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Comparison of approximations for differential equations

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Thanks

First of all, we thank ALLAH, almighty, for giving us the strength to survive, as well as the daring to overcome all difficulties..

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In order not to forget anyone, our sincere thanks go to all those who helped us to realize our memory.

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General Notations

$\delta_n(\epsilon)$ order functions.

$F(y, \dot{y})$ a numerical function with respect to y et $\dot{y}$.

$\dot{y} = \frac{dy}{d\theta}$ the prime derivative of y with respect to θ.

$\ddot{y} = \frac{d^2y}{d\theta^2}$ the second derivative of y with respect to θ.

$F_y = \frac{\partial F}{\partial y}$ the first derivative of F with respect to y.

$F_{yy} = \frac{\partial^2 F}{\partial y^2}$ the second derivative of F with respect to y.

$F_{\dot{y}} = \frac{\partial F}{\partial \dot{y}}$ the first derivative of F with respect to $\dot{y}$.

General Introduction

Approximation is a very important part of mathematics. This means removing a large number of numbers and converting them to an integer or ending with a decimal number. Researchers have resorted to approximations because of the difficulty of solving some problems, especially hard and nonlinear problems.

One of the theories which can be considered both old and new and which is currently gaining considerable popularity among researchers and mathematicians and mechanics is the perturbation theory. In current applications, some considerations only require the use of a small number of terms in the perturbation expansion, but applying simple perturbation is problematic if we want to compute a uniformly valid solution.
This work is divided into three chapters.

- The first chapter is devoted to basic notions and functional tools concerning the theory of perturbation used in this work.
- In the second chapter, we introduced the initial concepts and principle of some perturbation methods for calculating approximations with basic examples.
- In the last chapter, we analyzed and calculated the approximate solutions in different methods and compared them with each other and with the exact solution. we represented them graphically, and studied their behaviour.

Chapter 1

Preliminaries

In this chapter, we introduce some basic concepts of approximation and perturbation theory that we will use in Chapters 2 and 3.

1.1 Background and elementary examples

The discovery of perturbation theory began with the theory describing the motion of celestial bodies.
Where, there are significant deviations in its movement through the elliptical orbit, see [2, 6, 8].
We start with the following example that describes harmonic oscillation.

Example 1.1 *A simple example of a perturbation arising in a natural way is the following problem. Consider a harmonic oscillation, described by the equation* (1.1)

$$\ddot{x} + x = 0. \tag{1.1}$$

When deriving this equation, the effect of friction was neglected; However, in practice, friction will always be present.
If the oscillator is small to the point of friction, an optimized model of the oscillations is

given by the following equation

$$\ddot{x} + \varepsilon\dot{x} + x = 0. \tag{1.2}$$

The term $\varepsilon\dot{x}$ is called "friction term" or "damping term" and this particular simple form of the friction term has been based on certain assumptions concerning the mechanics of friction.

In all of our studies, the parameter ε takes a small enough, i.e

$$0 \leq \varepsilon << 1.$$

If we put $\varepsilon = 0$ in equation (1.2), we find the original equation (1.1). We call equation (1.1) the "unperturbed problem".

One of the interesting conclusions which we shall draw is, that in a great many problems, the introduction of small perturbations triggers off qualitatively and quantitatively behaviour of the solutions which diverges very much from the behaviour of the solutions of the unperturbed problem. We can observe this already in the simple example of the damped oscillator described by equation (1.2).

We will also look at some other examples.

Example 1.2 *Consider the initial value problem*

$$\dot{x} = -2x + 2\varepsilon, \quad x(0) = 1, \tag{1.3}$$

his solution is $x(t, \varepsilon) = \varepsilon + (1 - \varepsilon)e^{-2t}$.
The unperturbed problem is

$$\dot{y} = -2y, \quad y(0) = 1,$$

has the solution $y(t, \varepsilon) = e^{-2t}$. It is clear that

$$|x(t, \varepsilon) - y(t, \varepsilon)| = |\varepsilon - \varepsilon e^{-2t}|, 0 \leq \varepsilon,\, t \geq 0$$

The error, arising in approximating $x(t,\varepsilon)$ by $y(t,\varepsilon)$ is never bigger that ε, see figure 1.1.

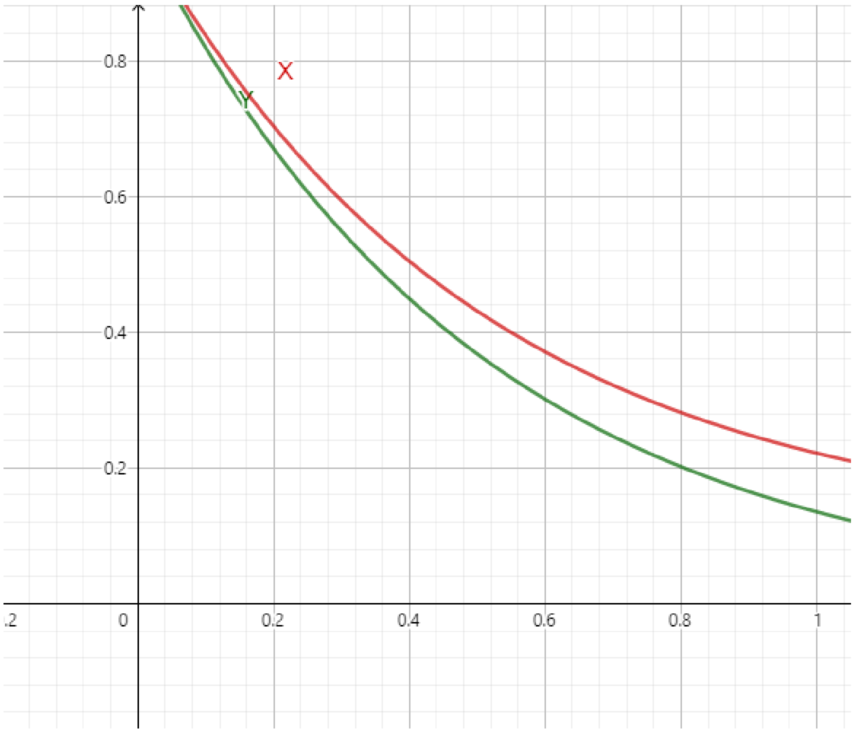

Figure 1.1: Solution of the equation (1.3) with $0 \leq x$.

Example 1.3 *The situation is very different in the problem*

$$\dot{x} = 2x + \varepsilon,\; x(0) = 1,$$

his solution is $x(t,\varepsilon) = -\varepsilon + (1+\varepsilon)e^{2t}$.

The unperturbed problem is

$$\dot{y} = 2y,\; y(0) = 1,$$

has the solution $y(t,\varepsilon) = e^{2t}$. *We find*

$$|x(t,\varepsilon) - y(t,\varepsilon)| = \varepsilon(|1 - e^{2t}|).$$

In the interval $0 \leq t \leq 1$*, the error in approximating* $x(t,\varepsilon)$ *by* $y(t,\varepsilon)$ *in terms of* ε*. This is not true for all* $t \geq 0$*, where the difference increases without bounds.*

In example 1.3, the solutions are not bounded for $t \geq 0$.

We shall show now that also in the case of bounded solutions the difference between the solutions of the perturbed and the unperturbed problem can be considerable.

Example 1.4 *Consider the following initial value problem*

$$\ddot{x} + (1+\varepsilon)^2 x = 0,\ x(0) = \frac{1}{1+\varepsilon},\ \dot{x}(0) = 0, \tag{1.4}$$

has the solution

$$x(t,\varepsilon) = \frac{1}{1+\varepsilon}\cos((1+\varepsilon)t).$$

The unperturbed problem is

$$\ddot{y} + y = 0,\ y(0) = 1,\ \dot{y}(0) = 0,$$

has the solution $y(t,\varepsilon) = \cos t$.
We should analyse the difference $x(t,\varepsilon) - y(t,\varepsilon)$. *The solutions are close for a long time, but if we wait long enough, take for example* $t = \pi/(2\varepsilon)$, *the difference becomes considerable, see figure 1.2.*

[h]

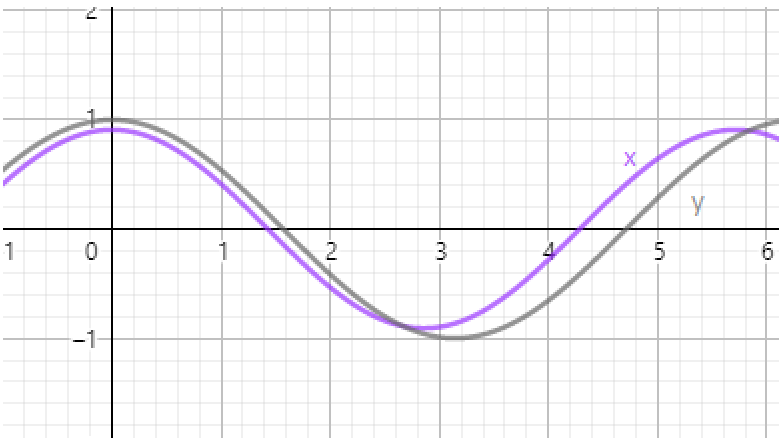

Figure 1.2: Solution of the equation (1.4), $0 \leq x \leq 20$.

These examples show that when calculating approximations for solutions to initial value problems, we must specify the time period for which we are looking for an approximation. In many cases, we would prefer this time period to be as large as possible.

We will now introduce a number of concepts that enable us to estimate vector functions in terms of the small parameter ε.

1.2 Basic material

Consider the vector function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the function $f(t, x, \varepsilon)$ is continuous in the variables $t \in \mathbb{R}$ and $x \in D \subset \mathbb{R}^n$, ε is a small parameter. The function f has to be expanded with respect to the small parameter ε.

1.2.1 Naive expansion

Consider the initial value problem

$$\dot{x} = f(t, x, \varepsilon),\ x(0) \text{ given },$$

$t \geq 0$, $x \in D \subset \mathbb{R}^n$. If we can expend $f(t, x, \varepsilon)$ in a Taylor series with respect to ε

$$f(t, x, \varepsilon) = f_0(t, x) + \varepsilon f_1(t, x) + \varepsilon^2 \cdots$$

then one might suppose that a similar expansion exists for the solution

$$x(t, \varepsilon) = x_0(t, \varepsilon) + \varepsilon x_1(t, \varepsilon) + \varepsilon^2 \cdots$$

Substitution of the expansion in the equation, equating coefficients of corresponding powers of ε and subsequently imposing the initial values produces a so-called formal expansion of the solution. It seems natural to expect that the formal expansion represents an asymptotic approximation of the solution. We shall show that this is correct, but only in a very restricted sense. First, we illustrate with a simple example.

Example 1.5 *Consider the oscillator with damping*

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0$$

with initial values $x(0) = 1$, $\dot{x}(0) = 0$. *The solution of the initial value problem is*

$$x(t,\varepsilon) = e^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}t\right) + \varepsilon\frac{1}{\sqrt{1-\varepsilon^2}}e^{-\varepsilon t}\sin\left(\sqrt{1-\varepsilon^2}t\right).$$

Ignoring the explicit solution we substitute

$$x(t,\varepsilon) = x_0(t,\varepsilon) + \varepsilon x_1(t,\varepsilon) + \varepsilon^2\cdots$$

Equating coefficients of equal powers of ε, *we find*

$$\begin{aligned}&\ddot{x}_0 + x_0 = 0\\ &\ddot{x}_n + x_n = -2\dot{x}_{n-1}, n = 1, 2, \cdots\end{aligned}$$

The initial values do not depend on ε, *so it is natural to put*

$$x_0(0) = 1 \quad , \quad \dot{x}_0(0) = 0$$

$$x_n(0) = 0 \quad , \quad \dot{x}_n(0) = 0, n = 1, 2, \cdots$$

Solving the equations and applying the initial conditions, we obtain

$$\begin{aligned}&x_0(t,\varepsilon) = \cos t\\ &x_1(t,\varepsilon) = \sin t - t\cos t, \ \textit{etc.}\end{aligned}$$

The formal expansion is

$$x(t,\varepsilon) = \cos t + \varepsilon(\sin t - t\cos t) + \varepsilon^2\cdots$$

This expansion corresponds with an oscillation with increasing amplitude.

Remark 1.2.1 *The expressions* $\varepsilon, \varepsilon^2, \cdots, \varepsilon^n, \cdots$ *are called order functions.*

We shall look for an expansion in the form

$$f(t,x,\varepsilon) = \sum_{n=0}^{N}\delta_n(\varepsilon)f_n(t,x) + \cdots,$$

in which $\delta_n(\varepsilon), n = 0, 1, 2, \cdots$ *are order functions, see [4, 9].*

Definition 1.2.1 *The function $\delta(\varepsilon)$ is continuous and positive in $(0, \varepsilon_0]$ and $\underset{\varepsilon \to 0}{lim} \delta(\varepsilon)$ exists, whereas $\delta(\varepsilon)$ is monotonically decreasing as ε tends to zero, $\delta(\varepsilon)$ is called an order function.*

In the case that f has a Taylor expansion, the order functions which have been used are

$$\{\varepsilon^n\}_{n=0}^{\infty}.$$

Other examples of order functions in $(0, 1]$ are

$$\varepsilon|\ln\varepsilon|, \sin\varepsilon, 2, e^{-1/\varepsilon}, ...$$

Very often we shall compare the magnitude of order functions with a characterisation like ε^{4n} goes faster to zero that ε^{2n}.

We compare the behaviour of these two order functions as ε tends to zero; for this comparison we shall use the Landau O-symbols.

Definition 1.2.2 *a. $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ as $\varepsilon \to 0$ if there exists a constant k such that $\delta_1(\varepsilon) \leq k\delta_2(\varepsilon)$ as $\varepsilon \to 0$.*

b. $\delta_1(\varepsilon) = o(\delta_2(\varepsilon))$ as $\varepsilon \to 0$ if

$$\lim_{\varepsilon \to 0} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = 0$$

Remark 1.2.2 *In a number of cases the following limit exists*

$$\lim_{\varepsilon \to 0} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = k$$

In this case we find with the definition $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$. So if this limit exists, we have in a simple way the O-estimate for order functions.

Example 1.6

1. *$\varepsilon^5 = o(\varepsilon^3)$ as $\varepsilon \to 0$.*

2. *$\varepsilon\cos(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$.*

3. $\varepsilon|\ln\varepsilon| = o(1)$ *as* $\varepsilon \to 0$.

In perturbation theory we usually omit as $\varepsilon \to 0$ *this is always the case to be considered. We are now able to compare functions of* ε*, but this does not apply to vector functions* $f(t, x, \varepsilon)$*. Consider for instance the function*

$$\varepsilon t \sin x,\ 0 \leq t \leq 1,\ x \in \mathbb{R},$$

we would estimate $\varepsilon t \sin x = O(\varepsilon)$ *on the domain* $[0, 1] \times \mathbb{R}$*, to do this, we extend our definitions.*

Definition 1.2.3 *Consider the vector function* $f(t, x, \varepsilon)$, $t \in I \subset \mathbf{R}$, $x \in D \subset \mathbf{R}^n$, $0 \leq \varepsilon \leq \varepsilon_0$.
a. $f(t, x, \varepsilon)$ *is* $O(\delta(\varepsilon))$ *if there exists a constant* k *such that* $\|f\| \leq k\delta(\varepsilon)$ *as* $\varepsilon \to 0$ *with* $\delta(\varepsilon)$ *an order function* ($\|.\|$ *is the sup norm on* $I \times D$*).*
b. $f(t, x, \varepsilon)$ *is* $o(\delta(\varepsilon))$ *as* $\varepsilon \to 0$ *if* $\lim\limits_{\varepsilon \to 0} \dfrac{\|f\|}{\delta(\varepsilon)} = 0$.

In the example given above we have now

$$\varepsilon t \sin x = O(\varepsilon), 0 \leq t \leq 1, x \in \mathbb{R}.$$

Note however that the $O(\varepsilon)$ estimate does not hold for the functions

$$\begin{aligned} &\varepsilon t \sin x, \quad t \geq 0,\ x \in \mathbb{R}; \\ &\varepsilon^2 t \sin x, \quad t \geq 0,\ x \in \mathbb{R}. \end{aligned}$$

As we are especially interested in initial value problems for differential equations, the variable t plays a special part. Often we shall study a solution or its approximation on an interval which we would like to take as large as possible. So in these cases we shall not fix the interval of time apriori. It turns out to be useful to characterise the size of the interval of time in terms of the small parameter ε.

1.2.2 Time scale

Definition 1.2.4 *Consider the vector function $f(t,x,\varepsilon)$, $t \geq 0$, $x \in D \subset \mathbb{R}^n$, and the order functions $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$, $f(t,x,\varepsilon) = O(\delta_1(\varepsilon))$ as $\varepsilon \to 0$ on the time scale $1/\delta_2(\varepsilon)$ if the estimate is valid for $x \in D$, $0 \leq \delta_2(\varepsilon)t \leq C$ with C a constant independent of ε, see [5, 7].*

Example 1.7 $f_1(t,x,\varepsilon) = \varepsilon t \cos x$, $t \geq 0$, $x \in \mathbb{R}$.
$f_1(t,x,\varepsilon) = O(\varepsilon)$ *on the time scale* 1,
$f_1(t,x,\varepsilon) = O(1)$ *on the time scale* $1/\varepsilon$.
$f_2(t,x,\varepsilon) = \varepsilon^2 t \cos x, t \geq 0, x \in \mathbb{R}$.
$f_2(t,x,\varepsilon) = O(\varepsilon)$ *on the time scale* $1/\varepsilon$,
$f_2(t,x,\varepsilon) = O\left(\varepsilon^{1/2}\right)$ *on the time scale* $1/\varepsilon^{3/2}$.

We are now able to introduce the concept of an asymptotic approximation of a function.

Definition 1.2.5 *Consider the functions $f(t,x,\varepsilon)$, $g(t,x,\varepsilon)$, $t \geq 0$, $x \in D \subset \mathbb{R}^n$, the function $g(t,x,\varepsilon)$ is an asymptotic approximation of $f(t,x,\varepsilon)$ on the time scale $1/\delta(\varepsilon)$ if $f(t,x,\varepsilon) - g(t,x,\varepsilon) = o(1)$ as $\varepsilon \to 0$ on the time scale $1/\delta(\varepsilon)$.*

Approximation of functions will take the form of asymptotic expansions like

$$f(t,x,\varepsilon) = \sum_{n=0}^{N} \delta_n(\varepsilon) f_n(t,x,\varepsilon) + o\left(\delta_N(\varepsilon)\right)$$

with $\delta_n(\varepsilon)$, $n = 0, \cdots, N$, order functions such that $\delta_{n+1}(\varepsilon) = o\left(\delta_n(\varepsilon)\right), n = 0, \cdots, N-1$. The expansion coefficients $f_n(t,x,\varepsilon)$ are bounded and $O(1)$ as $\varepsilon \to 0$ on the given t, x-domain.

Example 1.8 $f_1(t,x,\varepsilon) = x\cos(t + \varepsilon t)$, $t \geq 0$, $0 \leq x \leq 1$.
The function $x \sin t$ is an asymptotic approximation of $f_1(t,x,\varepsilon)$ on the time scale 1.
$f_2(t,x,\varepsilon) = \cos\varepsilon + \varepsilon^3 t e^{-\varepsilon t}$, $t \geq 0$.
The function ε is an asymptotic approximation of $f_2(t,x,\varepsilon)$ on the time scale $1/\varepsilon$.

The formal expansion can be characterised as follows.

Theorem 1.1 *Consider the initial value problem*

$$\dot{x} = f_0(t, x) + \varepsilon f_1(t, x) + \cdots + \varepsilon^m f_m(t, x) + \varepsilon^{m+1} R(t, x, \varepsilon)$$

with $x\left(t_0\right) = \eta$ *and* $|t - t_0| \leq h$, $x \in D \subset \mathbb{R}^n, 0 \leq \varepsilon \leq \varepsilon_0$. *Assume that in this domain we have*

a. $f_i(t, x), i = 0, \cdots, m$ *continuous in* t *and* $x, (m+1-i)$ *times continuously differentiable in* x*,*

b. $R(t, x, \varepsilon)$ *continuous in* t, x *and* ε*, Lipschitz-continuous in* x*.*

Substitution in the equation for x *the formal expansion*

$$x_0(t, \varepsilon) + \varepsilon x_1(t, \varepsilon) + \cdots + \varepsilon^m x_m(t, \varepsilon).$$

Taylor expansion with respect to powers of ε*, equating corresponding coefficients and applying the initial values* $x_0\left(t_0\right) = \eta$*,* $x_i(t, \varepsilon) = 0$*,* $i = 1, \cdots, m$ *produces an approximation of* $x(t, \varepsilon)$*:*

$$\|x(t, \varepsilon) - (x_0(t, \varepsilon) + \varepsilon x_1(t, \varepsilon) + \cdots + \varepsilon^m x_m(t, \varepsilon))\| = O\left(\varepsilon^{m+1}\right),$$

on the time scale 1.

Proof: See [9].

A natural question is whether, on taking the limit $m \to \infty$, while assuming that the righthand side of the equation can be represented by a convergent power series with respect to ε, we obtain in this way a convergent series for $x(t, \varepsilon)$. This question has been answered in a positive way by Poincaré.

1.2.3 Poincarè expansion theorem

Consider the initial value problem

$$\dot{x} = f(t, x, \varepsilon),\ x\,(t_0) = \eta.$$

Where we assume that $f(t, x, \varepsilon)$ can be expanded in a convergent Taylor series with respect to ε and x in a certain domain. The unperturbed problem is

$$\dot{x}_0 = f\,(t, x_0, 0)\,.$$

In many applications, for instance if we are looking for periodic solutions, we do not know precise initial conditions, we allow for this by admitting deviations of order μ

$$x\,(t_0) \equiv x_0\,(t_0) + \mu,$$

with μ a constant, at this point independent of ε. We translate

$$x = y + x_0(t, \varepsilon),$$

to find for y

$$\dot{y} = F(t, y, \varepsilon),\ y\,(t_0) = \mu,$$

with $F(t, y, \varepsilon) = f\,(t, y + x_0(t, \varepsilon), \varepsilon) - f\,(t, x_0(t, \varepsilon), 0)$.
The expansion properties of $f(t, x, \varepsilon)$ imply that $F(t, y, \varepsilon)$ possesses a convergent power expansion with respect to y and ε in a neighbourhood of $y = 0$, $\varepsilon = 0$. The proof of theorem (1.1), in making use of the Lipschitz-continuity of the derivatives to order m, can be viewed as contraction in a C^m-space. In the subsequent proof we use contraction in a Banach space of analytic functions.

Theorem 1.2 ***(Poincaré)***

Consider the initial value problem (1.7)

$$\dot{y} = F(t, y, \varepsilon),\ y\,(t_0) = \mu,$$

with $|t - t_0| \leq h$, $y \in D \subset \mathbb{R}^n$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \mu \leq \mu_0$.
If $F(t, y, \varepsilon)$ *is continuous with respect to* t, y *and* ε *and can be expanded in a convergent power series with respect to* y *and* ε *for* $\|y\| \leq \rho$, $0 \leq \varepsilon \leq \varepsilon_0$, *then* $y(t, \varepsilon)$ *can be expanded in a convergent power series with respect to* ε *and* μ *in a neighbourhood of* $\varepsilon = \mu = 0$, *convergent on the time scale* 1.

Proof: See [9].

Chapter 2

Perturbations methods

In this chapter, we will explain and compare some methods for solving differential equations.

2.1 Simple perturbation

2.1.1 Simple perturbation principle

The general procedure of the simple approximation consists in substituting for the equation, developing in powers of ϵ, and put all the coefficients of the powers of ϵ equal to zero. This gives a system of inhomogeneous linear differential equations that we can solve recursively, see [4] .
We give simple examples to illustrate this.

Example 2.1 *To illustrate the method, consider the following nonlinear differential equation:*

$$\ddot{y} + y + \epsilon(4\dot{y}^2 + y^2 - 1)\dot{y} = 0 \tag{2.1}$$

Examination of the first-integral shows that all its solutions are periodic. The substi-

tution of Eq. (2.22) into Eq. (2.1) gives

$$
\begin{aligned}
&(\ddot{y}_0. + \epsilon\ddot{y}_1 + \epsilon^2\ddot{y}_2 + \cdots) \\
&\quad + \left(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\right) + \epsilon(4(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 + \cdots)^2 \\
&\quad + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots)^2 - 1)((\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 + \cdots)) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\ddot{y}_0 + y_0) + \epsilon\left(\ddot{y}_1 + y_1 + (4\dot{y}_0^2 + y_0^2 - 1)\dot{y}_0\right) \\
&+ \epsilon^2\left(\ddot{y}_2 + y_2 + y_0\dot{y}_0 y_1 + (4\dot{y}_0 + y_0 - 1)y_1 + 8\dot{y}_0^2\dot{y}_1\right) + \cdots = 0.
\end{aligned}
$$

Setting the coefficients of the various powers of ϵ to zero leads to the following system of linear differential equations,

$$\ddot{y}_0 + y_0 = 0 \tag{2.2}$$

$$\ddot{y}_1 + y_1 = -\left((4\dot{y}_0^2 + y_0^2 - 1)\dot{y}_0\right) \tag{2.3}$$

$$
\begin{aligned}
\ddot{y}_2 + y_2 &= -\left(y_0\dot{y}_0 y_1 + (4\dot{y}_0 + y_0 - 1)y_1 + 8\dot{y}_0^2\dot{y}_1\right) \\
\vdots\ &\ \vdots\vdots \\
\ddot{y}_n + y_n &= F_n\left(y_0, y_1, \ldots, y_{n-1}\right)
\end{aligned}
$$

where F_n is polynomial function of its arguments.

Note that this system of equations can be solved recursively, i.e, the determination of y_k involves only knowledge of the functions y_m for $0 \leq m \leq k-1$.

To solve these equations, initial conditions have to be selected. The choice

$$y(0) = A, \quad \dot{y}_0 = 0$$

gives

$$
\begin{cases}
y_0(0) = A, & \textit{for } i \geq 1, \\
\dot{y}_k(0) = 0 & \textit{for } k \geq 0.
\end{cases} \tag{2.4}
$$

The solution of Eq. (2.2) is

$$y_0(t, \varepsilon) = A\cos t.$$

Substituting this result into the right-side of Eq. (2.3) gives

$$\ddot{y}_1 + y_1 = \left(4\dot{y}_0^2 + y_0^2 - 1\right)\dot{y} = -(-4A^2\sin^2(t,\varepsilon) + A^2\cos^2(t,\varepsilon) - 1)A\sin(t,\varepsilon),$$

$$\Rightarrow y_1(t,\varepsilon) = \left(\frac{2}{100}\right)(\cos 3t - \cos t).$$

Thus, to order ϵ, the "solution" to Eq. (2.1) is

$$y(t,\epsilon) = A\cos t + \epsilon(\left(\frac{2}{100}\right)(\cos 3t - \cos t)) \tag{2.5}$$

Example 2.2 *Consider the following nonlinear differential equation*

$$\ddot{y} + y = \epsilon(y^2 + 1)\dot{y}. \tag{2.6}$$

Suppose the solution of this equation of the following form

$$y(t,\varepsilon) = y_0(t,\varepsilon) + \epsilon y_1(t,\varepsilon) + \dots \tag{2.7}$$

It can be shown that all the solutions of (2.6) are periodic. The substitution of (2.7) in the equation (2.6) given

$$(\ddot{y}_0 + y_0) + \epsilon\left(\ddot{y}_1 + y_1 + \dot{y}_0 y_0^2 - \dot{y}_0\right) + \epsilon^2\left(\ddot{y}_2 + y_2 - 2y_0 y_1\dot{y}_0 - \dot{y}_1 - y_0^2\dot{y}\right) + ... = 0.$$

We equalize all the coefficients of the different powers of ϵ to zero, we then obtain the following system

$$\ddot{y}_0 + y_0 = 0. \tag{2.8}$$

$$\ddot{y}_1 + y_1 = \dot{y}_0 y_0^2 + \dot{y}_0 \tag{2.9}$$

$$\ddot{y}_2 + y_2 = 2y_0 y_1\dot{y}_0 + \dot{y}_1 - y_0^2\dot{y}$$

$$\dots\dots\dots$$

$$\ddot{y}_n + y_n = F_n\left(y_0, y_1, ..., y_{n-1}\right).$$

We can solve these equations with the initial conditions

$$y\left(0\right) = A, \qquad \dot{y} = 0,$$

so

$$\begin{cases} y_0\left(0\right) = A, \qquad y_i\left(0\right) = 0 \qquad for \qquad i \geq 1, \\ \dot{y}_k(0) = 0, \qquad\qquad for \qquad k \geq 0. \end{cases} \tag{2.10}$$

The solution of (2.8) *with the conditions (2.10) is*

$$y_0\left(t\right) = Acos\, t.$$

Substituting this result into the right side of the equation (2.9) we obtain

$$\ddot{y}_1 + y_1 = -A^3 \cos(t,\varepsilon)\sin(t,\varepsilon) - A\sin(t,\varepsilon) = -\frac{A^3}{2}\sin(2t) - A\sin(t,\varepsilon).$$

The particular solution of this equation is

$$y_1{}^P\left(t\right) = \frac{At}{2}\cos(t,\varepsilon) + \frac{A^3}{6}\sin(2t).$$

Therefore, the general solution is

$$y_1\left(t\right) = C_1 cos\, t + C_2 sin\, t\frac{At}{2}\cos(t,\varepsilon) + \frac{A^3}{6}\sin(2t).$$

The initial conditions, specified by the equations (2.10), allow to determine the arbitrary constants C_1 *and* C_2*, as* $C_1 = (-A^3/32)$ *and* $C_2 = 0$*, so* $y_1(t,\varepsilon)$ *becomes*

$$y_1\left(t\right) = \left[\left(-\frac{7}{32}A^3 - \frac{A}{2}\right)\sin t + \left(\frac{A}{2} + \frac{A^3}{8}\right)t\cos t + \frac{A^3}{32}\sin 3t\right]. \tag{2.11}$$

However, inspection of Eq (2.11) shows that $y_1(t,\varepsilon)$, the supposedly small correction term to the periodic function $y_0(t,\varepsilon)$, is not only nonperiodic, but, in addition, is unbounded as $t \to \infty$. Thus a direct, simple application of Eq (2.2) leads to serious difficulties if the goal is to calculate analytic periodic approximations to the solutions of nonlinear differential equations of the form given by Eq (2.11).

2.1.2 Regular perturbation method description

We consider the following wetted nonlinear wave equation

$$\begin{cases} \frac{d^2u}{dt^2} - u = -\varepsilon\,(u)^2\,, \\ u(0) = A,\ \frac{du}{dt}(0) = 0. \end{cases} \tag{2.12}$$

In order to show, the basic idea of regular perturbation method, consider the following differential equation

$$L_\varepsilon[u_\varepsilon(x)] = 0,\ x = (x_1, x_2, ..., x_n) \in \varOmega, \tag{2.13}$$

with boundary conditions

$$B_\varepsilon[u_\varepsilon(x)] = 0,\ x \in \partial\varOmega, \tag{2.14}$$

where L_ε is a genaral differential operator, B_ε is a boundary operator in $\partial\varOmega$.
In general, the equations (2.13) and (2.14) contain a very small parameter ε.
In this method, we use u_ε as a power series,

$$u_\varepsilon(x) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x). \tag{2.15}$$

Substituting (2.15) in (2.13) with (2.14), and collecting the coefficient of like powers of ε yields and equating the coefficient of each power of ε to zero. We obtain systems of recurrent boundary problems, easy to solve.
The approximate solution is given by

$$u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + ...$$

Example 2.3 *Let us suppose that the solution $u(t, \varepsilon)$ of the initial value problem* (2.12) *has the following form,*

$$u(t, \varepsilon) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(t, \varepsilon). \tag{2.16}$$

Putting (2.16) into (2.12), and collecting equal powers of ε, we obtain a system of recurrent initial value problems

$$u(t,\varepsilon) = \frac{d^2u_0}{dt^2} - u_0 + \varepsilon^1\left(\frac{d^2u_1}{dt^2} - u_1 - (u_0)^2\right) + \varepsilon^2\left(\frac{d^2u_2}{dt^2} - u_2 - 2(u_0)(u_1)\right) + \ldots.$$

So we have

$$\varepsilon^0 : \begin{cases} \dfrac{d^2u_0}{dt^2} - u_0 = 0, \\ u_0(0) = A,\ \dfrac{du_0}{dt}(0) = 0. \end{cases} \tag{2.17}$$

$$\varepsilon^1 : \begin{cases} \dfrac{d^2u_1}{dt^2} - u_1 = (u_0)^2, \\ u_1(0) = 0,\ \dfrac{du_1}{dt}(0) = 0. \end{cases} \tag{2.18}$$

$$\varepsilon^2 : \begin{cases} \dfrac{d^2u_2}{dt^2} - u_2 = 2(u_0)(u_1), \\ u_2(0) = 0,\ \dfrac{du_2}{dt}(0) = 0. \end{cases} \tag{2.19}$$

$$\cdots\cdots\cdots$$

Now we solve the equation (2.17).
Its characteristic equation is

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \begin{cases} -1 \\ 1 \end{cases} \Rightarrow u_0(t,\varepsilon) = c_1e^t + c_2e^{-t},$$

using the initial conditions, we find $c_1 = \frac{A}{2}$ and $c_2 = \frac{A}{2}$,
so $u_0(t,\varepsilon) = (\frac{A}{2})e^t + (\frac{A}{2})e^{-t} \Rightarrow u_0(t,\varepsilon) = A\cosh(t,\varepsilon)$*.*

After, we solve the equation (2.18).
The special solution of the equation (2.18) is
$u_{1_{p1}} = (\frac{A^2}{12})e^{2t}$ *and* $u_{1_{p2}} = (\frac{A^2}{12})e^{-2t}$*and* $-\frac{A^2}{2} \Rightarrow u_{1_p} = \frac{A^2}{12}(e^{2t+e^{-2t}}) - \frac{A^2}{2}$,
and $u_1(t,\varepsilon) = c_1e^t + c_2e^{-t} + \frac{A^2}{12}(e^{2t+e^{-2t}}) - \frac{A^2}{2}$.
Using the initial conditions we find $c_1 = \frac{A^2}{6}$ and $c_2 = \frac{A^2}{6}$,

so $u_1(t,\varepsilon) = \frac{A^2}{6}(e^{2t}+e^{2-t}) + \frac{A^2}{12}(e^{2t+e^{-2t}}) - \frac{A^2}{2}$

$\Rightarrow u_1(t,\varepsilon) = \frac{A^2}{3}\cosh(t,\varepsilon) + \frac{A^2}{6}\cosh(2t) - \frac{A^2}{2}$.

In the same way, we find

$u_2(t,\varepsilon) = -\frac{5}{12}A^3\cosh(t,\varepsilon) + \frac{A^3}{48}\cosh(3t) + \frac{A^3}{9}\cosh(2t) - \frac{5}{12}A^3 t\sinh(t,\varepsilon) - \frac{A^3}{3}$.

The second approximate solution of Eq (2.12) is

$$\begin{aligned} u(t,\varepsilon) =& A\cosh(t,\varepsilon) + \varepsilon\left(\frac{A^2}{3}\cosh(t,\varepsilon) + \frac{A^2}{6}\cosh(2t) - \frac{A^2}{2}\right) \\ &+ \varepsilon^2\left(-\frac{5}{12}A^3\cosh(t,\varepsilon) + \frac{A^3}{48}\cosh(3t) + \frac{A^3}{9}\cosh(2t) - \frac{5}{12}A^3 t\sinh(t,\varepsilon) - \frac{A^3}{3}\right). \end{aligned}$$

Remark 2.1.1 *We remark that the terms $y_1(t,0)$ and $y_2(t,0)$ are non periodic and unbounded as $t \to +\infty$. This leads to the notion of secular terms.*

2.1.3 Secular Terms

The calculations of the previous section show that using a solution of the form given by Eq (2.22) does not give an analytic periodic approximation to the actual periodic solution of Eq (2.20) if only a finite number of terms are retained.

The resulting approximation is aperiodic and becomes unbounded as $t \to \infty$. This lack of periodicity comes about because even if $y(t,\varepsilon)$ is periodic in t, the retention of only a finite number of terms, in its expansion, can give a function that is not periodic. For example, consider the function $\sin(1+\epsilon)t$. Its expansion in terms of ϵ is

$$\cos(1+\epsilon)t = \cos t - \epsilon t\sin t - \left(\frac{\epsilon^2 t^2}{2}\right)\cos t + \cdots \tag{2.20}$$

It is seen that the retention of a finite number of terms on the right-side of Eq. (2.21) gives a function that is not only aperiodic, but, is unbounded as $t \to \infty$.

Definition 2.1.1 *Terms such as $t^m cos\,(pt)$ or $t^m sin\,(nt)$ where $m,n \in \mathbb{N}^*$, $p \in \mathbb{N}$ are called secular terms.*

These expressions appear because the expansion (2.3) is not uniformly valid. The existence of such expressions destroys the periodicity of expansion (2.3) when only a finite number of terms is conserved. Therefore, to obtain a uniformly valid solution, we must look for an approximation that eliminates secular terms. A technique to avoid the presence of secular terms and allows for an approximation that is valid for all time has been developed by Lindstedt-Poincaré as described above in what follows.

2.2 Poincarè-Lindstedt method

2.2.1 Introduction

This section presents the simplest and perhaps one of the most useful of all approximation methods: the expansion of a solution to a differential equation in a series in a small parameter. This technique is known as the perturbation method, see [10, 11]. It will be used to construct uniformly valid periodic solutions to second-order nonlinear differential equations of the form

$$\ddot{y} + y = \epsilon F\left(y, \dot{y}\right), \quad \epsilon > 0 \tag{2.21}$$

where ϵ is a small positive parameter and, in general, F is assumed to be an analytic function of y and dy/dt. The starting point for the perturbation method is the assumption that a periodic solution to Eq. (2.21) can be written as

$$y(t, \epsilon) = \sum_{m=0}^{n} \epsilon^m y_m(t, \varepsilon) + O\left(\epsilon^{n+1}\right). \tag{2.22}$$

In general, this series is an asymptotic expansion. The original justification for this expansion was given by Poincarè [5].

The general procedure is to substitute Eq. (2.22) into Eq. (2.21), expand in powers of ϵ, and set the various coefficients of the powers of ϵ equal to zero. This leads to a set of linear inhomogeneous differential equations that can be solved recursively.

Example 2.4 *Consider the following nonlinear differential equation*

$$\ddot{y} + y = \epsilon(1 - 3\dot{y}^2)\dot{y}. \tag{2.23}$$

Suppose the solution of this equation of the following form

$$y(t, \varepsilon) = y_0(t, \varepsilon) + \epsilon y_1(t, \varepsilon) + \dots \tag{2.24}$$

It can be shown that all the solutions of (2.23) *are periodic. The substitution of* (2.24) *in the equation* (2.23) *gives*

$$(\ddot{y}_0 + y_0) + \epsilon\left(\ddot{y}_1 + y_1 - \dot{y}_0 + 3\dot{y}_0^3\right) + \epsilon^2\left(\ddot{y}_2 + y_2 - \dot{y}_1^2 + 9\dot{y}_1\dot{y}_0^2\right) + ... = 0.$$

We equalize all the coefficients of the different powers of ϵ *to zero, then we obtain the following system*

$$\ddot{y}_0 + y_0 = 0, \tag{2.25}$$

$$\ddot{y}_1 + y_1 = \dot{y}_0 - 3\dot{y}_0^3, \tag{2.26}$$

$$\ddot{y}_2 + y_2 = \dot{y}_1^2 - 9\dot{y}_1\dot{y}_0^2,$$

$$\dots\dots\dots$$

$$\ddot{y}_n + y_n = F_n\left(y_0, y_1, ..., y_{n-1}\right).$$

We can solve these equations with the initial conditions

$$y\left(0\right) = A, \qquad \dot{y} = 0,$$

so

$$\begin{cases} y_0\left(0\right) = A, \qquad y_i\left(0\right) = 0 \qquad \text{for} \qquad i \geq 1, \\ \dot{y}_k(0) = 0, \qquad\qquad \text{for} \qquad k \geq 0. \end{cases} \tag{2.27}$$

The solution of (2.25) *with the conditions* (2.27) *is*

$$y_0\,(t) = Acos\,t.$$

Substituting this result into the right side of the equation (2.26)*, we obtain*

$$\ddot{y}_1 + y_1 = \dot{y}_0 - 3\dot{y}_0^3 = -A\sin t + 3A^3\sin^3 t = (A^3\frac{9}{4} - A)\sin t - \frac{3}{4}A^3\sin 3t. \tag{2.28}$$

The particular solution of this equation is

$$y_1{}^P\,(t) = \left(\frac{1}{2}A - \frac{9}{8}A^3\right)t\cos t + \frac{3}{32}A^3\sin 3t.$$

Therefore, the general solution is

$$y_1\,(t) = C_1cos\,t + C_2sin\,t + \left(\frac{1}{2}A - \frac{9}{8}A^3\right)t\cos t + \frac{3}{32}A^3\sin 3t.$$

The initial conditions, specified by the equations (2.27)*, allow to determine the arbitrary constants* C_1 *and* C_2*, as* $C_1 = (-A^3/32)$ *and* $C_2 = 0$*, so* $y_1(t,\varepsilon)$ *becomes*

$$y_1\,(t) = \left[\left(\frac{27}{32}A^3 - \frac{A}{2}\right)\sin t + \left(\frac{A}{2} - \frac{27}{8}A^3\right)t\cos t + \frac{3A^3}{32}\sin 3t\right].$$

Thus, for the order ϵ*, the first-order perturbed solution of the equation (2.23) is*

$$y\,(t,\epsilon) = Acos\,t + \epsilon\left[\left(\frac{27}{32}A^3 - \frac{A}{2}\right)\sin t + \left(\frac{A}{2} - \frac{27}{8}A^3\right)t\cos t + \frac{3A^3}{32}\sin 3t\right] \tag{2.29}$$

However, the solution (2.29) *show that* $y_1(t,\varepsilon)$*, (the correction term of the periodic function* $y_0(t,\varepsilon)$ *is assumed to be small) is not only nonperiodic but also unbounded when* $t \to \infty$*. Thus, a direct application of* (2.23) *leads to difficulties if one wants to compute approximations of the analytic and periodic solutions of nonlinear differential equations of the form given by* (2.29)*.*

2.2.2 Poincarè-Lindstedt Method principle

The essence of this method consists in the introduction of a new independent variable that is linearly related to the old independent variable [2, 4]. This transformation allows the secular terms to be eliminated systematically. The fundamental idea came from the astronomer Lindstedt and is based on the observation that one of the consequences of the nonlinear terms in Eq (2.21) is to change the frequency of the linear system from $\omega_0 = \omega(0) = 1$ to $\omega(\epsilon) \neq 1$. To account for this change in frequency, a new variable $\theta = \omega t$ is introduced, and both y and ω are

expanded in powers of ϵ as follows:

$$\begin{aligned} y(\theta, \epsilon) &= y_0(\theta) + \epsilon y_1(\theta) + \cdots + \epsilon^n y_n(\theta) + \cdots \\ \omega(\epsilon) &= 1 + \epsilon\omega_1 + \cdots + \epsilon^n \omega_n + \cdots \end{aligned} \tag{2.30}$$

where, at this point, the ω_i are unknown constants. Introduce the following notation:

$$\dot{y} \equiv \frac{dy}{d\theta}, \quad \ddot{y} \equiv \frac{d^2 y}{d\theta^2}$$

$$F_y(y, \dot{y}) \equiv \frac{\partial F(y, \dot{y})}{\partial y}, \quad F_{\dot{y}} \equiv \frac{\partial F(y, \dot{y})}{\partial \dot{y}}.$$

If Eq (2.30) are substituted into Eq (2.21) and the coefficients of the various powers of ϵ are set equal to zero, then the following equations are obtained for the y_n :

$$\begin{aligned} &\ddot{y}_0 + y_0 = 0 \\ &\ddot{y}_1 + y_1 = -2\omega_1 \ddot{y}_0 + F(y_0, \dot{y}_0) \\ &\ddot{y}_2 + y_2 = -2\omega_1 \ddot{y}_1 - \left(\omega_1^2 + 2\omega_2\right) \ddot{y}_0 + F_y(y_0, \dot{y}_0) y_1 \quad + F_{\dot{y}}(y_0, \dot{y}_0)(\omega_1 \dot{y}_0 + \dot{y}_1), \\ &\vdots \quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ &\ddot{y}_n + y_n = G_n(\dot{y}_0, \dot{y}_1, \ldots, \dot{y}_{n-1}; \dot{y}_0, \dot{y}_1, \ldots \dot{y}_{n-1}) \end{aligned} \tag{2.31}$$

If F is a polynomial function of y and $\dot{y}$, then G_n is also a polynomial function of its arguments.

The periodicity condition for the new variable θ can be expressed as

$$y(\theta) = y(\theta + 2\pi).$$

The corresponding conditions for the $y_n(\theta)$ are

$$y_n(\theta) = y_n(\theta + 2\pi).$$

If Eq (2.30) is to be a periodic solution of Eq (2.21). Then the right-sides of Eq (2.31), must not contain constant multiples of either *sin* or *cos*. Therefore, if y_n is to be periodic, then two conditions must be satisfied at each step of the calculation. Detailed examination of Eq (2.31).

The initial conditions takes the form:

$$\begin{aligned} y(\theta,\varepsilon) &= A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots \\ \frac{dy(0,\varepsilon)}{d\theta} &= 0, \end{aligned} \tag{2.32}$$

where the A_n are, a priori, unknown constants. Thus, the periodicity requirement forces the right-side of Eq (2.31). At any given step in the calculation ω_n, A_{n-1} and y_n can be simultaneously determined.

The $(n+1)$th approximation to the solution of Eq. (2.21) according to the Poincarè-Lindstedt perturbation method is given by

$$y(\theta,\epsilon) = \sum_{m=0}^{n} \epsilon^m y_m(\theta) + O\left(\epsilon^{n+1}\right), \tag{2.33}$$

where

$$\theta = \omega t = \left[1 + \epsilon\omega_1 + \cdots + \epsilon^n \omega_n + O\left(\epsilon^{n+1}\right)\right] t.$$

The expansion given by Eq. (2.33) is an example of a so-called generalized asymptotic series.

2.2.3 Examples

Example 2.5 *The differential equation*

$$\ddot{y} + y + \epsilon\left(2\dot{y}^2 - 2y^2 - 1\right)\dot{y} = 0 \tag{2.34}$$

with initial conditions, $y(0) = A$ and $\dot{y}(0) = 0$, is known to have periodic solutions if the amplitude of the motion is not too large .
For this example, $F = -\left(2\dot{y}^2 - 2y^2 - 1\right)\dot{y}$ and the equations for $y_0(\theta), y_1(\theta)$, and $y_2(\theta)$ are

$$\begin{aligned}
&\ddot{y}_0 + y_0 = 0 \\
&\ddot{y}_1 + y_1 = -2\omega_1\ddot{y}_0 - \left(2\dot{y}_0^2 - 2y_0^2 - 1\right)\dot{y}_0 \qquad (2.35)\\
&\ddot{y}_2 + y_2 = -2\omega_1\dot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\dot{y}_0 + 4y_0\dot{y}_1 + (\omega_1\dot{y}_0 + \dot{y}_1)(\left(2\dot{y}_0^2 - 2y_0^2 - 1\right) + 4\dot{y}_0^2).
\end{aligned}$$

The solution for $y_0(\theta)$ is $y_0(\theta) = A\cos\theta$. Substituting this result into the second equation in (2.35) gives

$$\ddot{y}_1 + y_1 = 2\omega_1 A\cos\theta - \left(-2A^2\sin^2(\theta) - 2A^2\cos^2(\theta) - 1\right)$$

No secular term requires $\omega_1 = 0$ and $A = 1$. If the resulting differential equation is solved for $y_1(\theta)$, subject to the initial conditions $y_1(0) = \dot{y}_1(0) = 0$, then

$$y_1(\theta) = -\frac{1}{8}(3\sin(\theta) - \sin(3\theta)).$$

With these results, Eq. (2.35) becomes

$$\omega_2 = -\frac{5}{16},$$

The solution of The third equation in (2.35) with initial conditions $y_2(0) = \dot{y}_2(0) = 0$, is

$$y_2(\theta) = -\frac{82}{192}\cos(\theta) + \frac{33}{128}\cos(3\theta) - \frac{77}{384}\cos(5\theta).$$

Approximate solution to Eq. (2.34) is

$$y(\theta,\epsilon) = \cos\theta + \varepsilon(-\frac{1}{8}(3\sin(\theta) - \sin(3\theta))) + \varepsilon^2\left(-\frac{82}{192}\cos(\theta) + \frac{33}{128}\cos(3\theta) - \frac{77}{384}\cos(5\theta)\right),$$

where $\theta = \omega t$ and

$$\omega(\epsilon) = 1 - \varepsilon^2\left(\frac{5}{16}\right) + O\left(\varepsilon^3\right).$$

Example 2.6 *If* $F = -\left(3\dot{y}^2 - y^2 - 1\right)\dot{y}$*, then the corresponding differential equation is*

$$\ddot{y} + y + \epsilon\left(3\dot{y}^2 - y^2 - 1\right)\dot{y} = 0, \tag{2.36}$$

and the equations for $y_0(\theta), y_1(\theta)$*, and* $y_2(\theta)$ *are*

$$\begin{aligned}
&\ddot{y}_0 + y_0 = 0, \\
&\ddot{y}_1 + y_1 = -2\omega_1\ddot{y}_0 - \left(3\dot{y}_0^2 - y_0^2 - 1\right)\dot{y}_0 \\
&\ddot{y}_2 + y_2 = -2\omega_1\dot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\dot{y}_0 + 2y_0\dot{y}_1 + (\omega_1\dot{y}_0 + \dot{y}_1)(\left(3\dot{y}_0^2 - y_0^2 - 1\right) + 6\dot{y}_0^2).
\end{aligned} \tag{2.37}$$

If $y_0(\theta) = A\cos\theta$ *is substituted into the right-side of the second equation in*(2.37)*, we get*

$$\ddot{y}_1 + y_1 = 2\omega_1 A\cos\theta + \left(-3A^2\sin^2(\theta) - A^2\cos^2(\theta) - 1\right)\sin(\theta),$$

and

$$\omega_1 = 0.$$

After substitution, simplification and use of elementary terms $y_1(0) = \dot{y}_1(0) = 0$*, we get the following solution*

$$y_1(\theta) = -\left(\frac{1}{25}\right)(\cos(\theta) - \cos(3\theta)).$$

If $y_0(\theta), y_1(\theta)$*, and* ω_1 *are substituted into the right-side of the second equation in* (2.37)*, this*

$$\omega_2 = -\frac{7}{64}.$$

The general solution to the resulting differential equation for $y_2(\theta)$*, subject to* $y_2(0) = \dot{y}_2(0) = 0$*, is*

$$y_2(\theta) = -\frac{15}{100}\cos(\theta) + \frac{6}{100}\cos(3\theta) - \frac{1}{10}\cos(5\theta).$$

Therefore, to the third approximation, the solution to Eq. (2.36) *is*

$$\begin{aligned}
y(\theta,\epsilon) =& \frac{\sqrt{2}}{2}\cos(\theta) + \varepsilon(-\left(\frac{1}{25}\right)(\cos(\theta) - \cos(3\theta))) \\
&+ \varepsilon^2(-\frac{15}{100}\cos(\theta) + \frac{6}{100}\cos(3\theta) - \frac{1}{10}\cos(5\theta)) + O(\varepsilon^2)
\end{aligned}$$

where $\theta = \omega t$ *and*

$$\omega(\epsilon) = 1 + -\epsilon^2 \left(\frac{7}{64}\right) + O\left(\epsilon^2\right).$$

Proposition 2.2.1 *We consider the following equation*

$$\ddot{y} + y = G(\theta), \qquad y(0) = 0, \; \dot{y}(0) = 0, \; \textit{with} \quad G(\theta) \neq 0, \tag{2.38}$$

the solution of problem (2.38) is

$$y(\theta) = \int_0^\theta sin(\theta - \tau)\, G(\tau)\, d\tau. \tag{2.39}$$

Moreover, the problem (2.38) has a periodic solution $y_1(\theta, 0)$ *if and only if*

$$\begin{cases} \int_0^{2\pi} F(A\,cos\theta, -A\,sin\theta)\, sin\theta\, d\theta = 0, \\ 2\pi\omega_1 A + \int_0^{2\pi} F(A\,cos\theta, -A\,sin\theta)\, cos\theta\, d\theta = 0. \end{cases} \tag{2.40}$$

Proof: We know that the solution of equation (2.38) is $y(\theta) = C_1 cos\theta + C_2 sin\theta + y_p(\theta)$ such that $y_p(\theta) = C_1(\theta)\, cos\theta + C_2(\theta)\, sin\theta$. By variation of constants we find

$$\begin{cases} C_1'(\theta)\, cos\theta + C_2'(\theta)\, sin\theta = 0, \\ -C_1'(\theta)\, sin\theta + C_2'(\theta)\, cos\theta = G(\theta) \end{cases}$$

$$\Rightarrow \begin{cases} -C_1'(\theta) = -sin\theta\, G(\theta) \Rightarrow C_1(\theta) = -\int_0^\theta sin\tau\, G(\tau)\, d\tau, \quad C_1(0) = 0, \\ -C_2'(\theta) = cos\theta\, G(\theta) \Rightarrow C_1(\theta) = \int_0^\theta cos\tau\, G(\tau)\, d\tau, \quad C_2(0) = 0, \end{cases}$$

$$\Rightarrow y_p(\theta) = (-\int_0^\theta sin\tau\, G(\tau)\, d\tau) cos\theta + (\int_0^\theta cos\tau\, G(\tau)\, d\tau) sin\theta = \int_0^\theta (-sin\tau cos\theta + cos\tau sin\theta) G(\tau)\, d\tau$$

$\Rightarrow y_p(\theta) = \int_0^\theta sin(\theta - \tau)\, G(\tau)\, d\tau \Rightarrow y(\theta) = C_1 cos\theta + C_2 sin\theta + \int_0^\theta sin(\theta - \tau)\, G(\tau)\, d\tau$ with the initial values $y(0) = 0$, $\dot{y}(0) = 0$ we have $C_1 = C_2 = 0$, so we deduce that problem (2.38) admits (2.39) as a solution.

Moreover, the equation (2.38) gives

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 + G(\tau). \end{cases}$$

On the other hand, the condition of periodicity for the new variable θ can be expressed as $y(\theta) = y(\theta + 2\pi)$, so the corresponding conditions for $y_n(\theta)$ are $y_n(\theta) = y_n(\theta + 2\pi)\,,\ n = 1, 2, ...$

$$\Rightarrow \begin{cases} y_1(2\pi) = y_1(0) = 0 \\ y_2(2\pi) = y_2(0) = 0 \end{cases}$$

which yields to the periocity condition $\int_{\theta}^{\theta+2\pi} sin(\theta - \tau)\, G(\tau)\, d\tau = 0,$

$$\Rightarrow \begin{cases} \int_0^{2\pi} cos\theta G(\theta)\, d\theta = 0, \\ \int_0^{2\pi} sin\theta\, G(\theta)\ d\theta = 0. \end{cases} \tag{2.41}$$

According to equation (**??**) we have $G(\theta) = -2\omega_1\ddot{y_0} + F(y_0, \dot{y_0})\,,\ y_0 = A\,cos\theta$, we rewrite (2.41) as

$$\Rightarrow \begin{cases} \int_0^{2\pi} cos\theta\, [2\omega_1 A\,cos\theta + F(A\,cos\theta, -A\,sin\theta)]\, d\theta = 0, \\ \int_0^{2\pi} sin\theta\, [2\omega_1 A\,cos\theta + F(A\,cos\theta, -A\,sin\theta)]\, d\theta = 0, \end{cases}$$

$$\Rightarrow \begin{cases} 2\omega_1\pi A + \int_0^{2\pi} cos\theta F(A\,cos\theta, -A\,sin\theta)\, d\theta = 0, \\ \int_0^{2\pi} sin\theta F(A\,cos\theta, -A\,sin\theta)\, d\theta = 0, \end{cases}$$

which is required.

2.3 Averaging method

2.3.1 Averaging method principle

In this section we consider the equation

$$\ddot{x} + x = \varepsilon f(x, \dot{x}). \tag{2.42}$$

If $\varepsilon = 0$, the solutions are known, is a linear combination of $\cos t$ and $\sin t$. We can also write this linear combination as

$$r_0 \cos\left(\omega t + \phi_0\right).$$

The amplitude r_0 and the phase ϕ_0 are constants and they are determined by the initial values. We study the behaviour of the solutions for $\varepsilon \neq 0$, the solution still be written in this form where the amplitude r and the phase ϕ are now functions of time. So we put for the solution of equation (2.42)

$$x(t, \varepsilon) = r(t, \varepsilon)\cos(\omega t + \phi(t, \varepsilon)), \tag{2.43}$$

$$\dot{x}(t, \varepsilon) = -r(t, \varepsilon)\sin(\omega t + \phi(t, \varepsilon)). \tag{2.44}$$

Substitution of these expressions for x and $\dot{x}$ into equation (2.42) produces an equation for r and ϕ

$$\begin{aligned} -\dot{r}\sin(\omega t + \phi) - r\cos(\omega t + \phi)(1 + \dot{\phi}) + r\cos(\omega t + \phi) = \\ = \varepsilon f(r\cos(\omega t + \phi), -r\sin(\omega t + \phi)), \end{aligned}$$

or

$$-\dot{r}\sin(\omega t + \phi) - r\cos(\omega t + \phi)\dot{\phi} = \varepsilon f(r\cos(\omega t + \phi), -r\sin(\omega t + \phi)). \tag{2.45}$$

Another requirement is that differentiation of the righthandside of equation (2.43) must produce an expression which equals the righthand side of equation (2.44). So we find

$$\dot{r}\cos(\omega t + \phi) - r\sin(\omega t + \phi)(1 + \dot{\phi}) = -r\sin(\omega t + \phi),$$

or

$$\dot{r}\cos(\omega t + \phi) - \dot{\phi} r\sin(\omega t + \phi) = 0 \tag{2.46}$$

Equations (2.45) and (2.46) can be considered as two algebraic equations in $\dot{r}$ and $\dot{\phi}$, the solutions are

$$\begin{aligned}\dot{r} &= -\varepsilon \sin(\omega t + \phi) f(r\cos(\omega t + \phi), -r\sin(\omega t + \phi)) \\ \dot{\phi} &= -\frac{\varepsilon}{r}\cos(\omega t + \phi) f(r\cos(\omega t + \phi), -r\sin(\omega t + \phi))\end{aligned} \tag{2.47}$$

The transformation (2.43) – (2.44) $x,\ \dot{x} \to r, \phi$ presupposes of course that $r \neq 0$; in problems where $r\,(t_0) = 0$ we have to use a different transformation. The system of differential equations (2.47) is supplied with initial conditions, for instance at $t_0 = 0$ determined by

$$\begin{aligned}x(0) &= r(0)\cos\phi(0), \\ \dot{x}(0) &= -r(0)\sin\phi(0).\end{aligned}$$

The reasoning of Lagrange is now as follows. The righthand sides of system (2.47) can be expanded in the form

$$\varepsilon\,[g_0(r,\phi) + g_1(r,\phi)\sin t + h_1(r,\phi)\cos t + \ldots]$$

where the dots represent the higher order harmonics $g_n(r,\phi)\sin nt$ and $h_n(r,\phi)\cos nt$ $n = 2, 3, \ldots$ So this is what we are calling nowadays a Fourier expansion of the righthand sides. According to equations (2.47) the variables r and ϕ change slowly with time; the contribution of these changes is, averaged over the period 2π, zero, except for the term $g_0(r,\phi)$. So we omit all the terms which have average zero and we solve the simplified system of differential equations.
The idea of averaging is so natural, that for a long time the method has been used in many fields of application without people bothering about proofs of validity.

A survey of the theory of averaging and many new results can be found in [4]. We conclude this introduction with an explicit calculation.

Example 2.7 *Consider the equation for a nonlinear oscillator with damping*

$$\ddot{x} + x = -\varepsilon\left(\dot{x} + \alpha x^3\right). \tag{2.48}$$

The trivial solution is asymptotically stable. We shall apply the method of Lagrange. Transformation (2.47) *yields*

$$\dot{r} = -\varepsilon r \sin^2(\omega t + \phi) - \varepsilon r^3 \sin(\omega t + \phi) \cos^3(\omega t + \phi)$$
$$\dot{\phi} = -\varepsilon \cos(\omega t + \phi) \sin(\omega t + \phi) - \varepsilon r^3 \cos^4(\omega t + \phi)$$

The righthand sides are 2π *-periodic in* t*. We average now over* t*, keeping* r *and* ϕ *fixed. The result is*

$$\dot{r}_a = -\frac{1}{2}\varepsilon r_a, \dot{\phi}_a = \frac{3}{8}\varepsilon\alpha r_0^2$$

with index a *to indicate that we approximate* r *and* ϕ *by* r_a *and* ϕ_a*. The averagec equations are very easy to solve:*

$$r_a(t,\varepsilon) = r(0)e^{-\frac{1}{2}\epsilon t}, \phi_a(t,\varepsilon) = \phi(0) - (\frac{3}{8}\varepsilon\alpha r_0^2)e^{-\varepsilon t}.$$

As an approximation of $x(t,\varepsilon)$ *we propose*

$$x_a(t,\varepsilon) = r(0)e^{-\frac{1}{2}\epsilon t}\cos[t - (\frac{3}{8}\varepsilon\alpha r_0^2)e^{-\varepsilon t} + \phi(0)].$$

2.3.2 Averaging in the periodic case

We shall now consider the asymptotic validity of the averaging method. Consider the initial value problem

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon),\ x(0) = x_0. \tag{2.49}$$

We assume that $f(t,x)$ is T -periodic in t and we introduce the average

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt$$

In performing the integration y has been kept constant.
Consider now the initial value problem for the averaged equation

$$\dot{y} = \varepsilon f^0(y),\ y(0) = x_0. \tag{2.50}$$

The vector function $y(t,\varepsilon)$ represents an approximation of $x(t,\varepsilon)$ in the following theorem.

Theorem 2.1 *Consider the initial value problems* (2.49) *and* (2.50) *with* $x, y, x_0 \in D \subset \mathbb{R}^n, t \geq 0$. *Suppose that*

a. The vector functions f, g *and* $\partial f/\partial x$ *are defined, continuous and bounded by a constant* M *(independent of* ε *) in* $[0, \infty) \times D$,

b. g *is Lipschitz-continuous in* x *for* $x \in D$,

c. $f(t, x)$ *is* T*-periodic in* t *with average* $f^0(x)$, T *is a constant which is independent of* ε,

d. $y(t, \varepsilon)$ *is contained in an internal subset of* D.

Then we have $x(t, \varepsilon) - y(t, \varepsilon) = O(\varepsilon)$ *on the time scale* $1/\varepsilon$.

Proof: See [9].

Example 2.8 *An oscillator with nonlinear damping is*

$$\ddot{y}_2 + y + \epsilon \left|\frac{dy}{dt}\right| \frac{dy}{dt} = 0, \tag{2.51}$$

with

$$F = -\left|\frac{dy}{dt}\right| \frac{dy}{dt}.$$

For this case

$$\frac{d\phi}{dt} = 0 \Rightarrow \phi(t, \epsilon) = \phi_0,$$

and

$$\begin{aligned}\frac{dr}{dt} &= -\left(\frac{\epsilon r^2}{2\pi}\right) \int_0^{2\pi} |\sin\phi| \sin^2\phi \, d\phi \\ &= -\left(\frac{\epsilon r^2}{2\pi}\right) \left[\int_0^{\pi} \sin^3\phi \, d\phi - \int_{\pi}^{2\pi} \sin^3\phi \, d\phi\right] = -\left(\frac{4\epsilon r^2}{3\pi}\right).\end{aligned}$$

This last equation can be integrated to give

$$r(t, \epsilon) = \frac{r_0}{\left[1 + \left(\frac{4\epsilon r_0}{3\pi}\right) t\right]}.$$

Thus, the first approximation to the solution to Eq. (2.51) is

$$y = \frac{r_0 \cos(t+\phi_0)}{\left[1+\left(\frac{4tr_0}{3\pi}\right)t\right]}.$$

2.4 Renormalization group method

Renormalization group method is a method for finding the approximate solutions of ordinary differential equations in $\mathbb{R}^n$ in the form

$$\dot{y} = Fy + g(y,t,\varepsilon)$$

$$\dot{x} = Fx + \epsilon g_1(x,t) + \epsilon^2 g_2(x,t) + \cdots ; x \in \mathbb{R}^n \tag{2.52}$$

where ε is an infinitely small positive parameter, see [1].

For this system, we assume that

1. F be a square matrix $n \times n$, diagonalizable with imaginary eigenvalues,
2. The function $g(x,t,\varepsilon)$ is sufficiently differentiable in t, x and ε, the power series expansion of ε is given by the equation (2.52),
3. Each $g_i(x,t)$ is periodic in $t \in \mathbb{R}$ and polynomial in x.

Then we assume there is a similar expansion for the solution

$$x(t,\varepsilon) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots .$$

2.4.1 Examples

Example 2.9 *Solving the equation*

$$\ddot{y} + \varepsilon(2\dot{y}^2 + y^2 - 1)\dot{y} + y = 0,\ y(0) = A,\ \dot{y}(0) = 0,\ A \in \mathbb{R}. \tag{2.53}$$

We put

$$x = (z+\bar{z}) \quad \textit{and} \quad y = i(z-\bar{z}),$$

the equation(2.53) *becomes* $\begin{cases} \dot{z} = iz + \frac{\varepsilon}{2}[(z-\bar{z}) - (z+\bar{z})^2(z-\bar{z}) + 2(z-\bar{z})^3], \\ \dot{\bar{z}} = -i\bar{z} - \dfrac{\varepsilon}{2}[(z-\bar{z}) - (z+\bar{z})^2(z-\bar{z}) + 2(z-\bar{z})^3]. \end{cases}$ *It verifies the hypotheses (1-3) with*

$$f = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The two equations of the system being identical, the problem amounts to solving one of them, with

$$z(t,\varepsilon) = z_0 + \varepsilon z_1 + \cdots$$

$z(t,\varepsilon)$ *is the solution searched by the renormalization group method which diverges for* t *long because of the last term. For tho zero order, we have*

$$z_0 = z(t,0) = qe^{it} = qZ(t,\varepsilon),$$

with q *the integration constant by setting* $q = re^{i\theta}(\zeta)$ *with* $x = (z+\bar{z})$ *and* $y = i(z-\bar{z})$, *we find*

$$y(t,\varepsilon) = 2rcos(\theta(\zeta)+t) - \frac{r\varepsilon}{2}sin(\theta(\zeta)+t) + \frac{\varepsilon}{2}[1r^3 sin3(\theta(\zeta)+t) + 7r^3 sin(\theta(\zeta)+t)] + O(\varepsilon^2).$$

Suppose that $y_R(t,\varepsilon)$ *is a solution of* (2.53) *by renormalization group method, then* $y_R(t,\varepsilon)$ $= y(t,\varepsilon)$.

The renormalization group equation becomes

$$\begin{cases} \frac{dr}{d\zeta} = \frac{\varepsilon r}{2}(1-7r^2), \\ \frac{d\theta(\zeta)}{d\zeta} = 0. \end{cases}$$

It is easy to prove that this method has a stable periodic orbit (the limit cycle) of radius $r_s = \frac{\sqrt{7}}{7}$.

Chapter 3

Comparison approximation solutions by perturbation methods

In this chapter, we will analyze and compare all the results of the estimates obtained by the various methods of approximations previously studied, we follow the same methodology of [12]. We process this example.

$$\ddot{x} + \varepsilon(\dot{x}^2 - x^2 - 1)\dot{x} + x = 0,\ x(0) = A,\ \dot{x}(0) = 0. \tag{3.1}$$

3.1 Approximate solutions by simple perturbation method

We solve the equation (3.1) by using the simple perturbation method which is explained in the section 2.1.1.

Suppose that the second approximate solution is

$$y(t,\varepsilon) = y_0(t,\varepsilon) + \varepsilon y_1(t,\varepsilon) + \varepsilon^2 y_2(t,\varepsilon) + o(\varepsilon^2). \tag{3.2}$$

To determine $y_0(t,\varepsilon), y_1(t,\varepsilon)$ and $y_2(t,\varepsilon)$, substituting (3.2) into (3.1), and calculating , see figure 3.1, we find

Approximate solution of (3.1) in the orded 1 given by

$$y(t,\varepsilon) = A\cos(t,\varepsilon) + \varepsilon\left[\left(\frac{A^3}{16} - \frac{A}{2}\right)\sin(t,\varepsilon) + \left(\frac{A}{2} + \frac{A^3}{4}\right)\cos(t,\varepsilon) + \frac{A^3}{16}\sin(3t)\right].$$

Approximate solution of (3.1) in the orded 2 given by

$$\begin{aligned} y(t,\varepsilon) =& A\cos(t,\varepsilon) + \varepsilon\left[\left(\frac{A^3}{16} - \frac{A}{2}\right)\sin(t,\varepsilon) + \left(\frac{A}{2} + \frac{A^3}{4}\right)\cos(t,\varepsilon) + \frac{A^3}{16}\sin(3t)\right] \\ &+ \varepsilon^2\left[\left(\frac{A^5}{768} - \frac{5}{64}A^3\right)\cos(t,\varepsilon) + \left(-\frac{17}{64}A^5 + \frac{9}{32}A^3 - \frac{A}{8}\right)t\sin(t,\varepsilon)\right. \\ &+ \left(\frac{3}{32}A^5 + \frac{A}{8} - \frac{A^3}{4}\right)t^2\cos(t,\varepsilon) + \left(\frac{A^5}{128} + \frac{A^3}{64}\right)\cos(3t) + \frac{7A^5}{1536}\cos(5t) \\ &\left. + (\frac{-3A^5}{64} + \frac{3A^3}{32})t\sin(3t)\right]. \end{aligned}$$

see figure 3.1.

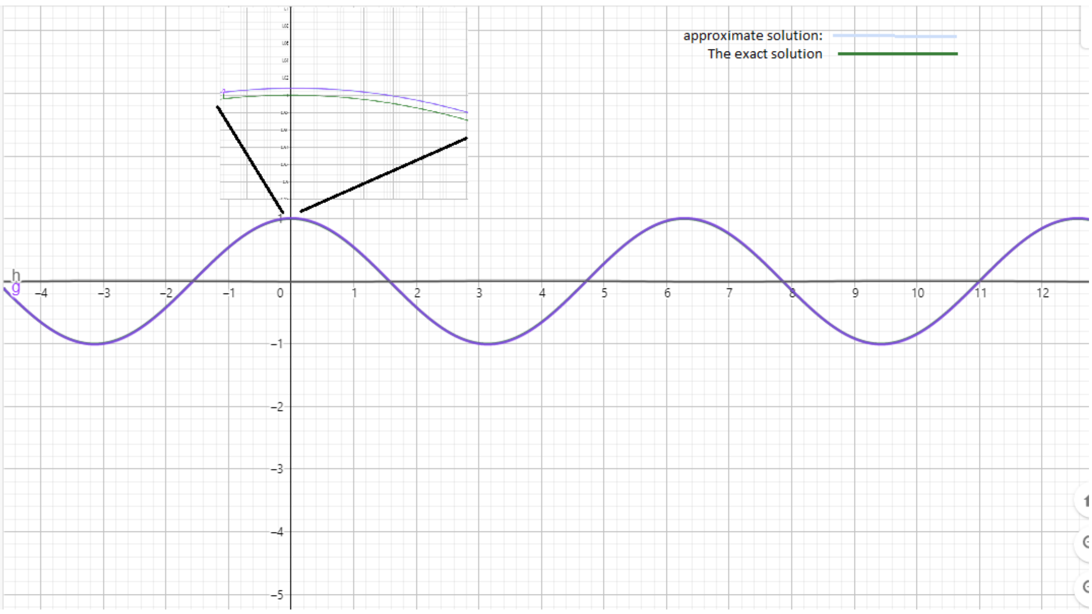

Figure 3.1: Comparison of a simple perturbation solution with $A = 1$ and $\varepsilon = 0.01$.

3.2 Approximate solutions by Poincarè-Lindstedt method

We solve the equation (3.1) by using the Poincarè-Lindstedt method which is explained in the section 2.2.

Substituting the solution $y_0(\theta) = A\cos(\theta)$ into Eq. (3.1) gives

$$\ddot{y}_1 + y_1 = 2\omega_1 A\cos(\theta) - \left(A^2\sin^2(\theta) + 2A^2\cos^2(\theta) - 1\right) A\sin(\theta)$$

has a periodic solution $y_1(0)$, if and only if,

$$\begin{cases} \int_0^{2\pi} A(1 - 2A^2\sin^2(\theta) - A^2\cos^2(\theta))A\sin^2(\theta)d\theta, \\ 2\pi\omega_1 A + \int_0^{2\pi} -A\left(1 - 2A^2\sin^2(\theta) - A^2\cos^2(\theta)\right)\sin(\theta)\cos(\theta)d\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2}\pi A^3 + \frac{3}{4}\pi A^3 - A\pi = 0, \\ 2\omega_1 A\pi = 0. \end{cases}$$

$$\Longrightarrow \begin{cases} A = \sqrt{2} \\ \omega_1 = 0 \end{cases}$$

So, we find $\ddot{y}_1 + y_1 = -\left(12\sin^2(\theta) + 2\cos^2(\theta)\right)\sqrt{2}\sin(\theta)$,

has the solution $y_1(\theta) = \frac{\sqrt{2}}{8}(3sin(\theta) - sin(3\theta))$

$$\ddot{y}_2 + y_2 = -2\omega_1\ddot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\ddot{y}_0 - 2\dot{y}_0\dot{y}_1 - 2\omega_1\dot{y}_0^2$$

After compensation, we find

$$\ddot{y}_2 + y_2 = 2\sqrt{2}\omega_2\cos(\theta) + \frac{3\sqrt{2}}{4}(cos(\theta) - cos(3\theta))\sin(\theta),$$

we have $\begin{cases} \omega_2 = \frac{5}{16}, \\ y_2 = \frac{19\sqrt{2}}{96}\cos(\theta) + \frac{33\sqrt{2}}{128}\cos(3\theta) - \frac{23\sqrt{2}}{384}\cos(5\theta). \end{cases}$

Therefore the approximate solution of (3.1) in orded 1, is given by

$$y(\theta,\varepsilon) = \sqrt{2}cos(\theta) + \varepsilon(\frac{\sqrt{2}}{8}(3sin(\theta) - sin(3\theta))) + O(\varepsilon). \tag{3.3}$$

Also, the approximate solution of (3.1) in orded 2, is given by

$$\begin{aligned} y(\theta,\varepsilon) =& \sqrt{2}cos(\theta) + \varepsilon(\frac{\sqrt{2}}{8}(3sin(\theta) - sin(3\theta))) \\ &+ \varepsilon^2(\frac{19\sqrt{2}}{96}\cos(\theta) + \frac{33\sqrt{2}}{128}\cos(3\theta) - \frac{23\sqrt{2}}{384}\cos(5\theta)) + O(\varepsilon^2) \end{aligned}$$

see figure 3.2.

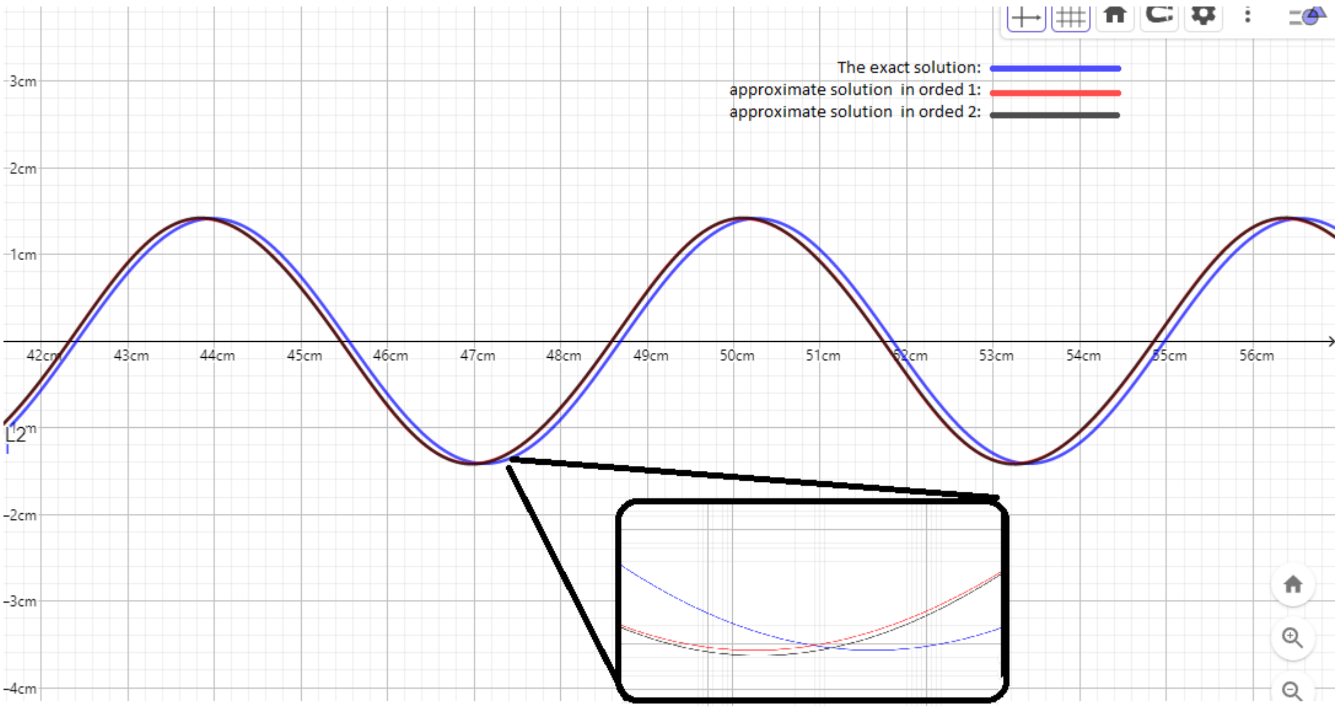

Figure 3.2: Comparison solutions with Poincarè-Lindstedt method for $A = 1$ and $\epsilon = 0.1$.

3.3 Approximate solutions by averaging method

We solve the equation (3.1) by using the averaging method which is explained in the section 2.3, see figure 3.3.

For $\varepsilon = 0$ small, Krylov and Boyolinbov posed the solution

$$x(t,\varepsilon) = A(t,\varepsilon)\, sin(\omega t + \Phi(t,\varepsilon)),$$

$$\dot{x}(t,\varepsilon) = A(t,\varepsilon)\omega\, cos(\omega t + \Phi(t,\varepsilon)).$$

Let $\theta = \omega t + \Phi$, we find

$$\begin{cases} \dot{A} = \dfrac{\varepsilon}{2\pi} \int_0^{2\pi} \left(A^3 cos^4(\theta) - A^3 sin^2(\theta) cos^2(\theta) - A cos^2(\theta)\right) d\theta, \\ \dot{\phi} = -\dfrac{\varepsilon}{2\pi A} \int_0^{2\pi} \left(A^3 sin(\theta) cos^3(\theta) - A^3 sin^3(\theta) cos(\theta) - A cos(\theta) sin(\theta)\right). \end{cases}$$

We recall that

$I_{m,n} = \int_0^{2\pi} sin^m(x) cos^n(x) dx = 0$, This is if m and n are an odd number approximate solution is

$$x(t,\varepsilon) = \frac{2}{[(\frac{4}{A_0^2} - 2)e^{\varepsilon t} - 1]^{\frac{1}{2}}} \sin(t).$$

see figure 3.3.

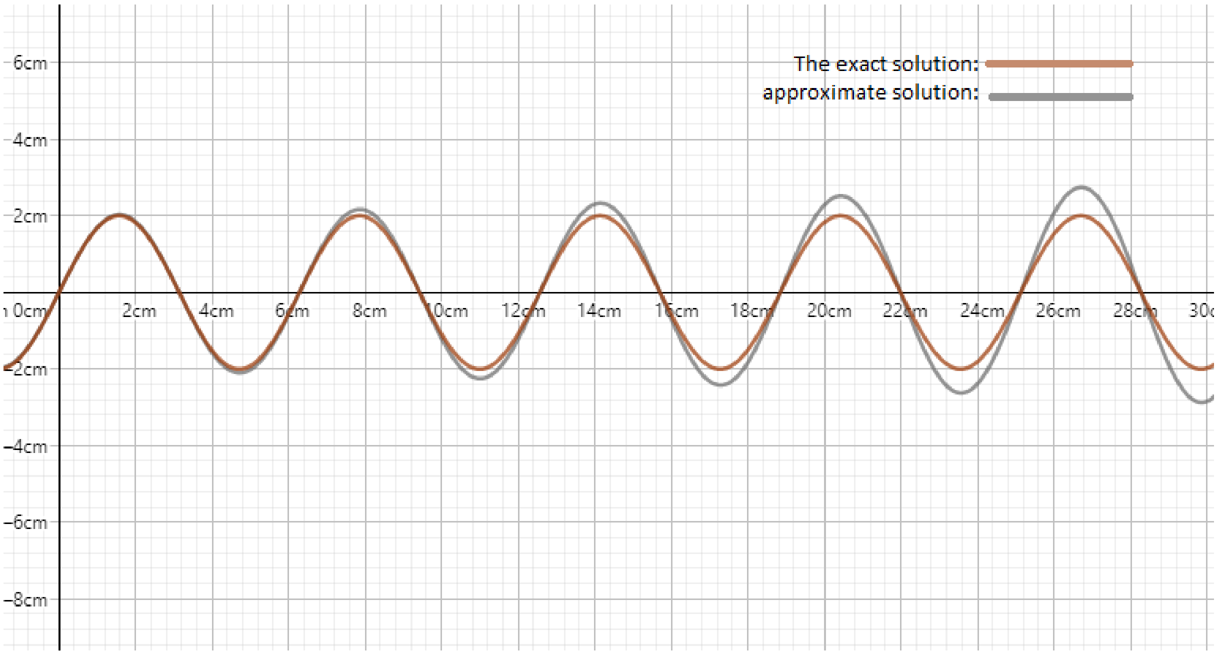

Figure 3.3: Solution of the equation (3.1) by averaging method $A = 1$ and $\epsilon = 0.1$.

3.4 Approximate solutions by renormalization group method

We solve the equation (3.1) by using the averaging method which is explained in the section 2.4, see figure 3.4. We put

$$x = (z + \bar{z}) \quad \text{and} \quad y = i(z - \bar{z}),$$

the equation(3.1) becomes $\begin{cases} \dot{z} = iz + \frac{\varepsilon}{2}[(z-\bar{z}) + (z+\bar{z})^2(z-\bar{z}) + (z-\bar{z})^3], \\ \dot{\bar{z}} = -i\bar{z} - \dfrac{\varepsilon}{2}[(z-\bar{z}) + (z+\bar{z})^2(z-\bar{z}) + (z-\bar{z})^3]. \end{cases}$ It verifies the hypotheses (1-3) with

$$f = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The two equations of the system being identical, the problem amounts to solving one of them, with

$$z(t,\varepsilon) = z_0 + \varepsilon z_1 + \cdots$$

$z(t,\varepsilon)$ is the solution searched by the renormalization group method which diverges for t long because of the last term. To zero order we have

$$z_0 = z(t,0) = qe^{it} = qz(t,\varepsilon),$$

with q the integration constant by setting $q = re^{i\theta}(\zeta)$ with $x = (z+\bar{z})$ and $y = i(z-\bar{z})$, we find

$$y(t,\varepsilon) = 2rcos(\theta(\zeta)+t) - \frac{r\varepsilon}{2}sin(\theta(\zeta)+t) + \frac{\varepsilon^2}{2}[r^3 sin3(\theta(\zeta)+t) + 2r^3 sin(\theta(\zeta)+t)] + O(\varepsilon^2).$$

Suppose that $y_R(t,\varepsilon)$ solution of (2.53) by renormalization group method, then $y_R(t,\varepsilon)$ $= y(t,\varepsilon)$.

The renormalization group equation becomes

$$\begin{cases} \frac{dr}{d\zeta} = \frac{\varepsilon r}{2}(1-2r^2), \\ \frac{d\theta(\zeta)}{d\zeta} = 0. \end{cases}$$

It is easy to prove that renormalization group equation has a stable periodic orbit (the limit cycle) of radius $r_s = \frac{\sqrt{2}}{2}$, see figure 3.4.

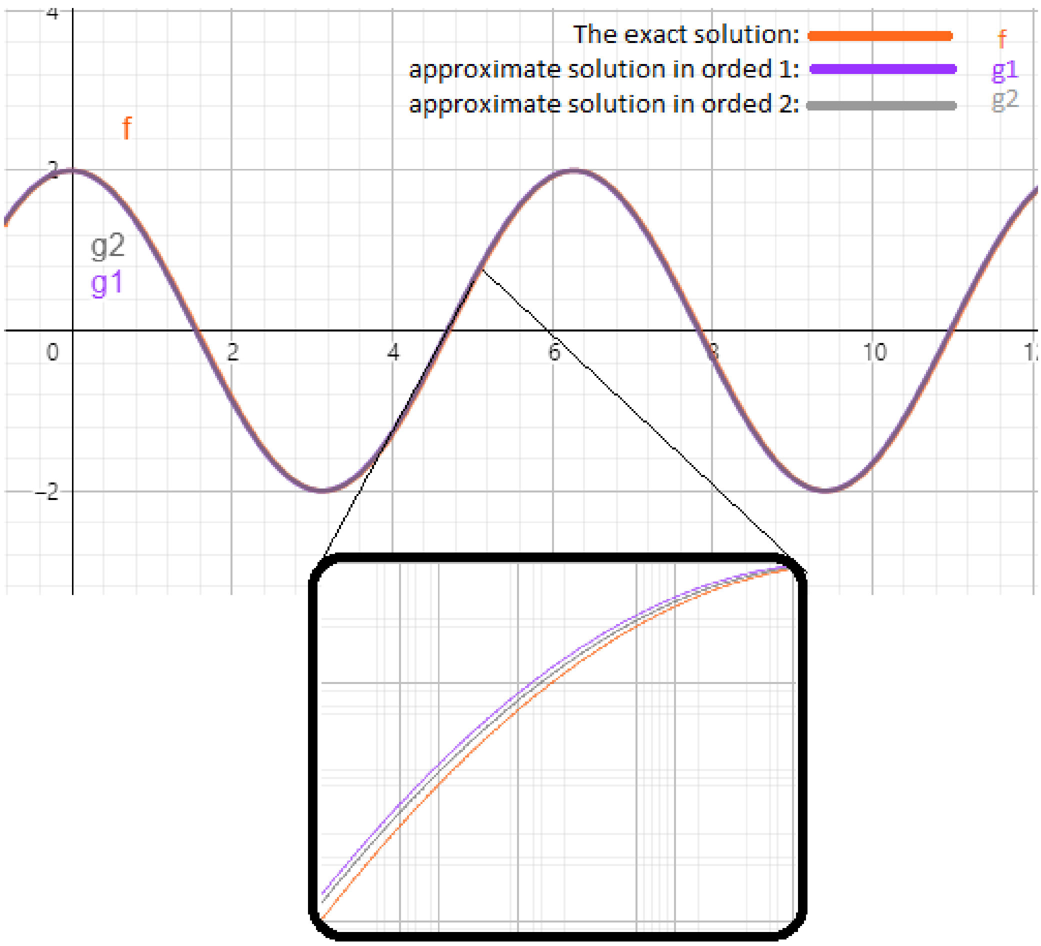

Figure 3.4: Comparison solutions with renormalization group method for $\varepsilon = 0.1,\ \mathrm{r} = 1$.

3.5 Conclusion

We conclude from these figures that the approximations are affected by the value of the small variable and are also affected by the approximation order, as well as by the approximation method used.

1. **In the simple perturbation method, the solution and the approximate solution are convergent along x and the first-order approximate solution and the second-order approximate solution are nearly identical, see figure 3.1.**

2. **As for the method of Lindstedt-Poincaré, we notice the intersection of the solution with the approximate solution, and that the higher the degree of the solution, and closer to the solution see figure 3.2, and this is also done in his renormalization group method, see figure 3.4.**

3. **As for Averaging method, the solution and the approximate solution apply in a small period, and then begin to diverge, see figure 3.3.**

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Résumé

Ce travail est consacré à l'étude et à la comparaison d'approximations analytiques de solutions de systèmes d'équations différentielles non linéaires du second ordre de différentes manières. Nous avons d'abord introduit toutes les propriétés et concepts de base utilisés dans notre étude, puis étudié les principes de différentes méthodes d'approximation. Enfin, nous appliquons ces méthodes pour résoudre et comparer des solutions approchées à des exemples de base, les représenter graphiquement et étudier leur comportement.

Mots clés: Approximation, Théorie des perturbations, solutions périodiques, méthodes d'approximation simple et moyenne, Lindstedt-Poincaré et des groupes de renormalisation.

Absract

This work is devoted to the study and comparison of analytical approximations to solutions of systems of nonlinear second order differential equations in different methods. We first introduced all the basic properties and concepts used in our study, and then studied the principles of different approximation methods. Finally, we apply these methods to solve and compare approximate solutions to basic examples, represent them graphically, and study their behaviour.

Keywords: Approximation, Theory of perturbation, periodic solution, simple, avergning , Lindstedt-Poincaré method and renormalization group methods.

الملخص

هذا العمل مخصص لدراسة و مقارنة التقريبات التحليلية لحلول أنظمة المعادلات التفاضلية اللاخطية من الدرجة الثانية بطرق مختلفة. قدمنا أولاً جميع الخصائص والمفاهيم الأساسية المستخدمة في دراستنا ، ثم درسنا مبادئ طرق التقريب المختلفة . أخيرًا ، طبقنا هذه الطرق لحل و مقارنة الحلول التقريبية لأمثلة أساسية وتمثيلها بيانيا ودراسة سلوكها.

كلمات مفتاحيه: : التقريب ، نظرية الإضطراب ، الحلول الدورية، طرق التقريب البسيط والمتوسط و لانديست بوانكارييه و طريقة مجموعة إعادة التطبيع.