $z_n\to z_*, h_n\in\mathcal{N}(z_n)$ and $h_n\to h_*$, we need to show that $h_*\in\mathcal{N}(z_*)$. Now $h_n\in\mathcal{N}(z_n)$ implies that there exists $v_n\in S_{\mathcal{H},z_n}$ such that

$$h_n(\omega) = \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v_n(\Lambda)d\Lambda+\frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^{\xi}(\xi-\Lambda)^{\beta-1}z_n(\Lambda)d\Lambda,$$

we have to show that there exists $v_*\in S_{\mathcal{H},z_*}$ such that for every $\omega\in[0,+\infty)$,

$$h_*(\omega) = \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda+\frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^{\xi}(\xi-\Lambda)^{\beta-1}z_*(\Lambda)d\Lambda.$$

By theorem 1.6.1, we prove that the set $\mathcal{V}=\{v_n\in S_{\mathcal{H},z_n}; n\geq 1\}$ is weakly compact in $\mathrm{L}^1(J;\mathbb{R})$

$$\int_A|v_n(\Lambda)|d\Lambda\leq\int_A|\varphi(\Lambda)|d\Lambda+\|z_n\|_{\mathbb{X}}\int_A|\psi(\Lambda)|d\Lambda$$
$$\leq|A|\big(\|\varphi\|_\infty+\|z_n\|_{\mathbb{X}}\|\psi\|_\infty\big)<\varepsilon,$$

we take $\delta=\frac{\varepsilon}{\|\varphi\|_\infty+\sup_{n\geq1}\|z_n\|_{\mathbb{X}}\|\psi\|_\infty}$, then the first conditions of theorem 1.6.1 is satisfied. Now we prove the second conditions of this theorem

$$\int_{b_\varepsilon}^{+\infty}v_n(\Lambda)d\Lambda\leq\int_{b_\varepsilon}^{+\infty}\varphi(\Lambda)d\Lambda+\|z_n\|_{\mathbb{X}}\times\int_{b_\varepsilon}^{+\infty}\psi(\Lambda)d\Lambda<\varepsilon.$$

Then by theorem 1.6.1 we conclude that $\mathcal{V}$ is weakly compact set in $\mathrm{L}^1(J,\mathbb{R})$.

Conclusion

There exist a subsequence $v_{n_k} \in S_{\mathcal{H},u_{n_k}}$ such that $v_{n_k} \rightharpoonup v_*$ in $\mathrm{L}^1(J,\mathbb{R})$, then

$$h_*(\omega) \;=\; \frac{1}{\Gamma(\varsigma)}\int_0^{\omega}(\omega-\Lambda)^{\varsigma-1}v_*(\Lambda)d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^{\xi}(\xi-\Lambda)^{\beta-1}z_*(\Lambda)d\Lambda, \tag{4.7}$$

because

$$\begin{aligned}\int_0^{\xi}(\xi-\Lambda)^{\beta-1}|z_n(\Lambda)-z(\Lambda)|d\Lambda \;\leq\; & \int_0^{\xi}|z_n(\Lambda)-z(\Lambda)|d\Lambda = \int_0^{\xi}(1+\Lambda^{\varsigma-1})^2\frac{|z_n(\Lambda)-z(\Lambda)|}{(1+\Lambda^{\varsigma-1})^2}d\Lambda \\ & \qquad\qquad\qquad\qquad\qquad (\text{see } 0<\xi<1) \\ & \leq \|z_n-z\|_{\mathbb{X}}\times\int_0^{\xi}(1+\Lambda^{\varsigma-1})^2 d\Lambda \to 0 \quad as \quad n\to\infty.\end{aligned}$$

Step 5. (A priori bounds on solutions.)

Let

$$\begin{aligned}K = \bar{B}(0,\frac{A}{1-B}) \;\cap\; & \Big\{h\in\mathbb{X};\Big|\frac{h(t_1)}{(1+t_1^{\varsigma-1})^2}-\frac{h(t_2)}{(1+t_2^{\varsigma-1})^2}\Big| \leq A(t_2-t_1)^{\varsigma-1}+\Lambda_1(t_1,t_2) \\ & +\frac{A}{1-B}\Big(\frac{\|\psi\|_1(t_2-t_1)^{\varsigma-1}}{\Gamma(\varsigma)}+\frac{c_\xi}{\Gamma(\varsigma)\Gamma(\beta)}\times\Big|\frac{t_1^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}-\frac{t_2^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}\Big| \\ & +\Lambda_2(t_1,t_2)\Big)\Big\},\end{aligned}$$

where

$$\begin{aligned}A = \frac{\|\varphi\|_1}{\Gamma(\varsigma)} \;\text{ and }\; B &= \frac{\|\psi\|_1}{\Gamma(\varsigma)}+\frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi+\frac{2\xi^{\varsigma}}{\varsigma}+\frac{\xi^{2\varsigma-1}}{2\varsigma-1}), \\ \Lambda_1(t_1,t_2) &= \frac{\|\varphi\|_1}{\Gamma(\varsigma)}\times\sup_{\Lambda\in[0,t_1]}\left|\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}-\frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}\right|, \\ \Lambda_2(t_1,t_2) &= \frac{\|\psi\|_1}{\Gamma(\varsigma)}\times\sup_{\Lambda\in[0,t_1]}\left|\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2}-\frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2}\right|.\end{aligned}$$

The set K is nonempty convex and compact set in $\mathbb{X}$, by theorem 1.3.3 the set valued map $\mathcal{N}$ has a fixed point, which represent a solution. □

Example 13. *Consider the following fractional differential inclusion*

$$\begin{cases}\mathcal{D}_{0^+}^{\frac{3}{2}}z(\omega)\in\mathcal{H}(\omega,z(\omega)), \; \textit{for almost } \omega\in[0,+\infty[, \\ z(0)=0, \\ \mathcal{D}_{0^+}^{\frac{1}{2}}z(0)=\int_0^{0.6}(0.6-\Lambda)z(\Lambda)d\Lambda,\end{cases} \tag{4.8}$$

where

$$\mathcal{H}(\omega,x)=\big[\varphi(\omega),\varphi(\omega)+\frac{|x|}{(1+\sqrt{\omega})^2}\psi(\omega)\big],$$

$$\psi(\omega)=\begin{cases}\sqrt{\omega}-\omega & \omega\in[0,1]\\ 0 & \omega>1\end{cases},\qquad \varphi(\omega)=\begin{cases}\cos\omega & \omega\in[0,\frac{\pi}{2}]\\ 0 & \omega>\frac{\pi}{2}\end{cases}$$

and

$$\frac{\|\psi\|_1}{\Gamma(\varsigma)}+\frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi+\frac{\xi^{\varsigma}}{\varsigma})\approx 0.8041.$$

Then the problem (4.18) *has a least one solution.*

Corollary 8. *On suppose that the all conditions of theorem 4.1.1 hold with* $\varphi=0$*. Then the function* $z=0$ *is a solution for* (4.1) *.*

Theorem 4.1.2. *Supposons that the conditions* $(B1)-(B3)$ *are hold*

$(B1)$ $\mathcal{H}:J\times\mathbb{R}\to\mathcal{P}_{c,cp}(\mathbb{R})$ *such that for every measurable function* $z:J\to\mathbb{R}$ *we have* $\omega\in J\mapsto\mathcal{H}(\omega,z(\omega))$ *is measurable.*

$(B2)$ *There are* $m\in\mathrm{L}^1(J,\mathbb{R})\cap\mathrm{L}^\infty(J,\mathbb{R})$ *and* φ *continuous nondecreasing such that*

(i) $\|\mathcal{H}(\omega,x)\|=\sup\{|v|:v\in\mathcal{H}(\omega,x)\}\leq m(\omega)\varphi(\frac{x}{(1+\omega^{\varsigma-1})^2})$

(ii) $\forall\varepsilon>0,\exists b_\varepsilon>0$ *such that* $\int_{b_\varepsilon}^{+\infty}m(\Lambda)d\Lambda<\varepsilon$.

$(B3)$ *There exist* $L>0$ *such that*

$$L>\frac{\|m\|_1\varphi(L)}{\Gamma(\varsigma)}+\frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi+\frac{\xi^{\varsigma}}{\varsigma})L.$$

Then the problem (4.1) *has a least one solution.*

Proof. To prove the problem (4.1) admits a solution, we have shown that $\mathcal{N}$ admits a fixed point.

From $(B1)$ and $(B2)(i)$ we can easily to show that $S_{\mathcal{H},z}\neq\emptyset$.

From step 1 in proof of theorem 4.1.1, we can see that $\mathcal{N}$ is convex valued. Now we prove that satisfied all hypothesis of theorem 1.3.2 the proof is given in four steps.

Step 1. ($\mathcal{N}$ bounded set into bounded set.)

Let $B_r = \{z \in \mathbb{X}, \|z\|_{\mathbb{X}} < r\}$ and let $h \in \mathcal{N}(z)$ with $z \in B_r$, then there exist $v \in S_{\mathcal{H},z}$ such that

$$h(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda \quad \text{for } i \in \{1,2\}.$$

For all $\omega \in J$ we have

$$\begin{aligned} \frac{|h(\omega)|}{1+\omega^{\varsigma-1}} &\leq \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} |v(\Lambda)| d\Lambda + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} |z(\Lambda)| d\Lambda \\ &\leq l = \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)} (\xi + \frac{2\xi^{\varsigma}}{\varsigma} + \frac{\xi^{2\varsigma-1}}{2\varsigma-1}) \|z\|_{\mathbb{X}}, \end{aligned}$$

and

$$\lim_{\omega \to +\infty} \frac{|h(\omega)|}{(1+\omega^{\varsigma-1})^2} \leq \lim_{\omega \to +\infty} \frac{l}{1+\omega^{\varsigma-1}} = 0.$$

Step 2 ($\mathcal{N}$ bounded set into equicontinuous set.)

$$\begin{aligned} \left|\frac{h(t_1)}{(1+t_1^{\varsigma-1})^2} - \frac{h(t_2)}{(1+t_2^{\varsigma-1})^2}\right| &= \frac{1}{\Gamma(\varsigma)} \int_{t_1}^{t_2} \frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} v(\Lambda) d\Lambda \\ &+ \frac{1}{\Gamma(\varsigma)} \int_0^{t_1} \left(\frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2} \right) v(\Lambda) d\Lambda \\ &+ \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi-\Lambda)^{\beta-1} z(\Lambda) d\Lambda \right] \left| \frac{\omega_2^{\varsigma-1}}{(1+\omega_2^{\varsigma-1})^2} - \frac{\omega_1^{\varsigma-1}}{(1+\omega_1^{\varsigma-1})^2} \right| \\ &\leq \frac{1}{\Gamma(\varsigma)} \times \frac{(t_2-t_1)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} \int_{t_1}^{t_2} |v(\Lambda)| d\Lambda \\ &+ \frac{1}{\Gamma(\varsigma)} \int_0^{t_1} |v(\Lambda)| d\Lambda \times \sup_{\Lambda \in [0,t_1]} \left| \frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2} \right| \\ &+ \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi-\Lambda)^{\beta-1} z(\Lambda) d\Lambda \right] \left| \frac{\omega_2^{\varsigma-1}}{(1+\omega_2^{\varsigma-1})^2} - \frac{\omega_1^{\varsigma-1}}{(1+\omega_1^{\varsigma-1})^2} \right| \\ &\leq \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} \times \frac{(t_2-t_1)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} \\ &+ \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} \times \sup_{\Lambda \in [0,t_1]} \left| \frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2} \right| \\ &+ \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi-\Lambda)^{\beta-1} z(\Lambda) d\Lambda \right] \left| \frac{\omega_2^{\varsigma-1}}{(1+\omega_2^{\varsigma-1})^2} - \frac{\omega_1^{\varsigma-1}}{(1+\omega_1^{\varsigma-1})^2} \right| \\ &\leq \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} \times \frac{(t_2-t_1)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} \\ &+ \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} \times \sup_{\Lambda \in [0,t_1]} \left| \frac{(t_2-\Lambda)^{\varsigma-1}}{(1+t_2^{\varsigma-1})^2} - \frac{(t_1-\Lambda)^{\varsigma-1}}{(1+t_1^{\varsigma-1})^2} \right| \\ &+ \frac{c_{\xi} \|z\|_{\mathbb{X}}}{\Gamma(\varsigma)\Gamma(\beta)} \times \left| \frac{\omega_2^{\varsigma-1}}{(1+\omega_2^{\varsigma-1})^2} - \frac{\omega_1^{\varsigma-1}}{(1+\omega_1^{\varsigma-1})^2} \right|, \end{aligned}$$

where $c_\xi = \int_0^\xi (\xi - \Lambda)^{\beta-1}(1 + \Lambda^{\varsigma-1})^2 d\Lambda$. As $t_2 \to t_1$ we have $|\frac{h(t_1)}{1+t_1^{\varsigma-1}} - \frac{h(t_2)}{1+t_2^{\varsigma-1}}| \to 0$.

Step 3. ($\mathcal{N}$ has a closed graph.)

Let $z_n \to z_*, h_n \in \mathcal{N}(z_n)$ and $h_n \to h_*$, we need to show that $h_* \in \mathcal{N}(z_*)$. Now $h_n \in \mathcal{N}(z_n)$ implies that there exists $v_n \in S_{\mathcal{H},z_n}$ such that

$$h_n(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_n(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z_n(\Lambda) d\Lambda, \quad (4.9)$$

we have to show that there exists $v_* \in S_{\mathcal{H},z_*}$ such that for every $\omega \in [0,1]$,

$$h_*(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z_*(\Lambda) d\Lambda. \quad (4.10)$$

By theorem 1.6.1, we prove that the set $\mathcal{V} = \{v_n \in S_{\mathcal{H},z_n}; n \geq 1\}$ is weakly compact in $\mathrm{L}^1(J;\mathbb{R})$

$$\int_A |v_n(\Lambda)| d\Lambda \leq |A| \|m\|_\infty \varphi(\|z_n\|_{\mathbb{X}}) < \varepsilon,$$

we take $\delta = \frac{\varepsilon}{\|m\|_\infty \varphi(\sup_{n\geq 1} \|z_n\|_{\mathbb{X}})}$, then the first conditions of theorem 1.6.1 is satisfied.

Now we prove the second conditions of this theorem, from (B2)(ii) there exist $b = b(\varepsilon)$ such that

$$\int_b^{+\infty} v_n(\Lambda) d\Lambda \leq \varphi(\|z_n\|_{\mathbb{X}}) \int_b^{+\infty} m(\Lambda) d\Lambda < \varepsilon.$$

Then by theorem 1.6.1 we conclude that $\mathcal{V}$ is weakly compact set in $\mathrm{L}^1(J,\mathbb{R})$.

Conclusion.

There exist a subsequence $v_{n_k} \in S_{\mathcal{H},u_{n_k}}$ such that $v_{n_k} \rightharpoonup v$ in $\mathrm{L}^1(J,\mathbb{R})$, then

$$h_*(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z_*(\Lambda) d\Lambda. \quad (4.11)$$

because

$$\begin{aligned}\int_0^\xi (\xi - \Lambda)^{\beta-1} |z_n(\Lambda) - z(\Lambda)| d\Lambda \leq \int_0^\xi |z_n(\Lambda) - z(\Lambda)| d\Lambda = \int_0^\xi (1 + \Lambda^{\varsigma-1})^2 \frac{|z_n(\Lambda) - z(\Lambda)|}{(1 + \Lambda^{\varsigma-1})^2} d\Lambda \\ \text{(see } 0 < \xi < 1) \\ \leq \|z_n - z\|_{\mathbb{X}} \times \int_0^\xi (1 + \Lambda^{\varsigma-1})^2 d\Lambda \to 0 \quad as \quad n \to \infty.\end{aligned}$$

Step 4. (a priori bounds on solutions.) Let $z \in \lambda\mathcal{N}(z)$ for $\lambda \in (0,1)$. Then for every $\omega \in J$,

$$\frac{|z(\omega)|}{1+\omega^{\varsigma-1}} \leq \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi + \frac{\xi^{\varsigma}}{\varsigma})\|z\|_{\mathbb{X}},$$

then

$$\|z\|_{\mathbb{X}} \leq \frac{\|m\|_1 \times \varphi(\|z\|_{\mathbb{X}})}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi + \frac{\xi^{\varsigma}}{\varsigma})\|z\|_{\mathbb{X}}.$$

Now, Let $\mathcal{V} = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < L\}$.

The set $\mathcal{V}$ is an open set in $\mathbb{X}$ with $0 \in \mathcal{V}$. From the steps 1, step 2, step 3 and step 4, together with theorem 1.3.2, we get that $\mathcal{N} : \bar{\mathcal{V}} \to \mathcal{P}_{c,cp}(\mathbb{X})$ is upper semicontinuous and completely continuous. From (B3) there is no $z \in \partial\mathcal{V}$ such that $z \in \lambda\mathcal{N}(z)$ for $\lambda \in (0,1)$. Therefore by theorem 1.3.2, we deduce that $\mathcal{N}$ has a fixed point $z \in \bar{\mathcal{V}}$, which represents a solution of the problem 4.1. This finishes the proof. □

Example 14. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{2}} z(\omega) \in \mathcal{H}(\omega, z(\omega)) \text{ for almost } \omega \in [0,+\infty[, \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{\frac{1}{2}} z(0) = \int_0^{0.6} (0.6-\Lambda) z(\Lambda) d\Lambda, \end{cases} \tag{4.12}$$

where

$$\mathcal{H}(\omega, x) = \left[\frac{m(\omega)\exp\frac{x}{(1+\sqrt{\omega})^2}}{2}, m(\omega)\exp\frac{x}{(1+\sqrt{\omega})^2}\right],$$

$$m(\omega) = \begin{cases} \frac{1}{7(1+\omega^2)} & \omega \in [0,1], \\ 0 & \omega > 1, \end{cases}$$

we take $L = 1$, *we have* $\quad L > \frac{\|m\|_1 \times \varphi(L)}{\Gamma(\varsigma)} + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\xi + \frac{\xi^{\varsigma}}{\varsigma})L \approx 0.9602.$

Then the problem (4.12) *has a least one solution in* $\mathbb{X}$.

4.1.2 Existence and uniqueness result in nonconvex case

Theorem 4.1.3. *Assume that the following assertions are met*

$(C1)$ $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ *such that for every measurable function* $z : J \to \mathbb{R}$ *we get* $\omega \mapsto \mathcal{H}(\omega, z(\omega))$ *is measurable.*

$(C2)$ *There is a function* $m_1 \in \mathrm{L}^\infty(J, \mathbb{R}^+)$ *such that*

(i) $\|\mathcal{H}(\omega, x)\| \leq m_1(\omega)$

(ii) $\forall \varepsilon > 0, \ \exists b_\varepsilon > 0$ *such that*

$$\int_{b_\varepsilon}^{+\infty} m_1(\Lambda) d\Lambda < \varepsilon.$$

$(C3)$ *There are* $m_2 \in \mathrm{L}^1(J, \mathbb{R}^+)$ *such that*

$$H_d(\mathcal{H}(\omega, x); \mathcal{H}(\omega, y)) \leq m_2(\omega) \frac{|x - y|}{(1 + \omega^{\varsigma - 1})^2}.$$

$(C4)$

$$\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{c_\xi}{\Gamma(\varsigma)\Gamma(\beta)} < 1 \quad \textit{with } c_\xi = \int_0^\xi (\xi - \Lambda)^{\beta - 1}(1 + \Lambda^{\varsigma - 1})^2 d\Lambda.$$

Then the problem (4.1) *has a unique solution in* $\mathbb{X}$.

Proof. Let $\mathcal{N}$ the set valued map defined (4.1.1), we prove that $\mathcal{N}$ has a unique fixed point, which represent the unique solution.

Let $z \in \mathbb{X}$, from $(C1)$ and $(C2)$(i) and lemma 1.4.1 the set $S_{\mathcal{H},z} \neq \emptyset$.

1. Proof the existence.

Step 1. $(\mathcal{N}(z) \in \mathcal{P}_{cl}(\mathbb{X}))$. Let (z_n) be a set in $\mathcal{N}(z)$ with $z_n \to \tilde{z}$ in $\mathbb{X}$. After that $\tilde{z} \in \mathbb{X}$ and there exist $v_n \in S_{\mathcal{H},z}$ with for $\omega \in J$

$$z_n(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda + \frac{\omega^{\varsigma - 1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta - 1} z(\Lambda) d\Lambda.$$

By $(C2)$ the set $\mathcal{V} = \{v_n \in S_{\mathcal{H},z}; n \geq 1\}$ satisfies all the conditions of theorem 1.6.1. Then we have $\mathcal{V}$ is weakly compact in $\mathrm{L}^1(J, \mathbb{R})$, then there exist $v \in \mathcal{V}$ such that

$$\tilde{z}(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma - 1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta - 1} z(\Lambda) d\Lambda.$$

Step 2. There exists $\gamma < 1$ such that

$$H_d(\mathcal{N}(z), \mathcal{N}(\tilde{z})) \leq \gamma \|z - \tilde{z}\|_{\mathbb{X}}.$$

Indeed. Let $z, \tilde{z} \in \mathbb{X}$ and $h_1 \in \mathcal{N}(z)$, then there exist $v_1 \in S_{\mathcal{H},z}$ such that for every $\omega \in J$

$$h_1(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - s)^{\varsigma-1} v_1(s) ds + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\eta (\eta - s)^{\beta-1} z(s) ds.$$

From $(C3)$, we deduce

$$H_d\Big(\mathcal{H}(\omega, z(\omega)); \mathcal{H}(\omega, \tilde{z}(\omega))\Big) \leq m_2(\omega) \frac{|z(\omega) - \tilde{z}(\omega)|}{(1 + \omega^{\varsigma-1})^2} \text{ for almost } \omega \in J.$$

Hence, there exists $w \in \mathcal{H}(\omega, \tilde{z}(\omega))$ such that

$$|v_1(\omega) - w| \leq m_2(\omega) \frac{|z(\omega) - \tilde{z}(\omega)|}{(1 + \omega^{\varsigma-1})^2} \text{ for almost } \omega \in J.$$

For almost $\omega \in J$, we consider the set-valued map defined as follows:

$$U(\omega) = \{w \in \mathbb{R} : |v_1(\omega) - w| \leq m_2(\omega) \frac{|z(\omega) - \tilde{z}(\omega)|}{1 + \omega^{\varsigma-1}}\} := \mathcal{B}(v_1(\omega), \zeta(\omega)),$$

where $\zeta(\omega) = m_2(\omega) \frac{|z(\omega) - \tilde{z}(\omega)|}{1 + \omega^{\varsigma-1}}$. Assumption $(C1)$ implies that the set-valued map $\omega \mapsto \mathcal{F}(\omega, \tilde{z}(\omega))$ is measurable. Since v_1 and ζ are measurable, it follows from proposition 1.4.1 that the multivalued U is measurable, and the set valued map $\omega \mapsto V(\omega) = U(\omega) \cap \mathcal{H}(\omega, \tilde{z}(\omega))$ is nonempty, measurable (see proposition 1.4.2) and closed set-valued, there are by lemma 1.4.1 an element v_2 which is a measurable selection for V. So

$$v_2(\omega) \in \mathcal{H}(\omega, \tilde{z}(\omega)),$$

and for almost $\omega \in J$, we have

$$|v_1(\omega) - v_2(\omega)| \leq m_2(\omega) \frac{|z(\omega) - \tilde{z}(\omega)|}{1 + \omega^{\varsigma-1}}.$$

For each $\omega \in J$, let us define

$$h_2(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_2(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} \tilde{z}(\Lambda) d\Lambda.$$

Hence

$$\begin{aligned} |h_1(\omega) - h_2(\omega)| \leq & \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} |v_1(\Lambda) - v_2(\Lambda)| d\Lambda \\ & + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} |z(\Lambda) - \tilde{z}(\Lambda)| d\Lambda, \end{aligned}$$

and

$$\frac{|h_1(\omega)-h_2(\omega)|}{(1+\omega^{\varsigma-1})^2} \leq \frac{1}{\Gamma(\varsigma)}\int_0^\omega |v_1(\Lambda)-v_2(\Lambda)|d\Lambda + \frac{c_\xi}{\Gamma(\varsigma)\Gamma(\beta)}\|z-\tilde{z}\|_{\mathbb{X}},$$

with $c_\xi = \int_0^\xi (\xi-\Lambda)^{\beta-1}(1+\Lambda^{\varsigma-1})^2 d\Lambda$. Then

$$\|h_1-h_2\|_{\mathbb{X}} \leq \left(\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{c_\xi}{\Gamma(\varsigma)\Gamma(\beta)}\right)\|z-\tilde{z}\|_{\mathbb{X}}.$$

We apply theorem 1.3.1 we get the set valued map $\mathcal{N}$ has a fixed point, which represent a solution.

2. Proof the uniqueness.

Let z_1, z_2 are two solution for (4.1), then there exist $v_1 \in S_{\mathcal{H},z_1}$ and $v_2 \in S_{\mathcal{H},z_2}$ such that

$$z_j(\omega) = \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}v_j(\Lambda)d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^\xi (\xi-\Lambda)^{\beta-1}z_j(\Lambda)d\Lambda,$$

for $1 \leq j \leq 2$. Then

$$\|z_1-z_2\|_{\mathbb{X}} \leq \left(\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{c_\xi}{\Gamma(\varsigma)\Gamma(\beta)}\right)\|z_1-z_2\|_{\mathbb{X}}.$$

It's obvious that $z_1 = z_2$. □

Example 15. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{2}} z(\omega) \in \mathcal{H}(\omega, z(\omega)) \quad \text{for almost } \ \omega \in J, \\ z(0) = 0, \\ \mathcal{D}_{0^+}^{\frac{1}{2}} z(0) = I_{0^+}^{1.7} z(\frac{1}{4}), \end{cases} \tag{4.13}$$

where

$$\mathcal{H}(\omega, x) = \{1\} \cup [\frac{3}{2}, \frac{3}{2} + \frac{m(\omega)}{(1+\sqrt{\omega})^2}|\sin x|],$$

and

$$m(\omega) = \begin{cases} \frac{1}{4} & \omega \in [0,1] \\ 0 & \omega > 1. \end{cases}$$

It's clearly that the set valued map $\mathcal{H}$ is measurable, and we take

$$m_1(\omega) = \frac{3}{2} + \frac{m(\omega)}{(1+\sqrt{\omega})^2},$$
$$m_2(\omega) = m(\omega),$$

$c_\xi < \frac{1}{2}$ *and* $\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{c_\xi}{\Gamma(\varsigma)\Gamma(\beta)} < 0.9031$.

After that we conclude the problem (4.13) has a unique solution.

Corollary 9. *If the conditions of precedent theorem are satisfied and the function* $m_1 = 0$. *Then the unique solution for (4.1) is* $z = 0$.

Corollary 10. *If the conditions of precedent theorem are satisfied and* $0 \in F(\omega, 0)$ *for almost* $\omega \in J$. *Then the unique solution for (4.1) is* $z = 0$.

4.2 Other technical for study the existence result in $\mathcal{C}(J;\mathbb{R})$

In this section we construct an other method to prove the existence to solution for (4.1) in the space $\mathcal{C}(J;\mathbb{R})$, for this let

$$\begin{cases} \mathcal{D}^{\varsigma}_{0^+} z(\omega) \in F(\omega, z(\omega)) \quad \text{for almost} \quad \omega \in [0,1], \\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0^+} z(0) = I^{\beta}_{0^+} z(\xi), \end{cases} \tag{4.14}$$

where $F(\omega, x) = \mathcal{H}(\omega, x)$ on $[0,1]$ and $x \in \mathbb{R}$.

Lemma 4.2.1. *For a given* $v \in \mathrm{L}^1([0,1], \mathbb{R})$, *a function* v *is a solution to*

$$\begin{cases} \mathcal{D}^{\varsigma}_{0^+} z(\omega) = v(\omega) \quad \textit{for almost} \quad \omega \in [0,1], 1 < \varsigma < 2, \\ z(0) = 0, \\ \mathcal{D}^{\varsigma-1}_{0^+} z(0) = I^{\beta}_{0^+} z(\xi), \end{cases} \tag{4.15}$$

if and only if

$$z(\omega) \quad = \quad \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - s)^{\varsigma-1} v(s) ds + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} z(s) ds. \tag{4.16}$$

Theorem 4.2.1. *Supposons that the conditions* $(B1), (B2)$ *and* $(B3)$ *are hold*

$(B1)$ $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ *is a Carathéodory set valued map such that for almost* $\omega \in (1, +\infty)$ *and all* $x \in \mathbb{R}$ *we have* $0 \in \mathcal{H}(\omega, x)$.

$(B2)$ *There are* $\psi \in L^{\infty}(J, \mathbb{R}_+)$ *and* $\phi \in C(J, \mathbb{R}_+)$ *nondecreasing such that*

$$\|\mathcal{H}(\omega, x)\| = \sup\{|v| : v \in \mathcal{H}(\omega, x)\} \leq \psi(\omega)\phi(|x|).$$

$(B3)$ *There exist* $\sigma > 0$ *such that*

$$\sigma > \frac{\|\psi\|_{\infty}\phi(\sigma)}{\Gamma(\varsigma)} + \frac{\xi^{\beta}}{\Gamma(\varsigma)\Gamma(\beta)}\sigma.$$

Then the problem (4.1) *has a least one solution in* $\mathcal{C}(J; \mathbb{R})$.

Proof. For show that the problem (4.1) has a solution, we show that the problem (4.14) has a solution. The prove of this result is done in five steps.

After that in step 6 we construct an element in $\mathcal{C}(J; \mathbb{R})$ depending on the solution for the problem (4.14) which represent a solution for the problem (4.1).

Now we prove the existence the solution for the problem (4.14). From $(B1)$ and proposition 1.4.4 and lemma 1.4.1 there exist a selection measurable for the set valued $\omega \in [0, 1] \mapsto F(\omega, z(\omega))$ and by $(B2)$ this selection belongs in $L^1([0, 1]; \mathbb{R})$ that means $S_{F,z} \neq \emptyset$ for all $z \in \mathcal{C}([0, 1]; \mathbb{R})$. Define an operator $\mathcal{N} : \mathcal{C}([0, 1]; \mathbb{R}) \to \mathcal{P}(\mathcal{C}([0, 1]; \mathbb{R}))$ by

$$\mathcal{N}(z) = \Bigg\{h \in \mathcal{C}([0, 1]; \mathbb{R}) : h(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda, v \in S_{F,z}\Bigg\},$$

where $S_{F,z} = \{v \in L^1([0, 1]; \mathbb{R}); v(\omega) \in F(\omega, z(\omega))$ for almost $\omega \in [0, 1]\}$.

We will show that $\mathcal{N}$ satisfies all hypotheses of the theorem 1.3.2.

Step 1. ($\mathcal{N}$ is convex valued.) Let $z \in \mathcal{C}([0, 1]; \mathbb{R})$, and $h_1, h_2 \in \mathcal{N}(z)$, then there exist $v_1, v_2 \in S_{F,z}$ such that

$$h_i(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v_i(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda \quad \text{for } i \in \{1, 2\}.$$

Then for $\lambda \in [0, 1]$ we have

$$\begin{aligned} \lambda h_1(\omega) + (1 - \lambda) h_2(\omega) &= \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} \big(\lambda v_1(\Lambda) + (1 - \lambda) v_2(\Lambda)\big) d\Lambda \\ &\quad + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda. \end{aligned}$$

Since $\mathcal{H}$ is convex set-valued, then $\lambda h_1 + (1-\lambda)h_2 \in \mathcal{N}(z)$.

Step 2. ($\mathcal{N}$ bounded set into bounded set.)

Let $B_r = \{z \in \mathcal{C}([0,1];\mathbb{R}), \|z\|_\infty < r\}$ and let $h \in \mathcal{N}(z)$ with $z \in B_r$, then there exist $v \in S_{F,z}$ such that

$$h(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda.$$

For all $\omega \in [0,1]$ we have

$$\begin{aligned} |h(\omega)| &\leq \frac{1}{\Gamma(\varsigma)} \int_0^\omega |v(\Lambda)| d\Lambda + \frac{\xi^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi |z(\Lambda)| d\Lambda \\ &\leq \frac{\|\psi\|_\infty \phi(\|z\|_\infty)}{\Gamma(\varsigma)} + \frac{\xi^\beta}{\Gamma(\varsigma)\Gamma(\beta)} \|z\|_\infty. \end{aligned}$$

Step 3. ($\mathcal{N}$ bounded set into equicontinuous set.)

Let B_r the set as defined in step 2, and $t_1, t_2 \in [0,1]$ where $t_1 \leq t_2$, then

$$\begin{aligned} |h(t_1) - h(t_2)| &= \Bigg| \frac{1}{\Gamma(\varsigma)} \int_{t_1}^{t_2} (t_2 - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda \\ &\quad + \frac{1}{\Gamma(\varsigma)} \int_0^{t_1} \Big((t_2 - \Lambda)^{\varsigma-1} - (t_1 - \Lambda)^{\varsigma-1} \Big) v(\Lambda) d\Lambda \\ &\quad + \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda \right] \times \left(t_1^{\varsigma-1} - t_2^{\varsigma-1} \right) \Bigg| \\ &\leq \frac{(t_2 - t_1)^{\varsigma-1}}{\Gamma(\varsigma)} \int_{t_1}^{t_2} |v(\Lambda)| d\Lambda + \frac{t_2^{\varsigma-1} - t_1^{\varsigma-1}}{\Gamma(\varsigma)} \int_0^{t_1} |v(\Lambda)| d\Lambda \\ &\quad + \left[\frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda \right] \times \left| t_1^{\varsigma-1} - t_2^{\varsigma-1} \right| \\ &\leq \frac{(t_2 - t_1)^{\varsigma-1}}{\Gamma(\varsigma)} \times \|m\|_\infty \phi(\|z\|_\infty) + \frac{(t_2^{\varsigma-1} - t_1^{\varsigma-1})}{\Gamma(\varsigma)} \times \|m\|_\infty \phi(\|z\|_\infty) \\ &\quad + \frac{\xi^\beta}{\Gamma(\varsigma)\Gamma(\beta)} \|z\|_\infty \\ &\leq \frac{2(t_2 - t_1)^{\varsigma-1}}{\Gamma(\varsigma)} \times \|m\|_\infty \phi(r) + \frac{\xi^\beta}{\Gamma(\varsigma)\Gamma(\beta)} r. \end{aligned}$$

As $t_1 \to t_2$ we get $h(t_1) \to h(t_2)$.

Step 4. ($\mathcal{N}$ has a closed graph.)

Let $z_n \to z_*, h_n \in \mathcal{N}(z_n)$ and $h_n \to h_*$, we need to show that $h_* \in \mathcal{N}(z_*)$. Now $h_n \in \mathcal{N}(z_n)$ implies that there exists $v_n \in S_{F,z_n}$ with

$$h_n(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v_n(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z_n(\Lambda) d\Lambda.$$

We have to show that there exists $v_* \in S_{F,z_*}$ such that for each $\omega \in [0,1]$,

$$h_*(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z_*(\Lambda) d\Lambda.$$

Step 5. (A priori bounds on solutions.) Let $z \in \mathcal{C}([0,1];\mathbb{R})$ such that $z \in \lambda\mathcal{N}(z)$ for $\lambda \in (0,1)$. Then for every $\omega \in [0,1]$ we have

$$|z(\omega)| \leq \frac{\|\psi\|_\infty \phi(\|z\|_\infty)}{\Gamma(\varsigma)} + \frac{\xi^\beta}{\Gamma(\varsigma)\Gamma(\beta)} \|z\|_\infty. \tag{4.17}$$

Let $\mathcal{V} = \{z \in \mathcal{C}([0,1];\mathbb{R}) : \|z\|_\infty < \sigma\}$.
It is clearly that $\mathcal{V}$ is an open set in $\mathcal{C}([0,1];\mathbb{R})$ and $0 \in \mathcal{V}$. From step 2, step 3 and step 4 we can conclude that the restriction of $\mathcal{N}$ in $\bar{\mathcal{V}}$ is upper semicontinuous, completely continuous and has a compact convex values. From the choice of $\mathcal{V}$, for all $z \in \partial\mathcal{V}$ and $\lambda \in (0,1)$ and by third condition in theorem we get $z \notin \lambda\mathcal{N}(z)$. Therefore, by theorem 1.3.2, we deduce that $\mathcal{N}$ has a fixed point $z \in \overline{\mathcal{V}}$,i.e., there exist $v \in S_{F,z}$ satisfies (4.16). which z represents a solution for (4.14)

Step 6. (Construct the solution for (4.1).)
Let v an element in $S_{F,z}$ such that v satisfies (4.16), and let

$$\tilde{v}(\omega) = \begin{cases} v(\omega) & \omega \in [0,1], \\ 0 & \omega > 1, \end{cases}$$

and

$$\tilde{z}(\omega) = \begin{cases} z(\omega) & \omega \in [0,1], \\ \frac{1}{\Gamma(\varsigma)} \int_0^1 (\omega - \Lambda)^{\varsigma-1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} z(\Lambda) d\Lambda & \omega > 1, \end{cases}$$

where z be a solution for (4.14).
It is clearly that $\tilde{v} \in S_{\mathcal{H},\tilde{z}} = \{\tilde{w} \in \mathrm{L}^1(J,\mathbb{R})\ ; \tilde{w}(\omega) \in \mathcal{H}(\omega, \tilde{z}(\omega))\ \text{ for almost }\ \omega \in J\}$ and

$$\tilde{z}(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^\omega (\omega - \Lambda)^{\varsigma-1} \tilde{v}(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^\xi (\xi - \Lambda)^{\beta-1} \tilde{z}(\Lambda) d\Lambda.$$

That means $\tilde{z} \in \mathcal{C}(J;\mathbb{R})$ is a solution for (4.1). $\square$

Example 16. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{2}} u(\omega) \in F(\omega, u(\omega)), \\ u(0) = 0, \\ \mathcal{D}_{0^+}^{\frac{1}{2}} u(0) = \int_0^{0.6} (0.6 - s) u(s) ds, \end{cases} \tag{4.18}$$

where

$$F(\omega, x) = [\varphi(\omega), \varphi(\omega) + \frac{|x|}{1+\sqrt{\omega}} m(\omega)],$$

$$m(\omega) = \begin{cases} \sqrt{\omega} - \omega & \omega \in [0,1] \\ 0 & \omega > 1 \end{cases} \quad \text{and} \quad \varphi(\omega) = \begin{cases} \cos\omega & \omega \in [0, \frac{\pi}{2}] \\ 0 & \omega > \frac{\pi}{2} \end{cases}$$

$\frac{\|m\|_1}{\Gamma(\varsigma)} + \frac{\eta^{\beta-1}}{\Gamma(\varsigma)\Gamma(\beta)}(\eta + \frac{\eta^\varsigma}{\varsigma}) \approx 0.8041.$

Then the problem (4.18) has a least one solution.

Theorem 4.2.2. *Assume that the following assertions are met*

$(C1)$ *$\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is a multivalued with for all $\omega > 1$ we get $0 \in \mathcal{H}(\omega, x)$ for each $x \in \mathbb{R}$, and*

- *$\mathcal{H}$ is integrably bounded and $\omega \mapsto \mathcal{H}(\omega, x)$ is measurable for each $x \in \mathbb{R}$.*
- *There are $m_2 \in \mathrm{L}^1(J, \mathbb{R}^+)$ such that*

$$H_d(\mathcal{H}(\omega, x); \mathcal{H}(\omega, y)) \leq m_2(\omega)|x - y|.$$

$(C4)$

$$\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{1}{\Gamma(\varsigma)\Gamma(\beta)} < 1.$$

Then the problem (4.1) has a unique solution in $\mathcal{C}(J; \mathbb{R})$.

Proof. We use the same method used in the theorem 4.2.1, let $z \in \mathcal{C}([0,1]; \mathbb{R})$, from $(C1)$ and proposition 1.4.4 the set $S_{F,z} \neq \emptyset$.

1. Proof the existence.

Step 1. $(\mathcal{N}(z) \in \mathcal{P}_{cl}(\mathcal{C}([0,1];\mathbb{R})))$. Indeed, let (z_n) be a sequence in $\mathcal{N}(z)$ such that $z_n \to \bar{z}$ in $\mathcal{C}([0,1];\mathbb{R})$. Then $\bar{z} \in \mathcal{C}([0,1];\mathbb{R})$ and there exist $v_n \in S_{F,z}$ such that for every $\omega \in [0,1]$

$$z_n(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma - 1} v_n(\Lambda) d\Lambda + \frac{\omega^{\varsigma - 1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta - 1} z(\Lambda) d\Lambda.$$

By $(C1)$ the set $\mathcal{V} = \{v_n \in S_{F,z}; n \geq 1\}$ satisfies all the conditions of theorem 1.6.2. Then we have $\mathcal{V}$ is weakly compact in $\mathrm{L}^1([0,1],\mathbb{R})$, then there exist $v \in \mathcal{V}$ such that

$$\bar{z}(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma - 1} v(\Lambda) d\Lambda + \frac{\omega^{\varsigma - 1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta - 1} z(\Lambda) d\Lambda.$$

Then $\bar{z} \in \mathcal{N}(z)$, i.e., $\mathcal{N}(z)$ is closed in $\mathcal{C}([0,1];\mathbb{R})$.

Step 2. There exists $\gamma < 1$ such that

$$H_d(\mathcal{N}(z), \mathcal{N}(\bar{z})) \leq \gamma \|z - \bar{z}\|_{\infty}.$$

Indeed. Let $z, \bar{z} \in \mathcal{C}([0,1];\mathbb{R})$ and $h_1 \in \mathcal{N}(z)$, then there exist $v_1 \in S_{F,z}$ such that for every $\omega \in [0,1]$

$$h_1(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - s)^{\varsigma - 1} v_1(s) ds + \frac{\omega^{\varsigma - 1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\eta} (\eta - s)^{\beta - 1} z(s) ds.$$

From $(C1)$, we deduce

$$H_d\Big(F(\omega, z(\omega)); F(\omega, \bar{z}(\omega))\Big) \leq m_2(\omega)|z(\omega) - \bar{z}(\omega)| \text{ for almost } \omega \in [0,1].$$

Hence, there exists $w \in F(\omega, \bar{z}(\omega))$ such that

$$|v_1(\omega) - w| \leq m_2(\omega)|z(\omega) - \tilde{z}(\omega)| \text{ for almost } \omega \in [0,1].$$

For almost $\omega \in [0,1]$, we consider the set-valued map defined as follows:

$$U(\omega) = \{w \in \mathbb{R} : |v_1(\omega) - w| \leq m_2(\omega)|z(\omega) - \tilde{z}(\omega)|\} := \mathcal{B}(v_1(\omega), \zeta(\omega)),$$

where $\zeta(\omega) = m_2(\omega)|z(\omega) - \tilde{z}(\omega)|$. Assumption $(C1)$ implies that the set-valued map $\omega \mapsto F(\omega, \bar{z}(\omega))$ is measurable (see proposition 1.4.4). Since v_1 and ζ are measurable, it follows from proposition 1.4.1 that the closed ball $\mathcal{B}$ is measurable. Moreover, the map $\omega \mapsto V(\omega) = U(\omega) \cap F(\omega, \bar{z}(\omega))$ is nonempty, measurable (see proposition 1.4.2) and closed set-valued, there exists by lemma 1.4.1 a function v_2 which is a measurable selection for V. So

$$v_2(\omega) \in F(\omega, \bar{z}(\omega)),$$

and for almost $\omega \in [0,1]$, we have

$$|v_1(\omega) - v_2(\omega)| \leq m_2(\omega)|z(\omega) - \bar{z}(\omega)|.$$

For each $\omega \in [0,1]$, let us define

$$h_2(\omega) = \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v_2(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} \bar{z}(\Lambda) d\Lambda.$$

Hence

$$\begin{aligned} |h_1(\omega) - h_2(\omega)| \ &\leq \ \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} |v_1(\Lambda) - v_2(\Lambda)| d\Lambda \\ &\quad + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} |z(\Lambda) - \bar{z}(\Lambda)| d\Lambda, \end{aligned}$$

and

$$\frac{|h_1(\omega) - h_2(\omega)|}{(1 + \omega^{\varsigma-1})^2} \leq \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} |v_1(\Lambda) - v_2(\Lambda)| d\Lambda + \frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \|z - \bar{z}\|_{\infty}.$$

Then

$$\|h_1 - h_2\|_{\infty} \leq \left(\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \right) \|z - \bar{z}\|_{\infty}.$$

We apply theorem 1.3.1 we get the set valued map $\mathcal{N}$ has a fixed point, which represent a solution.

2. Proof the uniqueness.

Let z_1, z_2 are two solution for (4.14), then there exist $v_1 \in S_{F,z_1}$ and $v_2 \in S_{F,z_2}$ such that

$$z_j(\omega) \ = \ \frac{1}{\Gamma(\varsigma)} \int_0^{\omega} (\omega - \Lambda)^{\varsigma-1} v_j(\Lambda) d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)} \int_0^{\xi} (\xi - \Lambda)^{\beta-1} z_j(\Lambda) d\Lambda,$$

for $1 \leq j \leq 2$. Then

$$\|z_1 - z_2\|_{\infty} \leq \left(\frac{\|m_2\|_1}{\Gamma(\varsigma)} + \frac{1}{\Gamma(\varsigma)\Gamma(\beta)} \right) \|z_1 - z_2\|_{\infty}.$$

It's obvious that $z_1 = z_2$.

3. Construct the solution on the problem 4.1.

Let v an element in $S_{F,z}$ such that v satisfies (4.16), and let

$$\tilde{v}(\omega) = \begin{cases} v(\omega) & \omega \in [0,1], \\ 0 & \omega > 1, \end{cases}$$

and

$$\tilde{z}(\omega) = \begin{cases} z(\omega) & \omega \in [0,1], \\ \frac{1}{\Gamma(\varsigma)}\int_0^1 (\omega-\Lambda)^{\varsigma-1} v(\Lambda)d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^\xi (\xi-\Lambda)^{\beta-1} z(\Lambda)d\Lambda & \omega > 1, \end{cases}$$

where z be a solution for (4.14).

It is clearly that $\tilde{v} \in S_{\mathcal{H},\tilde{z}} = \{\tilde{w} \in \mathrm{L}^1(J,\mathbb{R})\ ; \tilde{w}(\omega) \in \mathcal{H}(\omega, \tilde{z}(\omega))\ \text{ for almost } \ \omega \in J\}$ and

$$\tilde{z}(\omega) \quad = \quad \frac{1}{\Gamma(\varsigma)}\int_0^\omega (\omega-\Lambda)^{\varsigma-1}\tilde{v}(\Lambda)d\Lambda + \frac{\omega^{\varsigma-1}}{\Gamma(\varsigma)\Gamma(\beta)}\int_0^\xi (\xi-\Lambda)^{\beta-1}\tilde{z}(\Lambda)d\Lambda.$$

That means $\tilde{z} \in \mathcal{C}(J;\mathbb{R})$ is a solution for (4.1). Since the problem (4.14) has one solution, then the solution for the problem (4.1) is unique. $\square$

Conclusion and some perspectives

The aim of this thesis it is provide a study on existence and uniqueness for fractional differential inclusion in the Riemann-Liouville sense in different situations, for this we rely on three basic stages.

1. Construct the integral equation equivalent to fractional differential equation whose the second side is a selection in L^1 space.
2. Construct a multivalued with its element is the integral equation in terms of the selections of the inconnu function.
3. Prove that the multivalued has a fixed point, which represent a solution.
4. Example to ensure that proven theories are applicable.

In the future

1. Study the existence solution for the problem fractional inclusion in Lobesgue spaces in unbounded domain.
2. We will study the stability of the solution for the problem fractional in Banach space.
3. Study the existence solution for the problem fractional inclusion in other topological spaces (not necessary Banach space).

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