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Comparison between the different definitions of the essential spectrum and Applications

Submitted by:
- **BEYAT Kaouther**
- **LAIB Sohaib**

Supervised by:
- **Dr. ZAIZ Khaoula**

Board of Examiners::

MANSOUR Abdelouahab	President	Univ. of El-Oued
ZAIZ Khaoula	Supervisor	Univ. of El-Oued
GUESBA Messaoud	Examiner	Univ. of El-Oued

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Résumé

Ce travail s'inscrit dans le domaine de la théorie spectrale et plus particulièrement autour d'une notion relativement récente appelée spectre essentiel d'opérateurs linéaires non bornés sur un espace de Banach X.

Nous donnons un aperçu des résultats concernant divers spectres essentiels d'opérateurs linéaires non bornés sur un espace de Banach, et nous donnons quelques relations entre ce spectre essentiel, également une présentation de certaines propriétés spécifiques du spectre essentiel et enfin nous appliquons certains résultats aux opérateurs de transport .

Mots clés : Spectre essentiel, Opérateur de Fredholm, Opérateur non borné, Spectre, Opérateur de transport

Abstract

This work is part of the field of spectral theory and more particularly around a relatively recent notion called the essential spectrum of unbounded linear operators on a Banach space X.

We give a survey of results concerning various essential spectra of unbounded linear operators on a Banach space, and we give some relationships between this essential spectrum, also a presentation of some specific properties of the essential spectrum and finally we apply certain results in the transport operators.

Key words: Essential spectrum, Fredholm operator, Unbounded operator, Spectrum, Transport operator.

الملخص

يدخل هذا العمل في مجال النظرية الطيفية وبشكل ادق حول فكرة حديثة نسبياً تسمى الطيف الأساسي للمؤثرين الخطيين غير المحدودين فيX فضاء بناخ .

نقدم دراسة للنتائج المتعلقة بمختلف الأطياف الأساسية للمؤثرين الخطيين غير المحدودين في فضاء بناخ، ونقدم بعض العلاقات بين هذا الطيف الأساسي ، وكذلك عرضًا لبعض الخصائص المحددة للطيف الأساسي ، وأخيرًا نطبق نتائج معينة في مؤثري النقل .

كلمات مفتاحية: الطيف الأساسي ، مؤثر فريدهولم ، مؤثر غير محدود ، طيف ، مؤثر النقل.

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Finally, all those who were not quoted and contributed to the investigation, From near or far, for this work received my thanks

DEDICATION:

I dedicate this work to my great strength in the life of my mother, my father, my brothers and my sisters and my fiancée...

Laib sohaib

I dedicate this work to the source of light in my life to the two people dearest in my heart to my father and mother. And my only sister Meriem and also to my brothers, Abdel Kader Abd Rahmane Kossai

Beyat kaouther

Contents

General ratings

$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$
$L(X)$	bounded linear operators from X into X,
A^*	The adjoint operator of A,
$R(A)$	The range of A,
$N(A)$	The kernel of A,
$\sigma(A)$	The spectrum of A ,
$\sigma_p(A)$	The point spectrum of A,
$\sigma_r(A)$	The residual spectrum of A,
$r(A)$	Spectral radius of A,
$\sigma_c(A)$	The continuous spectrum of A,
$\sigma_{ap}(A)$	The approximate point spectrum of A,
$\rho(A)$	The resolvent of A,
$\mathcal{F}(X,Y)$	The set of Fredholm perturbation,
$\alpha(A)$	The nullity of A is defined as the dimension of $N(A)$,
$\beta(A)$	The de ficiency of A is de finedas the codimension of $R(A)$,
$\Phi_{\pm}$	The set of semi − Fredholm operators,
I	denote the identity operator in X
Φ_{+}	The set of upper semi − Fredholm operators,
Φ_{-}	The set of lower semi − Fredholm operators,

$D(A)$	domain of A,
$\sigma_{e1}(A)$	essential spectrum of Gustafson,
$\sigma_{e2}(A)$	essential spectrum of Weidmann,
$\sigma_{e3}(A)$	essential spectrum of Kato,
$\sigma_{e4}(A)$	essential spectrum of Wolf,
$\sigma_{e5}(A)$	essential spectrum of Schechter,
$\sigma_{e6}(A)$	essential spectrum of Browder,
$\sigma_{e7}(A)$	essential spectrum of Rakovcèvic,
$\sigma_{e8}(A)$	essential spectrum of Schmoeger,

General Introduction

This thesis is related to spectral theory which is an inclusive term for theories extending the theory of eigenvectors and eigenvalues of a single square matrix to a much broader theory of the structure of operators in a variety mathematical spaces.

The theory of essential spectra of linear operators in Banach space is a modern section of spectral analysis widely used in the mathematical and physical sense when solving a number of applications that can be formulated in terms of operators linear. In the spectral theory are a large number of essential spectra defined for an individual operator, which have been introduced and studied extensively.

The original definition of the essential spectrum goes back to H. Weyl around 1909, when he defined the essential spectrum of a self-adjoint operator A on a Hilbert space as the set of all points of the spectrum of A except isolated eigenvalues of finite multiplicity. With the development of research, it appeared many ways to define the essential spectrum most of them are enlargement to the continuous spectrum. If A is non-selfadjoint operator several definitions are studied after by Gustafson, Weidmann, Kato, Wolf, Schechter and other, where they given specific properties of the essential spectrum can't be hold for the spectrum. See the references [17, 20, 39] and others ....

Our main objective in this thesis is to present the most various definitions of the essential spectrum founded in the mathematical literature, which beginning with the fundamental work of Weyl, is becoming more and more a special branch of spectral theory producing results and problems of its own. And present their application to transport operators.

This thesis is composed of an introduction, three chapters and a list of bibliographies.

The first chapter is devoted to the preliminaries where we present some fundamental and important properties of the Banach space, as well as the fundamental properties of the bounded and unbounded linear operators in the Banach space, After we present some basic notions such as the spectrum and the resolvent...etc. In the last part of this chapter, we introduce the notion of Fredholm operators.

In the second chapter, we present the definitions of essential spectrum and its properties, so we give the relation between these definitions, finally we saw the essential spectrum of the sum and product of two operators.

In the last chapter, we presented predefined examples in which we will calculate the essential spectrum of the operators, also we will apply some results seen in the second chapter.

Chapter 1

Preliminaries

The objective of this chapter is to recall some notions and results that will be used throughout this work. For more details, references to the literature will be systematically given.

1.1 Banach space

Definition 1.1.1. Let X be a vector space over $\mathbb{K}$ (over $K = \mathbb{R}$ or $K = \mathbb{C}$). A norm on X is a map $\|\cdot\| : X \to \mathbb{R}_+$ that satisfies the following three properties.

(N1) $\|x\| = 0 \Leftrightarrow x = 0.$ (definiteness)

(N2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}.$ (homogeneity)

(N3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X.$ (triangle inequality)

Definition 1.1.2. Let $(X, \|\cdot\|)$ be a normed space.

1. A sequence $(x_n)_{n\in\mathrm{N}}$ in X is said to converge to $a \in X$ if

$$\forall\varepsilon > 0 \exists N_0 \in \mathbf{N} : \forall n \geq N_0 : \|x_n - a\| < \varepsilon.$$

In this case we $\lim_{n\to\infty} x_n = a$ or $x_n \to a$ as $n \to \infty$.

2. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if

$$\forall\varepsilon > 0 \exists N_0 \in \mathbb{N} : \forall n, m \geq N_0 : \|x_n - x_m\| < \varepsilon.$$

Remark 1.1.1. $+ : X \times X \to X$ given by $+(x, y) = x + y$ and $\cdot : \mathbb{K} \times X \to X$ given by $\cdot(\lambda, x) = \lambda x$ are continuous. Moreover, $\| \; \| : X \to \mathbb{R}_+$ is continuous.

Example 1.1.1. (1) If $V = \mathbb{R}^n$. We define

$$\|x\|_p = \left(\sum_{i=1}^{n} \| \, x_i|^p \right)^{\frac{1}{p}}$$

and $\|x\|_\infty = \max_{i=1,\ldots,n} \|x_i \,|$. Then, $(\mathbb{R}^n, \|\|_p)$ is Banach space.

(2) $\ell_p(\mathbb{C}) = \{(x_n)_n : \sum_n |x_n|^p < \infty\}$ is a Banach space with respect to

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{\frac{1}{p}}.$$

(3) If $\|\|$ is a norm on $\mathbb{R}^n$, then $(\mathbb{R}^n, \|\|)$ is a Banach space.

(4) $(C[0,1], \|\|_1)$, where

$$\|f\|_1 = \int_0^1 |f(s)|ds$$

is a normed space, but not a Banach space.

1.1.1 Topological Properties

Definition 1.1.3. For $x \in X$ and $r > 0$, the (open) ball with center x and radius r is the set

$$B(x,r) := \{y \in X : \|x - y\| < r\}.$$

The closed ball with center in x and radius r is the set

$$\overline{B(x,r)} := \{y \in X : \|x - y\| \leq r\}.$$

Definition 1.1.4. A set $X \subset Y$ is called open if for every $x \in X$ there exists $r > 0$ such that $B(x,r) \subseteq X$.

Definition 1.1.5. A set $X \subseteq Y$ is called closed if the limit of every sequence $\{x_n\}_{n\geq 1} \subset X$, which converges in Y belongs to X. I.e., if $\{x_n\}_{n\geq 1} \subset X$ and $x_n \to x$ in Y then $x \in X$.

We mention two basic claims which follow from these definition.

Definition 1.1.6. A set A in a normed linear space is open if all points of A are interior points of A. A set A in a normed linear space is closed if it contains all of its boundary points.

Definition 1.1.7. Let $(X, \|\cdot\|)$ be a normed space.
A subset $A \subset X$ is called bounded in X if there exists an $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.
Similarly, a sequence (x_n) in X is called bounded if $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.

Proposition 1.1.1. [27] *A complete normed space is called a Banach space.*

Claim 1.1.1. A set $B \subseteq Y$ is open ,if and only if, the set $B^c = Y - B$ is closed.

Claim 1.1.2. [15] The space X and the collection of all open subsets of X give a topological space. I.e,

(i) The sets X and $\emptyset$ are open in X.

(ii) If $U_1, U_2 \ldots$ are open subsets of X then $\bigcup_n U_n$ is also an open subset of X.

(iii) If $U_1, \ldots, U_N$ are open subsets of X then $\bigcap_{n=1}^N U_n$ is also an open subset of X.

Note that an infinite intersection of open sets is not necessarily open. From this claim, we immediately have:

Claim 1.1.3. (i) The sets X and $\emptyset$ are closed in X.

(ii) If $K_1, K_2 \ldots$ are closed subsets of X then $\bigcap_n K_n$ is also a closed subset of X.

(iii) If $K_1, \ldots, K_N$ are closed subsets of X then $\bigcup_{n=1}^{N} K_n$ is also a closed subset of X

We finish this section with an equivalent definition for convergence of a sequence.

Definition 1.1.8. Let X and Y be normed space , and a map $T : X \to Y$ between normed spaces is continuous at $x_0 \in X$ if for every neighborhood V of $T(x_0)$ there is a neighborhood U of x_0 such that $T(U) \subset V$. The map T is continuous on X if it is continuous at each $x \in X$.

However, if T is linear, then if T is continuous at a single point, it is continuous on all of X.

Proposition 1.1.2. [9] Let X, Y be normed spaces over the same field and assume that $T : X \to Y$ is linear. Then the following are equivalent:

1. T is continuous at 0 .
2. T is continuous on X.
3. T is continuous at some point of X.
4. Y is a normed space, then each of the above is equivalent to
5. $T(U)$ is a bounded subset of Y for some neighborhood U of 0 .

Definition 1.1.9. Let $M = (X, \|\cdot\|)$ be a normed space. Let $Y \subseteq X$ be a subset of X. Suppose:

$$\forall x \in X : \forall \epsilon \in \mathbb{R}_+ : \exists y \in Y : \|x - y\| < \epsilon$$

Then Y is dense in X.

Definition 1.1.10. Let $(X, \|\cdot\|)$ be a normed space.

1. A subset $A \subset X$ is called dense in X if $\overline{A} = X$.
2. The normed space $(X, \|\cdot\|)$ is called separable if there exists a countable subset
$$A \subset X \text{ such that } \overline{A} = X.$$
3. A normed space that is not separable is called inseparable.

Proposition 1.1.3. A normed space $(X, \|\cdot\|)$ is separable if and only if there exists a countable set A such that the linear span of A is dense in X, i.e., if $X = \overline{(\text{span}(A))}$.

Definition 1.1.11. Let X and Y be normed spaces. The compact topology on $L(X, Y)$ is the topology generated by sets of the form $S(K, U)$, defined as follows: fix $K \subset X$ compact and $U \subset Y$ open, and let

$$S(K, U) = \{f \in L(X, Y) \mid f(K) \subset U\}.$$

If Y is a metric space, then convergence in the compact-open topology is the same as uniform convergence on compact sets, i.e. sequence f_n of functions from X to Y converges to a function f if for all compact $K \in X$, $f_n(x)$ converges uniformly to $f(x)$ for all $x \in K$.

Theorem 1.1.1. In a finite dimensional normed space X, any subset M of X is compact if and only if M is closed and bounded.

Lemma 1.1.1. Let A and B be two nonempty subsets of X.
Then A, B is a separation of X if and only if $X = A \bigcup B$ and $A \cap \bar{B} = \emptyset = \bar{A} \cap B$.

Definition 1.1.12. X is called connected if X has no a separation.

Theorem 1.1.2. [15]The following statements are equivalent:

1. X is connected.
2. The only open and closed subsets of X are $\emptyset$ and X.
3. Every continuous function of X onto discrete space $\{0; 1\}$ is constant.

Theorem 1.1.3. [27] Let X be connected and $f : X \longrightarrow Y$a continuous function. Then $f(X)$ (with subspace topology) is a connected space.

Example 1.1.2.

1. Every space with the trivial topology is connected.
2. A discrete space is connected if and only if it is a singlemember set.

1.2 Linear operator

Definition 1.2.1. If X and Y are two linear spaces over the same scalar field, a linear operator from X to Y is a rule A that associates with each $x \in X$ a unique element $Ax \in Y$ such that

$$\forall x, y \in X \text{ and } \forall \alpha, \beta \in \mathbb{K} \ , \ A(\alpha x + \beta y) = \alpha Ax + \beta Ay.$$

Linear operators are also referred to as linear transformations or linear operators.

Definition 1.2.2. The null space (or kernel) of linear operator $A : X \to Y$, denoted

$$N(A) = \{x \in X : Ax = 0\}.$$

The range of A (also called the image of X under A), denoted

$$R(A) = \{y \in Y, x \in X : Ax = y\}.$$

Definition 1.2.3. linear operator $A : X \to Y$

1. A is injective (or one-to-one) if $\quad Ax = Ay \Longleftrightarrow x = y$.
2. A is surjective (or onto) if $\quad R(A) = Y$.

Theorem 1.2.1. Linear operator $A : X \longrightarrow Y$ is injective if, and only if, $N(A) = 0$.

Definition 1.2.4. If $N(A) = 0$, then we define $A^{-1} : R(A) \to X$ as $A^{-1}z = x$ where $Ax = z$.

Theorem 1.2.2. [33] Let $A : X \to Y$ be linear with $N(A) = 0$. Then $A^{-1} : R(A) \to X$ is linear.

Theorem 1.2.3. [33] Let A be a linear operator, then

(a) The range $R(A)$ is a subspace.

(b) If $\dim \mathrm{D(A)} = \mathrm{n}$, then $\dim R(\mathrm{A}) \leq \mathrm{n}$.

(c) The null space N(A) is a subspace.

Lemma 1.2.1. Let X, Y and Z be vector spaces and let $\mathrm{A} : \mathrm{X} \to \mathrm{Y}$ and $\mathrm{S} : \mathrm{Y} \to \mathrm{Z}$ be bijective linear operators. Then $(\mathrm{SA})^{-1}$ exists and $(\mathrm{SA})^{-1} = \mathrm{A}^{-1}\,\mathrm{S}^{-1}$.

1.2.1 Bounded operators

Definition 1.2.5. Let X and Y be two normed spaces. We denote both the X and Y norms by $\|\cdot\|$. A linear operator $T : X \to Y$ is bounded if there is a constant $M \geq 0$ such that

$$\|Ax\| \leq M\|x\| \quad \text{for all } x \in X.$$

If $A : X \to Y$ is a bounded linear operator, then we define the operator norm or uniform norm $\|A\|$ of A by

$$\|A\| = \inf\{M \mid \|Ax\| \leq M\|x\| \text{ for all } x \in X\}.$$

We denote the set of all linear bounded operator $T : X \to Y$ by $L(X, Y)$, and the set of all bounded linear operator $A : X \to Y$. we write $L(X, X) = L(X)$.
Equivalent expressions for $\|A\|$ are:

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}; \quad \|A\| = \sup_{\|x\| \leq 1} \|Ax\|; \quad \|A\| = \sup_{\|x\| = 1} \|Ax\|.$$

Example 1.2.1. Let $X = C([0, 1])$ with the maximum norm, and

$$k : [0, 1] \times [0, 1] \to \mathbb{R}$$

be a continuous function. We define the linear integral operator $K : X \to X$ by

$$Kf(x) = \int_0^1 k(x, y) f(y) dy.$$

Then K is bounded

$$\|K\| = \max_{0 \leq x \leq 1} \left\{ \int_0^1 |k(x, y)| dy \right\}.$$

This expression is the "continuous" analog of the maximum row sum for the ∞-norm of a matrix.

Remark 1.2.1. For linear operator, boundedness is equivalent to continuity.

Theorem 1.2.4. [20] A linear operator is bounded if and only if it is continuous.

Theorem 1.2.5. [20] **(Bounded linear transformation)** Let X be a normed linear space and Y a Banach space. If M is a dense linear subspace of X and

$$T : M \subset X \longrightarrow Y$$

is a bounded linear operator, then there is a unique bounded linear operator $A^* : X \longrightarrow Y$ such that $A^*x = Tx$ for all $x \in M$. Moreover,$\|A^*\| = \|A\|$

Definition 1.2.6. Let X be a linear space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if there are constants $c > 0$ and $C > 0$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \text{for all } x \in X.$$

Theorem 1.2.6. [26] Two norms on a linear space generate the same topology if, and only if, they are equivalent.

Theorem 1.2.7. [16] **(Open mapping)** Suppose that $T : X \longrightarrow Y$ is a one-to-one, onto bounded linear map between Banach spaces X, Y . Then $T^{-1} : Y \longrightarrow X$ is bounded.

1.2.2 Unbounded operator

Definition 1.2.7. Let X and Y be two Banach spaces. An unbounded linear operator from X into Y is a linear map $A : D(A) \subset X \to Y$ defined on a linear subspace $D(A) \subset X$ with values in Y. The set $D(A)$ is called the domain of A.

One says that A is bounded (or continuous) if $D(A) = X$ and if there is a constant $c \geq 0$ such that

$$\|Au\| \leq c\|u\| \quad \forall u \in X.$$

The norm of a bounded operator is defined by

$$\|A\|_{L(X,Y)} = \mathrm{Sup}_{u \neq 0} \frac{\|Au\|}{\|u\|}.$$

Remark 1.2.2. It may of course happen that an unbounded linear operator turns out to be bounded. This terminology is slightly inconsistent, but it is commonly used and does not lead to any confusion. Here are some important definitions and further notation:

$$\begin{aligned}
&\text{Graph of } A = G(A) = \{[u, Au]; u \in D(A)\} \subset X \times Y, \\
&\text{Range of } A = R(A) = \{Au; u \in D(A)\} \subset Y, \\
&\text{Kernel of } A = N(A) = \{u \in D(A); Au = 0\} \subset X.
\end{aligned}$$

A map A is said to be closed if $G(A)$ is closed in $X \times Y$.

Remark 1.2.3. In order to prove that an operator A is closed, one proceeds in general as follows. Take a sequence (u_n) in $D(A)$ such that $u_n \to u$ in X and $Au_n \to f$ in Y. Then check two facts:

(i) $u \in D(A)$,

(ii) $f = Au$.

Note that it does not suffice to consider sequences (u_n) such that $u_n \to 0$ in X and $Au_n \to f$ in Y (and to prove that $f = 0$).

Remark 1.2.4. If A is closed, then $N(A)$ is closed; however, $R(A)$ need not be closed.

Definition 1.2.8. A densely defined linear operator A from one normed space, X, to another one, Y is a linear operator that is defined on a dense linear subspace $\mathrm{D}(A)$ of X and takes values in Y, written

$$T : \mathrm{D}(A) \subseteq X \to Y.$$

Sometimes this is abbreviated as $T : X \to Y$ when the context makes it clear that X might not be the set-theoretic domain of A.

Remark 1.2.5. In practice, most unbounded operators are closed and are densely defined, i.e., $D(A)$ is dense in X.

Example 1.2.2. Consider the space $C^0([0,1];\mathbb{R})$ of all real-valued, continuous functions defined on the unit interval;
Let $C^1([0,1];\mathbb{R})$ denote the subspace consisting of all continuously differentiable functions.
Equip $C^0([0,1];\mathbb{R})$ with the supremum norm $\|\cdot\|_\infty$; this makes $C^0([0,1];\mathbb{R})$ into a real Banach space.
The differentiation operator D given by

$$(\mathrm{D}u)(x) = u'(x)$$

is a densely defined operator from $C^0([0,1];\mathbb{R})$ to itself, defined on the dense subspace $C^1([0,1];\mathbb{R})$.
The operator D is an example of an unbounded linear operator, since

$$u_n(x) = e^{-nx} \quad \text{has} \quad \frac{\|\mathrm{D}u_n\|_\infty}{\|u_n\|_\infty} = n.$$

This unboundedness causes problems if one wishes to somehow continuously extend the differentiation operator D to the whole of $C^0([0,1];\mathbb{R})$.

Definition 1.2.9. Let $A : \mathrm{D}(A) \subseteq X_0 \to X_1$ be a linear operator.
Then, the graph norm of A is defined by $\mathrm{D}(A) \ni x \mapsto \|x\|_A := \sqrt{\|x\|^2 + \|Ax\|^2}$.

Lemma 1.2.2. Let $A : \mathrm{D}(A) \subseteq X_0 \to X_1$ be a linear operator.
Then, A is closed if, and only if, $\mathrm{D}(A)$ equipped with the graph norm is a Banach space.
if i e,so for all $(x_n)_n$ in $\mathrm{D}(A)$ convergent in X_0 such that $(Ax_n)_n$ is convergent in X_1 we have

$$\lim_{n\to\infty} x_n \in \mathrm{D}(A) \text{ and } A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n.$$

Lemma 1.2.3. Let $A : \mathrm{D}(A) \subseteq X_0 \to X_1$ be a closed linear operator.
Then A is bounded if, and only if, $\mathrm{D}(A) \subseteq X_0$ is closed.

Proof. First of all note that boundedness of A is equivalent to the fact that the graph norm and the X_0-norm on $\mathrm{D}(A)$ are equivalent.
Hence, the closedness and boundedness of A implies that $\mathrm{D}(A) \subseteq X_0$ is closed.
On the other hand, the embedding

$$\iota : (\mathrm{D}(A), \|\cdot\|_A) \hookrightarrow (\mathrm{dom}(A), \|\cdot\|_{X_0})$$

is continuous and bijective. Since the range is closed, the open mapping theorem implies that ι^{-1} is continuous.
This yields the equivalence of the graph norm and the X_0-norm and, thus, the boundedness of A. $\square$

For unbounded operators, obtaining a precise description of the domain may be difficult. However, there may be a subset of the domain which essentially (or approximately) describes the operator. This gives rise to the following notion of a core.

Definition 1.2.10. Let $A \subseteq X_0 \times X_1$. A set $D \subseteq \mathrm{D}(A)$ is called a core for A provided $\overline{A \cap (D \times X_1)} = \bar{A}$

Proposition 1.2.1. [20] Let $A \in L(X_0, X_1)$, and $D \subseteq X_0$ a dense linear subspace. Then D is a core for A.

Corollary 1.2.1. Let $A : \mathrm{D}(A) \subseteq X_0 \to X_1$ be a densely defined, bounded linear operator. Then there exists a unique $B \in L(X_0, X_1)$ with $B \supseteq A$.
In particular, we have $B = \bar{A}$ and

$$\|B\| = \sup_{x \in \mathrm{D}(A), x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

we denote $\mathcal{C}(X.Y)$ the set of all closed operatpors.

The main result concerning operators with closed range is the following.

Theorem 1.2.8. ([16]) Let $A \in \mathcal{C}(X, Y)$. By $\mathcal{C}(X, Y)$ we denote the set of all closed, densely defined linear operators from X into Y The following properties are equivalent:

1. $R(A)$ is closed,
2. $R(A^*)$ is closed,
3. $R(A) = N(A^*)^{\perp}$,
4. $R(A^*) = N(A)^{\perp}$

Theorem 1.2.9. Let $\mathrm{T} : \mathrm{D(T)} \to \mathrm{Y}$ be a linear operator, where $\mathrm{D(T)} \subseteq \mathrm{X}$ and X and Y are normed spaces then,

1. T is continuous if and only if T is bounded.
2. If T is continuous at a single point, it is continuous.

Remark 1.2.6. ([16]) Let $A \in \mathcal{C}(X, Y)$ and $Y = M \oplus Y_0$ does not imply that M is closed.
Then there exists a one-dimensional subspace Y_0 such that $Y = M \oplus Y_0$ (recall that $Y/N(f)$ is one-dimensional).
But $M = N(f)$ cannot be closed because f is continuous if and only if $f^{-1}(0)$ is closed.

$$\mathrm{D}(Y/M) < \infty \Rightarrow M \text{ is closed.} \tag{1.1}$$

However Theorem 1.2.9 asserts that if M is a range of a closed limear operator then RT is defined by

$$\begin{aligned} D(RT) &= \{x \in D(T) \mid Tx \in D(R)\} \\ (RT)x &= R(Tx) \text{ for } x \in D(RT) \end{aligned} \tag{1.2}$$

is true. Of course, it is true that

$$M \text{ is closed; } \dim(Y/M) < \infty \Rightarrow M \text{ is complemented.}$$

Definition 1.2.11. An operator $A \in \mathcal{C}(X, Y)$ is said to be bounded below if A is injective and has closed range.

Theorem 1.2.10. ([19])$A \in \mathcal{C}(X, Y)$ is bounded below if and only if there erists $c > 0$ such that

$$\|Ax\| \geq c\|x\| \text{ for all } x \in D(A). \tag{1.3}$$

The next result shows that the properties to be bounded below or to be surjective are dual each other.

Theorem 1.2.11. [2] Let $A \in \mathcal{C}(X, Y)$, then

1. A is bounded below (respertively, surjective) if and only if T^* is surjective (respectively, bounded below).
2. If A is bounded. belou (respectively, surjective) then $\lambda I - A$ is surjective (respectively, bounded below) for all $|\lambda| < \gamma(A)$.

1.2.3 The adjoint of operator

Recall that when X is a Banach space, the dual space $X^* := L(X, \mathbb{C})$, consists of the bounded linear functionals x^* on X; it is a Banach space with the norm

$$\|x\|_{X^*} = \inf\{|x^*(x)| : x \in X, \|x\| = 1\}.$$

When $A : X \longrightarrow Y$ is densely defined, we can define the adjoint operator $A^* : Y^* \rightarrow X^*$ as follows: The domain $D(A^*)$ consists of the $y^* \in Y^*$. for which the linear functional

$$x \mapsto y^*(Ax), x \in D(A). \tag{1.4}$$

is continuous (from X to $\mathbb{C}$). This means that there is a constant c (depending on y^*) such that

$$|y^*(Ax)| \leq c\|x\|_X, \text{ for all } x \in D(A).$$

Since $D(A)$ is dense in X, the mapping extends by continuity to X, so there is a uniquely determined $x^* \in X^*$ so that

$$y^*(Ax) = x^*(x), \text{ for } x \in D(A). \tag{1.5}$$

Since x^* is determined from y^*, we can define the operator A^* from Y^* to X^* by:

$$A^*y^* = x^*, \text{ for } y^* \in D(A^*). \tag{1.6}$$

Definition 1.2.12. Let $A \in L(X)$.

1. If $A = A^*$ we say that A is self-adjount operator.
2. If $A^*A = AA^*$ we say that is normal operator.

Theorem 1.2.12. ([19]) Let $A \in \mathcal{C}(X,Y)$. Then there is an adjoint operator $A^* : Y^* \longrightarrow X^*$, uniquely defined by (1.4)-(1.6). Moreover, A^* is closed.
If A is a bounded operator then A^* is also a bounded operator from Y^* into X^* and, moreower, $\|A^*\| = \|A\|$.

For nonempty sets $M \subseteq X$ and $N \subseteq X^*$ we define the annihilators

$$\begin{aligned} M^{\perp} &= \{f \in X^* : f(x) = 0 \quad \text{for all } x \in M\} \\ N^{\perp} &= \{x \in X : f(x) = 0 \quad \text{for all } x \in N\}. \end{aligned}$$

Even if M and N are not subspaces, and $M^{\perp}$ and $N^{\perp}$ are closed subspaces of X^* and X respectively. We have $M^{\perp} = X^*$ (resp. $N^{\perp} = X$) if and only if $M = \{0\}$ (resp. $N = \{0\}$).

Definition 1.2.13. Let A an operator on a vector space X. The nullity of A is the positive integer

$$\alpha(A) = \dim N(A).$$

The deficiency of A is the positive integer

$$\beta(A) = \operatorname{codim} R(A).$$

Let $\Delta(X)$ denote the set of all linear operators on vector space X for which $\alpha(A)$ and $\beta(A)$ are both finite. The index of $A \in \Delta(X)$ is the integer

$$\mathrm{i}(A) = \alpha(A) - \beta(A).$$

By the index theorem we have

$$\mathrm{i}(AB) = \mathrm{i}(A) + \mathrm{i}(B), \quad \forall A, B \in \Delta(X).$$

In the next theorem we establish the basic relationships between the quantities $\alpha(A)$, $\beta(A)$, $a(A)$ and $\mathrm{i}(A)$.

Definition 1.2.14. An operator $A : X \to Y$ is said to be compact if $A(B)$ is relatively compact in Y for every bounded subset $B \subset X$.
Equivalently, for every bounded sequence $(x_n)_n$ in X there exists a convergent subsequence of $(Ax_n)_n$ in Y.

The set of all compact operators from X into Y is denoted by $K(X,Y)$.
If $X = Y$, we write $\mathcal{K}(X) := \mathcal{K}(X,X)$.
If A is compact, then $A(\overline{B(0,1)})$ is bounded and thus A is bounded; i.e. $\mathcal{K}(X,Y) \subset L(X,Y)$.
A special class of compact operators is the space of operators of finite-rank defined by

$$\mathcal{F}(X,Y) = \{A \in L(X,Y) : \mathrm{D}\, R(A) < \infty\}.$$

If $X = Y$, we write $\mathcal{F}(X) := \mathcal{K}(X,X)$.

Theorem 1.2.13. An operator $A \in (X,Y)$ is compact if and only if $A^* \in (Y^*,X^*)$ is compact.

The following classical theorem extends fundamental results on matrices known from Linear Algebta.

Theorem 1.2.14. ([19]) . Let $K \in K(X)$. Then

1. $R(I - K)$ is closed.
2. $a(I - K) = d(I - K) < \infty$.
3. $\alpha(I - K) = \beta(I - K) < \infty$.

1.3 Fredholm operator

We now introduce some important classes of operators in Fredholm [1] theory. Let X and Y be Banach spaces and let A be an operator from X into Y. For $A \in \mathcal{C}(X, Y)$, the nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y.

Definition 1.3.1. Let X, Y be Banach spaces. The set of Fredholm upper semi-Fredholm and lower semi-Fredholm operators from X into Y are defined respetively

$$\Phi(X, Y) = \{A \in \mathcal{C}(X, Y) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y, \beta(A) < \infty\},$$

$$\Phi_+(X, Y) = \{A \in \mathcal{C}(X, Y) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

the set of lower semi-Fredholm operators from X into Y is defined by

$$\Phi_-(X, Y) = \{A \in \mathcal{C}(X, Y) : \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

the set of semi-Fredholm operators from X into Y is defined by

$$\Phi_\pm(X, Y) = \Phi_+(X, Y) \cup \Phi_-(X, Y),$$

the set of Fredholm operators from X into Y is defined by

$$\Phi(X, Y) = \Phi_+(X, Y) \cap \Phi_-(X, Y),$$

the set of bounded Fredholm operators from X into Y is defined by

$$\Phi^b(X, Y) = \Phi(X, Y) \cap L(X, Y).$$

If $A \in \Phi_\pm(X, Y)$, clearly, ind A is an integer or $\pm\infty$. If $X = Y$ then $\Phi_+(X, Y), \Phi_-(X, Y), \Phi_\pm(X, Y), \Phi(X$ and $\Phi^b(X, Y)$ are replaced, respectively, by $\Phi_+(X), \Phi_-(X), \Phi_\pm(X), \Phi(X)$ and $\Phi^b(X)$.
Observe that in the case $X = Y$ the class $\Phi(X)$ is non-empty since the identity trivially is a Fredholm operator.
But for certain different Banach spaces X, Y no bounded Fredholm operators from X to Y may exist (see [13]).

Theorem 1.3.1. ([33, 38]) Suppose that X, Y and Z are Banach spaces.

(i) If $A \in \Phi_+(X, Y)$ and $B \in \Phi_+(Y, Z)$, then $BA \in \Phi_+(X, Z)$ and $\mathrm{i}(BA) = \mathrm{i}(A) + \mathrm{i}(B)$,

(ii) If $A \in \Phi_-(X, Y)$ and $B \in \Phi_-(Y, Z)$, then $BA \in \Phi_-(X, Z)$ and $\mathrm{i}(BA) = \mathrm{i}(A) + \mathrm{i}(B)$,

(iii) If $A \in \Phi(X, Y)$ and $B \in \Phi(Y, Z)$, then $BA \in \Phi(X, Z)$ and $\mathrm{i}(BA) = \mathrm{i}(A) + \mathrm{i}(B)$

(iv) If $BA \in \Phi_+(X, Z)$, then $B \in \Phi_+(Y, Z)$,

(v) If $BA \in \Phi_-(X, Z)$, then $B \in \Phi_-(Y, Z)$,

(vi) If $BA \in \Phi(X, Z)$, then $B \in \Phi_-(Y, Z)$ and $A \in \Phi_+(X, Y)$.

[1] It is named in honor of Erik Ivar Frodholm, Swedish mathematician, April 7,1866 - August 17, 1927.

The converse of (i)-(iii) in Theorem 1.3.1 is not true in general. To see this, consider the following operators on ℓ^2 :

$$Ax = (0, x_1, 0, x_2, 0, \dots)$$
$$Bx = (x_2, x_3, x_4, \dots)$$

Then A and B are not Fredholm, but $BA = I$.
However, if $BA = AB$ then we have if $BA \in \Phi(X)$ then $A \in \Phi(X)$ and $B \in \Phi(X)$. Because $N(A) \subset N(BA)$ and $R(BA) = R(AB) \subset R(A)$ By Theorem 1.2.8

$$A \in \Phi_+(X,Y) \Leftrightarrow A^* \in \Phi_-(Y^*, X^*)$$
$$A \in \Phi_-(X,Y) \Leftrightarrow A^* \in \Phi_+(Y^*, X^*)$$
$$A \in \Phi(X,Y) \Leftrightarrow A^* \in \Phi(Y^*, X^*)$$

If $A \in \Phi_\pm(X)$, then $\mathrm{i}\,(A^*) = -\,\mathrm{i}(A)$.

Example 1.3.1. If U is the unilateral shift operator on ℓ^2, then

$$\mathrm{i}(U) = 1 \quad \text{and} \quad \mathrm{i}\,(U^*) = -1.$$

With U and U^*, we can build a Fredholm operator whose index is equal to an arbitrary prescribed integer. Indeed if

$$A = \begin{pmatrix} Up & 0 \\ 0 & U^{*q} \end{pmatrix} : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$$

then A is Fredholm, $\alpha(A) = q, \beta(A) = p$, and hence $\mathrm{i}\,(A) = q - p$.
By [20] we have the following important stability property of semi-Fredholm operators.

Theorem 1.3.2. Suppose that $A \in \Phi_\pm(X,Y)$. If $B \in L(X,Y)$ such that $D(A) \subset D(B)$ and $\|B\| < \gamma(A)$ then $A + B \in \Phi_\pm(X,Y)$ and

$$\alpha(A+B) \le \alpha(A), \quad \beta(A+B) \le \beta(A), \quad \mathrm{i}(A+B) = \mathrm{i}(A).$$

This theorem extended to the unbounded case by Theorem 1.17 in the following way

Theorem 1.3.3. Theorem 1.3.2 is true if B is A-bonded with $a < (1-b)\gamma(A)$.

The following theorem establishes an important characterization of Fredholm operators.

Theorem 1.3.4. (Atkinson characterization of Fredholm operators) Let $A \in \Phi(X,Y)$ if and only there exist $U_1, U_2 \in L(Y,X)$ and finite-dimensional operators $K_1 \in \mathcal{F}(X), K_2 \in \mathcal{F}(Y)$ such that

$$U_1 A = I - K_1 \text{ on } D(A) \text{ and } AU_2 = I - K_2 \text{ on } Y.$$

For a proof of this classical result we refer to [38]. It should be noted that in the characterization above the ideal $\mathcal{F}(X)$ may be replaced by the ideal $\mathcal{K}(X)$ of all compact operators. In particular, $A \in \Phi^b(X)$ if and only if A is invertible in $L(X)$ modulo the ideal of finite-dimensional operators $\mathcal{F}(X)$.

Definition 1.3.2. Let X and Y be two Banach spaces and let $F \in L(X,Y)$.

(i) The operator F is called Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$.

(ii) F is called an upper (resp. lower) semi-Fredholm perturbation if $U+F \in \Phi_+(X, Y)$ (resp. $U + F \in \Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (resp. $U \in \Phi_-(X, Y)$).

We denote by $\mathcal{F}(X, Y)$ the set of Fredholm perturbations and by $\mathcal{F}_+(X, Y)$ (resp. $\mathcal{F}_-(X, Y)$) the set of upper semiFredholm (resp. lower semi-Fredholm) perturbations.

Lemma 1.3.1. [14] Let $A \in \mathcal{C}(X, Y)$ and $F \in \mathcal{L}(X, Y)$. Then

(i) If $A \in \Phi^b(X, Y)$ and $F \in \mathcal{F}^b(X, Y)$, then $A + F \in \Phi^b(X, Y)$ and $i(A + F) = i(A)$.

(ii) If $A \in \Phi^b_+(X, Y)$ and $F \in \mathcal{F}^b_+(X, Y)$, then $A + F \in \Phi^b_+(X, Y)$ and $i(A + F)i(A)$.

(iii) If $A \in \Phi^b_-(X, Y)$ and $F \in \mathcal{F}^b_-(X, Y)$, then $A + F \in \Phi^b_-(X, Y)$ and $i(A + F)i(A)$.

Proof. Statement (i) follows from [14, Proposition 3]. (ii) Let $A \in \Phi^b_+(X, Y)$ and $F \in \mathcal{F}^b_+(X, Y)$. The fact that $A + F \in \Phi^b_+(X, Y)$ follows from Definition 1.3.2 To prove that $i(A + F) = i(A)$, we discuss two cases.

Case 1: If $A \in \Phi^b_+(X, Y)\backslash\Phi^b(X, Y)$, then $i(A) = -\infty$. Hence, $i(A + F) = -\infty$. Otherwise, $A + F \in \Phi^b(X, Y)$ and therefore $A \in \Phi^b_+(X, Y)$, since $F \in \mathcal{F}^b_+(X, Y) \subset \mathcal{F}^b(X, Y)$. What is contradictory.

Case 2: If $A \in \Phi^b(X, Y)$, then the result follows from the assertion (i). Statement (iii) can be checked in the same way as (ii). □

Lemma 1.3.2. [4] Let $A \in \mathcal{C}(X, Y)$ and $F \in \mathcal{L}(X, Y)$. Then

(i) If $A \in \Phi^b(X, Y)$ and $F \in \mathcal{F}^b(X, Y)$, then $A + F \in \Phi^b(X, Y)$ and $i(A + F) = i(A)$.

(ii) If $A \in \Phi^b_+(X, Y)$ and $F \in \mathcal{F}^b_+(X, Y)$, then $A + F \in \Phi^b_+(X, Y)$ and $i(A + F)i(A)$.

(iii) If $A \in \Phi^b_-(X, Y)$ and $F \in \mathcal{F}^b_-(X, Y)$, then $A + F \in \Phi^b_-(X, Y)$ and $i(A + F)i(A)$.

Proposition 1.3.1. (i) Φ_U is open.

(ii) $i(\lambda - U)$ is constant on any component of Φ_U.

(iii) $\alpha(\lambda - U)$ and $\beta(\lambda - U)$ are constant on any component of Φ_U except on a discrete set of points at which they have larger values.

Theorem 1.3.5. For $T \in L(X)$ the following are equivalent:

(1) T is Fredholm;

(2) $\exists H \in L(X)$ such that $1 - TH$ and $1 - HT$ are of finite rank;

(3) $\exists H \in L(X)$ such that $1 - TH$ and $1 - HT$ are compact;

where 1 denotes the identity operator on X.

1.4 Spectrum

1.4.1 Spectrum of bounded operator

Definition 1.4.1. Let $A : X \to X$ be a bounded linear operator on a normed complex space X. The spectrum

$$\sigma(A) = \{\forall \lambda \in \mathbb{C}, A - \lambda I \text{ is not invertible.}\}$$

Definition 1.4.2. Let A be a closable linear operator in a Banach space X. The resolvent set and the spectrum of A are, respectively, defined as

$$\rho(A) = \left\{\lambda \in \mathbb{C} \ \text{ such that } A - \lambda \ \text{ is injective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\right\}.$$

We recall $\sigma(A) = \mathbb{C} \backslash \rho(A)$ the spectrum of A as

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is not injective}\}$$

and the point spectrum, continuous

$$\sigma_c(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective } \overline{R(\lambda - A)} = X, R(\lambda - A) = X\}$$

and the residual spectrum are defined as:

$$\sigma_r(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is injective} \overline{R(\lambda - A)} \neq X\}$$

Theorem 1.4.1. The spectrum $\sigma(T)$ of any bounded linear operator T is a closed subset of contained in $|\lambda| \leq \|T\|$.

Proof. Define the map $F : \mathrm{C} \to B(X)$ by $F(\lambda) = T - \lambda I$. We have $\|F(\lambda) - F(\mu)\| = |\lambda - \mu|$, so that F is continuous. We have $\sigma(T) = F^{-1}(B(X) \backslash G)$ where G denotes the set of invertible operators. Since G is open, we deduce that $B(X) \backslash G$ and $\sigma(T)$ are closed.

Suppose that $|\lambda| > \|T\|$. Then $\|\lambda^{-1}T\| < 1$, and it follows that $I - \lambda^{-1}T$ is invertible. Hence, $T - \lambda I = -\lambda\left(I - \lambda^{-1}T\right)$ is also invertible. This show that $\lambda \notin \sigma(T)$. □

Example 1.4.1.

1. Let $X = C([a, b]), \phi \in X$, and $A_\phi : X \to X$ be the multiplication operator: $A_\phi(f) = \phi f$. It is easy to check that if ϕ is monotone, A_ϕ has no eigenvalues. We claim that
$$\sigma\left(A_\phi\right) = \phi([a, b])$$
Indeed, let $\lambda \in \phi([a, b])$. If B is the inverse of $A_\phi - \lambda I$, then $(A_\phi - \lambda I)B = I$, and for every $f \in X, (\phi - \lambda)B(f) = f$. Thke to $\in [a, b]$ such that $\phi\left(t_0\right) = \lambda$. Then $f\left(t_0\right) = 0$, but this cannot hold for for all $f \in X$. On the other hand, if $\lambda \notin \phi([a, b])$, then $(\phi - \lambda)^{-1}$ is continuous on $[a, b]$, and $A_{(\phi-\lambda)^{-1}}$ defines the inverse of $A_\phi - \lambda I = A_{\phi-\lambda}$.

2. Let $X = \ell^2$ and $S : (x_1, x_2, \ldots,) \mapsto (0, x_1, x_2, \ldots)$ be the shift operator on X. We claim that that S has no eigenvalues. Indeed, if $Sx = \lambda x$, then $\lambda x_1 = 0$ and $\lambda x_i - x_{i-1} = 0$ for all $i > 1$. For here we deduce that $x = 0$, so that S has no eigenvectors. Next we show that
$$\sigma(S) = \{|\lambda| \leq 1\}.$$

Since $\|S\| = 1$, it is clear that $\sigma(S)$ is contained in this set. To prove the opposite inclusion, we show that if $|\lambda| \leq 1$, then $S - \lambda I$ is not onto. When $\lambda = 0$, is easy to check that $y = (1, 0, \ldots)$ not in the image of $S - \lambda I$. Now let $\lambda \neq 0$. We suppose that there exists $x \in X$ such that $(S - \lambda I)x = y$. Then $-\lambda x_1 = 1$ and $x_{i-1} - \lambda x_i = 0$. We deduce that $x_i = -1/\lambda^i$. However, since $|\lambda| \geq 1$, this x is not in ℓ^2. Hence, $S - \lambda I$ is not onto as claimed.

Proposition 1.4.1. We have

1. For all $\lambda, \mu \in \mathbb{C}$ $\qquad |j(\lambda) - j(\mu)| \leq |\lambda - \mu|.$
2. If $\lambda \in \rho(A)$ then $j(\lambda I - A) = \|R(\lambda, A)\|^{-1}$.

1.4.2 Spectrum of unbounded operator

Definition 1.4.3. Let X be a Banach space and $T : D(T) \to X$ be a linear operator defined on domain $D(T) \subseteq X$.
A complex number λ is said to be in the resolvent set (also called regular set) of T if the operator

$$T - \lambda I : D(T) \to X$$

has a bounded everywhere-defined inverse, i.e. if there exists a bounded operator

$$S : X \to D(T)$$

such that

$$S(T - \lambda I) = I_{D(T)}, (T - \lambda I)S = I_X$$

A complex number λ is then in the spectrum if λ is not in the resolvent set.
For λ to be in the resolvent (i.e. not in the spectrum), just like in the bounded case,

$$T - \lambda I \text{ must be bijective,}$$

since it must have a two-sided inverse. As before, if an inverse exists, then its linearity is immediate, but in general it may not be bounded, so this condition must be checked separately.
By the closed graph theorem, boundedness of $(T - \lambda I)^{-1}$ does follow directly from its existence when T is closed. Then, just as in the bounded case, a complex number λ lies in the spectrum of a closed operator

$$T \text{ if and only if } T - \lambda I \text{ is not bijective.}$$

Note that the class of closed operators includes all bounded operators.

Definition 1.4.4. Let A be a closable linear operator in a Banach space X.
The resolvent set and the spectrum of A are, respectively, Then, according to the closed graph we deduce that

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ such that } A \text{ is bijective } \}$$

and hence,

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

Proposition 1.4.2. Let A be closed operator on X and $\lambda \in \rho(A)$. Then the following assertions hold.

1. $\sigma(R(\lambda, A)) = \left\{\frac{1}{\lambda-\mu}; \mu \in \sigma(A)\right\}$.
2. $\sigma_i(R(\lambda, A)) = \left\{\frac{1}{\lambda-\mu}; \mu \in \sigma_i(A)\right\}$ for $i = p, ap, r, c$.
3. If A is unbounded, then $0 \in \sigma(R(\lambda, A))$.

There are some overlapping parts of the spectrum which are commonly used too. For instance, the defect spectrum (or the surjectivity spectrum) $\sigma_{su}(A)$ and the compression spectrum $\sigma_{co}(A)$ are defined by

$$\sigma_{su}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not surjective }\}$$

and

$$\sigma_{co}(A) := \{\lambda \in \mathbb{C} : R(\lambda I - A) \text{ is not dense in } X\}.$$

By the closed range theorem we easily know that the approximate point spectrum and the surjectivity spectrum are dual to each other, in the sense that $\sigma_{ap}(A) = \sigma_{su}(A^*)$ and $\sigma_{ap}(A^*) = \sigma_{su}(A)$. Moreover, this two sets form a (not necessarily disjoint) subdivision

$$\sigma(A) = \sigma_{ap}(A) \cup \sigma_{su}(A)$$

of the spectrum and $\sigma_{co}(A) \subset \sigma_{su}(A)$. Moreover, we note that

$$\sigma_{\infty\infty}(A) = \sigma_{p2}(A) \cup \sigma_{p3}(A) \cup \sigma_r(A).$$

Remark 1.4.1. The spectrum of an unbounded operator is in general a closed, possibly empty, subset of the complex plane.
lf the operator T is not closed, then $\sigma(T) = \mathbb{C}$.

Point spectrum
If an operator is not injective (so there is some nonzero x with $T(x) = 0$), then it is clearly not invertible. So if λ is an eigenvalue of T, one necessarily has $\lambda \in \sigma(T)$. The set of eigenvalues of T is also called the point spectrum of T, denoted by $\sigma_{\mathrm{p}}(T)$.

Approximate point spectrum
More generally, by the bounded inverse theorem, T is not invertible if it is not bounded below; that is, if there is no $c \mid > 0$ such that $\|Tx\| \geq \|c \mid \|x\|$ for all $x \in X$. So the spectrum includes the set of approximate eigenvalues, which are those λ such that T - λ is not bounded below; equivalently, it is the set of λ for which there is a sequence of unit vectors $x_1, x_2, \ldots$ for which

$$\lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0$$

The set of approximate eigenvalues is known as the approximate point spectrum, denoted by $\sigma_{\mathrm{ap}}(T)$. It is easy to see that the eigenvalues lie in the approximate point spectrum. For example, consider the right shift R on $l^2(\mathbb{Z})$ defined by

$$R : e_j \mapsto e_{j+1}, \quad j \in \mathbb{Z}$$

where $(e_j)_{j\in\mathbb{N}}$ is the standard orthonormal basis in $l^2(\mathbb{Z})$. Direct calculation shows R has no eigenvalues, but every λ with $|\lambda| = 1$ is an approximate eigenvalue; letting x_n be the vector

$$\frac{1}{\sqrt{n}}\left(\ldots, 0, 1, \lambda^{-1}, \lambda^{-2}, \ldots, \lambda^{1-n}, 0, \ldots\right)$$

one can see that $\|x_n\| = 1$ for all n, but

$$\|Rx_n - \lambda x_n\| = \sqrt{\frac{2}{n}} \to 0.$$

Since R is a unitary operator, its spectrum lies on the unit circle. Therefore, the approximate point spectrum of R is its entire spectrum. This conclusion is also true for a more general class of operators. A unitary operator is normal.

Continuous spectrum

The set of all λ for which $T - \lambda I$ is injective and has dense range, but is not surjective, is called the continuous spectrum of T, denoted by $\sigma_{\mathrm{c}}(T)$. The continuous spectrum therefore consists of those approximate eigenvalues which are not eigenvalues and do not lie in the residual spectrum. That is,

$$\sigma_{\mathrm{c}}(T) = \sigma_{\mathrm{ap}}(T) \backslash \left(\sigma_{\mathrm{r}}(T) \cup \sigma_{\mathrm{p}}(T)\right)$$

For example, $A : l^2(\mathbb{N}) \to l^2(\mathbb{N}), e_j \mapsto e_j/j, j \in \mathbb{N}$, is injective and has a dense range, yet $\operatorname{Ran}(A) \subsetneq l^2(\mathbb{N})$. .Indeed, if $x = \sum_{j\in\mathbb{N}} c_j e_j \in l^2(\mathbb{N})$ with $c_j \in \mathbb{C}$ such that $\sum_{j\in\mathbb{N}} |c_j|^2 < \infty$, one does not necessarily have $\sum_{j\in\mathbb{N}} |jc_j|^2 < \infty$, and then $\sum_{j\in\mathbb{N}} jc_je_j \notin l^2(\mathbb{N})$

Compression spectrum

The set of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ does not have dense range is known as the compression spectrum of T and is denoted by $\sigma_{\mathrm{cp}}(T)$.

Residual spectrum

The set of $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is injective but does not have dense range is known as the residual spectrum of T and is denoted by $\sigma_{\mathrm{r}}(T)$:

$$\sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{cp}}(T) \backslash \sigma_{\mathrm{p}}(T)$$

An operator may be injective, even bounded below, but still not invertible. The right shift on $l^2(\mathbb{N}), R : l^2(\mathbb{N}) \to l^2(\mathbb{N}), R : e_j \mapsto e_{j+1}, j \in \mathbb{N}$, is such an example. This shift operator is an isometry, therefore bounded below by 1 . But it is not invertible as it is not surjective ($e_1 \notin \operatorname{Ran}(R)$), and moreover $\operatorname{Ran}(R)$ is not dense in $l^2(\mathbb{N})$ $\left(e_1 \notin \overline{\operatorname{Ran}(R)}\right)$.

Peripheral spectrum

The peripheral spectrum of an operator is defined as the set of points in its spectrum which have modulus equal to its spectral radius. [50]

Discrete spectrum

The discrete spectrum is defined as the set of normal eigenvalues. Equivalently, it can be characterized as the set of isolated points of the spectrum such that the corresponding Riesz projector is of finite rank.

Definition 1.4.5. Let X be a banach space and let A be a bounded.
linear operotor in $L(X)$. we will define a function $f(A)$ of A by cauchy's type integral.

$$f(A) = (2\pi i)^{-1} \int_C f(\lambda)(\lambda - A)^{-1} d\lambda$$

Where C is circumference of sufficiently small radius.
For this purpose, we denote by $F(A)$ the family of all complex- valued functions $f(\lambda)$ wihich are holomrphic in some neigh borhood of the spectrum $\sigma(A)$ of A.

Theorem 1.4.2. If $f \in F(A)$ then $f(\sigma(A)) = \sigma(f(A))$.

1.4.3 Spectral radius

Definition 1.4.6. The spectral radius of an operator A is defined by

$$r(A) = max\{|\lambda| : \lambda \in \sigma(A)\}$$

Let $L(X)$ denote the algebra of all bounded linear operators on a complex Banach space X. For $A \in L(X)$, let $r(A)$ and $\|A\|$ denote the spectral radius and the usual operator norm of A, respectively. It is well known that for every $A \in L(X)$, we have

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}, \tag{1.7}$$

a special case of the spectral mapping theorem, which asserts that

$$r(A^n) = (r(A))^n \text{ for every positive integer } n,$$

and a commutativity property, which asserts that

$$r(AB) = r(BA) \text{ for every } A, B \in L(X). \tag{1.8}$$

The property (1.8) is an immediate consequence of the fact that the spectra of the operators AB and BA have the same nonzero elements.

It follows easily from the spectral radius formula (1.7) that if $A, B \in L(X)$ are such that $AB = BA$, then

$$r(A+B) \leq r(A) + r(B)$$

and

$$r(AB) \leq r(A)r(B).$$

Corollary 1.4.1. If $A, B \in L(X)$, then

$$r(AB \pm BA) \leq \frac{1}{2}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\|A^2\|\|B^2\|}\right). \tag{1.9}$$

Proof. The inequality (1.9) follows from Theorem 1 by letting $A_1 = B_2 = A, B_1 = B$, and $A_2 = \pm B$ □

Corollary 1.4.2. Let $T \in L(X)$, then

$$r(T) \leq \|T\|$$

.

Proof. If $|\lambda| > \|T\| \Longrightarrow \lambda \in \rho(T)$, then

$$\forall \lambda \in \sigma(T) : |\lambda| \leq \|T\|$$

. Thus $\sup_{\lambda\in\sigma(T)} |\lambda| \leq \|T\|$, therefore $r(T) \leq \|T\|$. □

Proposition 1.4.3. [34] Let $T, S \in L(X), \alpha \in \mathbb{C}$ and $n \in \mathbb{N}$. Then :

1. $r(\alpha T) = |\alpha| r(T)$.
2. $r(T^n) = r(T)^n$.

3. $r(TS) = r(ST)$.
4. $r(T^*) = r(T)$.
5. If $T \geq 0$, then $r(\sqrt{T}) = \sqrt{r(T)}$.

Example 1.4.2. Consider the matrix

$$A = \begin{bmatrix} 9 & -1 & 2 \\ -2 & 8 & 4 \\ 1 & 1 & 8 \end{bmatrix}$$

whose eigenvalues are $5, 10, 10$; by definition, $\rho(A) = 10$. In the following table, the values of $\left\|A^k\right\|^{\frac{1}{k}}$ for the four most used norms are listed versus several increasing values of k (note that, due to the particular form of this matrix, $\|\cdot\|_1 = \|\cdot\|_\infty$):

$\boldsymbol{k}$	$\|\cdot\|_1 = \|\cdot\|_\infty$	$\|\cdot\|_F$	$\|\cdot\|_2$
1	14	15.362291496	10.681145748
2	12.649110641	12.328294348	10.595665162
3	11,934831919	11,532450664	10,500980846
4	11.501633169	11.151002986	10.418165779
5	11.216043151	10.921242235	10.351918183
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	10.604944422	10.455910430	10.183690042
11	10.548677680	10.413702213	10.166990229
12	10.501921835	10.378620930	10.153031596

1.5 The Riesz projection and the singularities of the resolvent

Let A a closed operator and its spectrum $\sigma(A)$ decomposes into two non-empty disjoint closed subsets σ and τ. Suppose that σ is bounded, we can surrounding σ by a Jordan contour [2] positively oriented Γ. Hence we can associate a Riesz integral with σ

$$P_\sigma(A) = \frac{1}{2\pi i} \oint_\Gamma (\lambda - A)^{-1} d\lambda, \tag{1.10}$$

This projection is called the spectral projection associated with the spectral set σ. Since $(\lambda - A)^{-1}$ is analytic operator function on the resolvent set of A, a standard argument of complex function theory shows that the definition of P_σ does not depend on the particular choice of the contour Γ. A fundamental properties of the spectral projection P_σ are given in the next results.

Proposition 1.5.1. Let A be a closed operator on X such that its spectrum $\sigma(A)$ is the disjoint union of two non-empty closed subsets σ and τ with σ is bounded. Let P_σ be as defined in (1.10). Then

(i) P_σ is a projection.

[2] A subset Γ of $\mathbb{C}$ is a Jordan contour if there exists a finite number of pairwise disjoint closed simple rectifiable Jordan curves positively oriented $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ such that $\Gamma = \cup_{i=1}^n \Gamma_i$.

(ii) P_σ commutes with A on $D(A)$;

(iii) $X = R(P_\sigma) \oplus N(P_\sigma), \sigma(AP_\sigma) = \sigma$ and $\sigma(A(I - P_\sigma)) = \sigma(A) \backslash \sigma$.

Proposition 1.5.2. Let A be a closed operator on X and λ_0 an isolated point of the spectrum of A. Then

(i) $N(\lambda_0 - A) \subseteq R(P_{\lambda_0})$.

(ii) If X is a Hilbert space and A is self-adjoint, then P_{λ_0} is the orthogonal projection onto $N(\lambda_0 - A)$.

Remark 1.5.1. Let $A \in \mathcal{C}(X), \lambda$ and μ two different isolated points of $\sigma(A)$ then

(i) $P_\lambda P_\mu = P_\mu P_\lambda = 0$.

(ii) $P_{\{\lambda,\mu\}} = P_\lambda + P_\mu$.

Let $A \in \mathcal{C}(X)$ and λ_0 an isolated point of $\sigma(A)$. The Laurent series for the resolvent $(\lambda I - A)^{-1}$ in neighborhood of the isolated singularity λ_0 is given by

$$(\lambda I - A)^{-1} = \sum_{n=-\infty}^{+\infty} (\lambda - \lambda_0)^n A_n \tag{1.11}$$

where

$$A_n = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} \frac{1}{(\lambda - \lambda_0)^{n+1}} R(\lambda, A) d\lambda \tag{1.12}$$

and Γ_{λ_0} is a positively oriented small circle enclosing λ_0 but no other point of $\sigma(A)$. The coefficients A_n defined in (1.19) stisfay some useful identities given in the next proposition.

Proposition 1.5.3. The coefficients A_n given by (1.12) are bounded operators and satisfies the following proprieties

(i) $AA_n = A_nA$ on $D(A)$ for all $n \in \mathbb{Z}$,

(ii) $A_nA_m = (1 - \eta_n - \eta_m) A_{n+m+1}$ where $\eta_n = 1$ if $n \geq 0$ and $\eta_n = 0$ if $n < 0$.

(iii) $A_{-1} = I - (\lambda_0 I - A) A_0$,

(iv) $A_{n-1} = (A - \lambda_0 I) A_n$ for each $n \neq 0$,

(v) $A_0 = (A - \lambda_0 I)^n A_n$ and $A_{-n} = (A - \lambda_0 I)^{n-1} A_{-1}$ for all $n \geq 1$.

Theorem 1.5.1. If λ_0 is an isolated point in the spectrum of a closed operator A, then Laurent series (1.18) around λ_0 is equivalent to

$$(\lambda I - A)^{-1} = \frac{1}{\lambda - \lambda_0} P_{\lambda_0} + \sum_{n=1}^{\infty} \frac{1}{(\lambda - \lambda_0)^{n+1}} D^n - \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n S^{n+1}$$

with the residue operator A_{-1} is a projection coincides with the Riesz projection associated to λ_0 and $SP_{\lambda_0} = P_{\lambda_0} S = 0, DS = SD = 0, D = DP_{\lambda_0} = P_{\lambda_0} D$.

Now we characterize the isolated points of the spectrum that are poles of the resolvent.

Theorem 1.5.2. Let λ_0 be an isolated point in the spectrum of a closed operator A and let

$$(\lambda I - A)^{-1} = \frac{1}{\lambda - \lambda_0} P_{\lambda_0} + \sum_{n=1}^{\infty} \frac{1}{(\lambda - \lambda_0)^{n+1}} D^n - \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n S^{n+1}$$

be the Laurent series around λ_0. Then the following statements are equivalents

1. λ_0 is a pole of the resolvent of order m.
2. The operator $D = (\lambda_0 I - A) P_{\lambda_0}$ is a nilpotent operator of order m.
3. $a(\lambda_0 I - A) = d(\lambda_0 I - A) = m < \infty$.

Corollary 1.5.1. Let A be a closed operator. If λ_0 is a pole of order m of the resolvent around λ_0. Then λ_0 is an eigenvalue of A. Moreover

$$X = N\left((\lambda_0 I - A)^n\right) \oplus R\left((\lambda_0 I - A)^n\right) \text{ for all } n \geq m.$$

and the Laurent series (1.32) around λ_0 is equivalent to

$$(\lambda I - A)^{-1} = \frac{1}{\lambda - \lambda_0} P_{\lambda_0} + \sum_{n=1}^{m-1} \frac{1}{(\lambda - \lambda_0)^{n+1}} D^n - \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n S^{n+1}; \tag{1.13}$$

Remark 1.5.2. Let λ_0 a pole of order m of the resolvent around λ_0. From Proposition 1.5.1 it follows that $X = R(P_{\lambda_0}) \oplus N(P_{\lambda_0})$. There are two numbers measuring, roughly speaking, the number of eigenvectors belonging to $N(\lambda_0 - A)$. They are called multiplicities: The algebraic multiplicity of the eigenvalue λ_0 is defined as the dimension of the space $R(P_{\lambda_0})$ and equal m the order of the pole λ_0. The geometric multiplicity of λ_0 is defined as the dimension of $N(\lambda_0 - A)$ and equal $\alpha(\lambda_0 - A)$. In general, we have $\alpha(\lambda_0 - A) \leq m$.

Chapter 2

Essential Spectra of linear operator

2.1 Definitions and notations

Let X be a Banach space and $\mathcal{L}(X)$ be the set of all bounded linear operators from X into X. For $T \in \mathcal{L}(X)$ we denote by $R(T)$ its range, $N(T)$ its null space, $\sigma(T)$ its spectrum and T^* the adjoint of T. Let I denote the identity operator in X. An operator $T \in \mathcal{L}(X)$ is said to be semiregular if $R(T)$ is closed and $N(T^n) \subseteq R(T)$, for all $n \geq 0$. T admits a generalized Kato decomposition, if we can write $T = T_1 \oplus T_0$ where T_0 is a quasi-nilpotent operator and T_1 is a semi-regular one. If we assume in the definition above that T_0 is nilpotent, T is said to be of Kato type.

2.1.1 Gustafson,Weidmann essential spectrum

Definition 2.1.1. (Gustafson) Let X be a Banach space.Let $A \in \mathcal{C}(X)$, then

$$\sigma_{e1}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+A},$$

Definition 2.1.2. (Weidmann) Let X be a Banach space.Let $A \in \mathcal{C}(X)$, then

$$\sigma_{e2}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_-(X)\} := \mathbb{C}\backslash\Phi_{-A}.$$

Remark 2.1.1. Let $\Phi^b(X, Y)$ denote the set $\Phi(X, Y) \cap L(X, Y)$. In particular, it is shown that $\mathcal{F}^b(X, Y)$ is a closed subset of $L(X, Y)$ and if $X = Y$, then $\mathcal{F}^b(X) := \mathcal{F}^b(X, X)$ is a closed two-sided ideal of $L(X)$.
The result of this lemma remains valid if we replace $\Phi^b(X, Y)$ by $\Phi(X, Y)$ and $\mathcal{F}^b(X, Y)$ by $\mathcal{F}(X, Y)$.

Theorem 2.1.1. [17] Let $A, B \in \mathcal{C}(X)$ such that $\rho(A) \cap \rho(B) \neq \emptyset$.

(i) If for some $\lambda \in \rho(A) \cap \rho(B)$ the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{F}^b_+(X)$, then

$$\sigma_{e1}(A) = \sigma_{e1}(B).$$

(ii) If for some $\lambda \in \rho(A) \cap \rho(B)$ the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{F}^b_-(X)$, then

$$\sigma_{e2}(A) = \sigma_{e2}(B)$$

Proof. Without loss of generality, we suppose that $\lambda = 0$. Hence $0 \in \rho(A) \cap \rho(B)$. Therefore, we can write for $\mu \neq 0$

$$\mu - A = -\mu\left(\mu^{-1} - A^{-1}\right) A.$$

Since, A is one to one and onto, then

$$\alpha(\mu - A) = \alpha\left(\mu^{-1} - A^{-1}\right) \text{ and } R(\mu - A) = R\left(\mu^{-1} - A^{-1}\right).$$

This shows that $\mu \in \Phi_{+A}$ (resp. Φ_{-A}) if and only if $\mu^{-1} \in \Phi_{+A^{-1}}$ (resp. $\Phi_{-A^{-1}}$), in this case we have $i(\mu - A) = i\left(\mu^{-1} - A^{-1}\right)$. Therefore, it follows from Lemma 1.3.1 that $\Phi_{+A} = \Phi_{+B}$ (resp. $\Phi_{-A} = \Phi_{-B}$) and $i(\mu - A) = i(\mu - B)$, for each $\mu \in \Phi_{+A}$ (resp. Φ_{-A}), since $A^{-1} - B^{-1} \in \mathcal{F}_+^b(X)$ (resp. $\in \mathcal{F}_-^b(X)$). This proves the first part of (i) and (ii). The second part of these assertions follows from [24, Theorem 3.2] . □

Example 2.1.1.

$$\lambda = 0 \in \sigma_{\mathrm{e}\ 1}(A) \text{ for the operator } A : l^2(\mathbb{N}) \to l^2(\mathbb{N}), A : e_j \mapsto e_j/j, j \in \mathbb{N}$$

(because the range of this operator is not closed: the range does not include all of $l^2(\mathbb{N})$ although its closure does).

$$\lambda = 0 \in \sigma_{\mathrm{e}\ 1}(N) \text{ for } N : l^2(\mathbb{N}) \to l^2(\mathbb{N}), N : v \mapsto 0 \text{ for any } v \in l^2(\mathbb{N})$$

(because both kernel and cokernel of this operator are infinite-dimensional).

$\lambda = 0 \in \sigma_{\mathrm{e}\ 2}(B)$ for the operatort $B : l^2(\mathbb{N}) \to l^2(\mathbb{N}), B : e_j \mapsto e_{j/2}$ if j is even and $e_j \mapsto 0$ when j is odd

(kernel is infinite-dimensional; cokernel is zero-dimensional).
Note that $\lambda = 0 \notin \sigma_{\mathrm{e}1}(B)$.

2.1.2 Kato essential spectrum

Definition 2.1.3. Let X be a Banach space.let $A \in \mathcal{C}(X)$ then

$$\sigma_{e3}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi_{\pm}(X)\} := \mathbb{C}\backslash\Phi_{\pm A},$$

We show that these results are also valid on $C(\wp)$ where $\wp$ is a compact Hausdorff space.

Proposition 2.1.1. [2] Let X be one of the spaces $L_p(\mu), p \in \{1\} \cup (2, \infty)$, or $C(\wp)$ and assume that $A \in \mathcal{C}(X)$.

If $S \in \mathscr{S}(X)$, then

$$\sigma_{ei}(A) = \sigma_{ei}(A + S), \quad i = 2, 3.$$

Note that $\mathscr{S}(X)$. is a closed subspace of $L(x)$.

Proposition 2.1.2. Let $M, N \in \mathcal{F}(X)$. For every $x \in X$ and $0 < \epsilon < 1$ there exists $x_0 \in X$ such that $(x - x_0) \in M$ and

(1)

$$\operatorname{dist}(x_0, N) \geq \left((1 - \epsilon)\frac{1 - \delta(M, N)}{1 + \delta(M, N)}\right)\|x_0\|.$$

Example 2.1.2.

$$\lambda = 0 \in \sigma_{\mathrm{e}\ 3}(J) \text{ for the operator } J : l^2(\mathbb{N}) \to l^2(\mathbb{N}), J : e_j \mapsto e_{2j}$$

(kernel is zerodimensional, cokernel is infinite-dimensional).
Note that $\lambda = 0 \notin \sigma_{\mathrm{e}2}(J)$.

2.1.3 Wolf, Schechter essential spectrum

Definition 2.1.4. (Wolf) Let X be a Banach space , the Wolf essential spectrum of an operator $A \in \mathcal{C}(X)$ is defined by

$$\sigma_{e4}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi(X)\} := \mathbb{C} \backslash \Phi_A,$$

where $\mathcal{K}(X)$ stands for the ideal of all compact operators on X.

Definition 2.1.5. (Schechter) Let X be a Banach space and let $A \in \mathcal{L}(X)$ then

$$\sigma_{e5}(A) := \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K).$$

Theorem 2.1.2. Let $A \in \mathcal{C}(X)$. If the complement of $\sigma_{e4}(A)$ is connected and $\rho(A)$ is not empty, then

$$\sigma_{e4}(A) = \sigma_{e5}(A).$$

Proof. Since the inclusion $\sigma_{e4}(A) \subset \sigma_{e5}(A)$ is known, it suffices to show that

$$\sigma_{e5}(A) \subset \sigma_{e4}(A)$$

which is equivalent to

$$C\sigma_{e4}(A) \cap \sigma_{e5}(A) = \emptyset.$$

Suppose that

$$C\sigma_{e4}(A) \cap \sigma_{e5}(A) \neq \emptyset,$$

let

$$\lambda_0 \in C\sigma_{e4}(A) \cap \sigma_{e5}(A).$$

Since $\rho(A) \neq \emptyset$, then there exists $\lambda_1 \in \mathbf{C}$ such that $\lambda_1 \in \rho(A)$ and consequently

$$\lambda_1 - A \in \Phi(X) \text{ and } i\,(\lambda_1 - A) = 0.$$

Moreover, $C\sigma_{e4}(A)$ is connected, it follows from Proposition 1.3.1 (ii) that $i(\lambda - A)$ is constant on any component of Φ_A. Therefore

$$i\,(\lambda_1 - A) = i\,(\lambda_0 - A) = 0.$$

In this way we see that $\lambda_0 \notin \sigma_{e5}(A)$, which is a contradiction. This proves

$$C\sigma_{e4}(A) \cap \sigma_{e5}(A) = \emptyset,$$

and completes the proof of theorem. □

Theorem 2.1.3. (N. Moalla [31]) Let $A \in \Phi(X)$, then for all $K \in \mathscr{S}_n(X)$, we have

(i) $K + A \in \Phi(X)$ and $i(K + A) = i(A)$,

(ii) $\sigma_{e5}(A + K) = \sigma_{e5}(A)$.

Remark 2.1.2. If $A \in L(X)$ and the complement of $\sigma_{e4}(A)$ is connected, then

$$\sigma_{e4}(A) = \sigma_{e5}(A)$$

$$\sigma_{e5}(A + K) = \sigma_{e5}(A) \text{ for all } K \in \mathcal{K}(X).$$

Remark 2.1.3. If X satisfies the Dunford-Pettis property, and if A is a closed, densely defined, and linear operator on X, then

$$\sigma_{e5}(A+K) = \sigma_{e5}(A) \text{ for all } K \in \mathcal{K}(X).$$

Theorem 2.1.4. Let $A \in \mathcal{C}(X)$ such that $\rho(A)$ is not empty.
If $\mathbb{C}\backslash\sigma_{e4}(A)$ is connected, then $\sigma_{e4}(A) = \sigma_{e5}(A)$.

Proposition 2.1.3. . Let $A \in \mathcal{C}(X)$. Then, $\lambda \notin \sigma_w(A)$ if, and only if, $\lambda - U \in \Phi(X)$ and $i(\lambda - A) = 0$.

Definition 2.1.6. Let $A \in L(X)$.

The spectral radius of $A \in L(X)$ by

$$r_e(A) = \sup\{|\lambda| : \lambda \in \sigma_{ei}(A)\}.$$

Corollary 2.1.1. ([30] and [29]). Let $\Phi : L(\mathscr{H}) \to L(\mathscr{H})$ be a linear map surjective up to compact operators.
Then Φ preserves the essential spectrum (the semi-Fredholm spectrum if and only if $\Phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$, and the induced map $\varphi : \mathcal{C}(\mathscr{H}) \to \mathcal{C}(\mathscr{H})$ is either a continuous automorphism or a continuous antiautomorphism.

Proof. It suffices to prove the 'only if' part. Assume that Φ preserves the essential spectrum. We first prove that Φ is unital modulo a compact operator Let $A \in L(\mathscr{H})$ such that $r_e(A) = 0$.
There are two operators $S, K \in L(\mathscr{H})$ such that K is compact and $A = \Phi(S) + K$. We have

$$\begin{aligned}
\sigma_{ei}(A + \Phi(I) - A) &= \sigma_{ei}(A + \Phi(I)) - 1 \\
&= \sigma_{ei}(\Phi(S) + K + \Phi(I)) - 1 \\
&= \sigma_{ei}(\Phi(S) + \Phi(I)) - 1 \qquad\qquad i = 4 \text{ and } 5 \\
&= \sigma_{ei}(S + I) - 1 \\
&= \sigma_{ei}(S) \\
&= \sigma_{ei}(\Phi(S)) \\
&= \sigma_{ei}(A) \\
&= \{0\}.
\end{aligned}$$

By Lemma 2.1.2 we see that $\Phi(I) - I$ is a compact operator. Note that, since Φ preserves the essential spectrum (the semi-Fredholm spectrum), it preserves the essential spectral radius as well. It follows from Theorem 3.1 in [30] that $\Phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$ and that there are a scalar λ of modulus 1 and either a continuous automorphism or a continuous antiautomorphism ψ such that $\varphi = \lambda\psi$. What was shown above implies that φ is unital. As ψ is unital as well, we see that $\lambda = 1$ and φ is either a continuous automorphism or a continuous antiautomorphism. $\square$

Example 2.1.3.

$$\lambda = 0 \in \sigma_{\mathrm{e}\,4}(R) \text{ where } R : l^2(\mathbb{N}) \to l^2(\mathbb{N}) \text{ is the right shift operator}$$

$$R : l^2(\mathbb{N}) \to l^2(\mathbb{N}), R : e_j \mapsto e_{j+1} \text{ for } j \in \mathbb{N}$$

(its kernel is zero, its cokernel is onedimensional).
Note that $\lambda = 0 \notin \sigma_{\mathrm{e}\,3}(R)$.

Lemma 2.1.1. Let $A \in \Phi_+(X)$. Then the following statements are equivalent

(i) $i(A) \leqslant 0$

(ii) A can be expressed in the form $A = U + K$ where $K \in \mathcal{K}(X)$ and $U \in \mathcal{C}(X)$ an operator with closed range and $\alpha(U) = 0$.

This lemma is well known for bounded upper semi-Fredholm operators. The proof is a straightforward adaption of the proof of Theorem 3.9 in [49].

The results of the next proposition was established in [35] and [42] for bounded linear operators.
We will improve it for closed, densely defined linear operators. This result is a characterization of the essential approximate point spectrum (resp. the essential defect spectrum) by means of upper semi-Fredholm (resp. lower semi-Fredholm) operators.

2.1.4 Browder essential spectrum

Definition 2.1.7. Let X be a Banach space.Let $A \in \mathcal{C}(X)$ them

$$\sigma_{e6}(A) := \mathbb{C}\backslash\rho_6(A),$$

$\rho_6(A) := \{\lambda \in \rho_5(A)$ such that all scalars near λ are in $\rho(A)\}$ with $\rho_5(A)$
$\rho_5(A) := \{\lambda \in \Phi_A$ such that $i(\lambda - A) = 0\}$

Proposition 2.1.4. Let $A \in \mathcal{C}(X, Y)$ and let S be a non-null bounded linear operator from X into Y. Then, we have the following results:

(i) $\Phi_{A,S}$ is open.

(ii) $i(\lambda S - A)$ is constant on any component of $\Phi_{A,S}$.

(iii) $\alpha(\lambda S - A)$ and $\beta(\lambda S - A)$ are constant on any component of $\Phi_{A,S}$, except on a discrete set of points on which they have larger values.

Proposition 2.1.5. Let $A \in \mathscr{C}(X)$ and $S \in \mathscr{L}(X)$ such that $S \neq A$ and $S \neq 0$. Then, for any $\mu, \lambda \in \rho_{6,S}(A)$, we have

$$R_{b,S}(A, \lambda) - R_{b,S}(A, \mu) = (\lambda - \mu) R_{b,S}(A, \lambda) S R_{b,S}(A, \mu) + \mathscr{M}(\lambda, \mu),$$

where $\mathscr{M}(\lambda, \mu)$ is a finite rank operator with the following expression

$$\mathscr{M}(\lambda, \mu) = R_{b,S}(A, \lambda) \left[(A - (\lambda S + I)) P_{\lambda,S} - (A - (\mu S + I)) P_{\mu,S}\right] R_{b,S}(A, \mu)$$

is a finite rank operator with $\operatorname{rank}(\mathscr{M}(\lambda, \mu)) = \operatorname{rank}(P_{\lambda,S}) + \operatorname{rank}(P_{\mu,S})$ in case $\lambda \neq \mu$.

Proof. We have

$$R_{b,S}(A, \lambda) - R_{b,S}(A, \mu) = R_{b,S}(A, \lambda) \left[A_{\mu,S} - A_{\lambda,S}\right] R_{b,S}(A, \mu),$$

where

$$A_{\mu,S} := (A - \mu S)(I - P_{\mu,S}) + P_{\mu,S}$$

So,

$$\begin{aligned} A_{\mu,S} - A_{\lambda,S} &= [(A-\mu S)(I-P_{\mu,S}) + P_{\mu,S}] - [(A-\lambda S)(I-P_{\lambda,S}) + P_{\lambda,S}] \\ &= [(A-(\lambda S+I))P_{\lambda,S} - (A-(\mu S+I))P_{\mu,S}] + (\lambda-\mu)S. \end{aligned}$$

Therefore,

$$R_{b,S}(A,\lambda) - R_{b,S}(A,\mu) = (\lambda-\mu)R_{b,S}(A,\lambda)SR_{b,S}(A,\mu) + \mathscr{M}(\lambda,\mu).$$

□

Proposition 2.1.6. Let X and Y be two complex Banach spaces. Let $A \in \mathscr{C}(X), S \in \mathscr{L}(X), B \in \mathscr{L}(Y,X)$, and $C : X \longrightarrow Y$ be two linear operators. Then,

(i) $R_{b,S}(A,\mu)B$ is continuous for some $\mu \in \rho_{6,S}(A)$ if, and only if, it is continuous for all $\mu \in \rho_{6,S}(A)$.

(ii) $C(A-\mu S)^{-1}$ is bounded for some $\lambda \in \rho_S(A)$ if, and only if, $CR_{b,S}(A,\mu)$ is bounded for some (hence for every) $\mu \in \rho_{6,S}(A)$.

(iii) If B and C satisfy the conditions (i) and (ii), respectively, and B is densely defined, then $C\mathscr{M}(\lambda,\mu), \overline{\mathscr{M}(\lambda,\mu)B}$, and $\overline{C\mathscr{M}(\lambda,\mu)B}$ are operators of finite rank for any $\mu, \lambda \in \rho_{6,S}(A)$.

Proof. From the resolvent identity, we have for any $\mu, \lambda \in \rho_S(A)$,

$$(A-\lambda S)^{-1}C = (A-\mu S)^{-1}C + (\lambda-\mu)(A-\lambda S)^{-1}S(A-\mu S)^{-1}C \tag{2.1}$$

and for any $\mu, \lambda \in \rho_{6,S}(A)$,

$$R_{b,S}(A,\lambda)B = R_{b,S}(A,\mu)B + (\lambda-\mu)R_{b,S}(A,\lambda)SR_{b,S}(A,\mu)B + \mathscr{M}(\lambda,\mu)B, \tag{2.2}$$

and

$$CR_{b,S}(A,\lambda) = CR_{b,S}(A,\mu) + (\lambda-\mu)CR_{b,S}(A,\lambda)SR_{b,S}(A,\mu) + C\mathscr{A}(\lambda,\mu). \tag{2.3}$$

(i) Since S is bounded, then

$$R_{b,S}(A,\lambda)SR_{b,S}(A,\mu)B$$

is bounded. According to Proposition 2.1.5 that the operator

$$(A-(\lambda S+I))P_{\lambda,S} - (A-(\mu S+I))P_{\mu,S}$$

is bounded. Thus,

$$\mathcal{A}(\lambda,\mu)B$$

has finite dimensional range, and in view of Eq. (2.2), we have

$$R_{b,S}(A,\lambda)B - R_{b,S}(A,\mu)B$$

is bounded. Hence, $R_{b,S}(A,\mu)B$ is continuous for some $\mu \in \rho_{6.S}(A)$ if, and only if, it is continuous for all $\mu \in \rho_{6.S}(A)$. (ii) If $CR_{b,S}(A,\lambda)$ is bounded for some $\lambda \in \rho_{6,S}(A)$, then clearly $CR_{b,S}(A,\mu)$ is also bounded for any μ and, it follows from the Eq. (2.3) that

$$CR_{b,S}(A,\mu)$$

is bounded for any μ. By using Eq. (2.1) we have $C(A-\mu S)^{-1}$ is bounded for every $\mu \in \rho_S(A)$. (iii) According to Proposition 2.1.5 that the operator $\mathscr{M}(\lambda,\mu)$ is a finite rank operator. So, $C\mathscr{A}(\lambda,\mu)$ and $\mathscr{A}(\lambda,\mu)B$ are a finite rank operator. Hence, it is clear that

$$\overline{\mathscr{M}(\lambda,\mu)B}$$

is of finite rank if B is densely defined. Since

$$\begin{aligned} C\mathscr{A}(\lambda,\mu)B =& CR_{b,S}(A,\lambda)\left[(A-(\lambda S+I))P_{\lambda,S}\right. \\ & \left.(A-(\mu S+I))P_{\mu,S}\right]R_{b,S}(A,\mu)B \\ =& CR_{b,S}(A,\lambda)\left[(A-\lambda S)\left(I-P_{\lambda,S}\right)+P_{\lambda,S}\right]R_{b,S}(A,\mu)B \end{aligned}$$

and if B and C satisfy the conditions (i) and (ii), respectively, then

$$\overline{\mathscr{M}(\lambda,\mu)B}$$

will again be continuous and densely defined with finite-dimensional range. □

Remark 2.1.4. Let X be a Banach space, $T \in L(X), \delta(T)$ the resolvent set of T,

$$H=\{\lambda \in \mathbb{C} : \lambda - T \text{ is Fredholm }\}$$

and

$$H_r=\{\lambda \in \mathbb{C} : \lambda - T \text{ is Fredholm and } i(\lambda-T)=r\},$$

where r is a fixed constant.

Proposition 2.1.7. [2] Whenever $H_r \cap \delta(T) \neq \varnothing$ and $\lambda \in H_r$, then $\lambda \in \sigma_{e6}(T)$.

Proof. Suppose $H_r \cap \delta(T) \neq \varnothing$; let $\lambda_0 \in H_r \cap \delta(T)$ then $T-\lambda_0$ is invertile, therefore

$$\dim\ker\left(T-\lambda_0\right)=\dim\operatorname{coker}\left(T-\lambda_0\right)=0,$$

which gives $i\left(T-\lambda_0\right)=0$. But

$$\lambda_0 \in H_r \text{ and hence } i(T-\lambda)=0 \text{ for all } \lambda \in H_r.$$

Now from Theorem 3.3 of Gohberg and Krein [15, p. 205] it follows that

$$\dim\ker(T-\lambda)=0 \text{ for all } \lambda \in H_{\mathrm{r}}\ ,$$

except at possible isolated points $\{\lambda_j\}$. At these isolated points, $T-\lambda$ has closed range, trivial kernel fone-to-one), and trivial cokernel (range dense). Therefore $T-\lambda$ is invertible except for these possible isolated points $\{\lambda_j\}$.
This implies

$$\lambda \notin \sigma_{e6}(T) \text{ when } T-\lambda \text{ is invertible.}$$

Next, one verifies that $\lambda_j \notin \sigma_{e6}(T)$ by considering the definition of $\sigma_{e6}(T)$; observe for λ_j as above

(1) range of $T-\lambda_j$ is closed since $T-\lambda_j$ is Fredholm,

(2) λ_j is not a limit point of $\sigma(T)$, and

(3) $[\cup_{n\geqslant 1}$ null space $(T-\lambda_j)^n]$ is finite dimensional.

(This corresponds to C_{λ_j} of Theorem 4.2 of Gohberg and Krein [15, p. 212] which shows $\lambda_j \notin \sigma_{e6}(T)$. □

2.1.5 Rakočević ,Schmoeger essential spectrum

Definition 2.1.8. (Rakočević) Let X be a Banach space.let $A \in \mathcal{L}(X)$ them

$$\sigma_{e7}(A) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{ap}(A+K),$$

$$\sigma_{ap}(A) := \{\lambda \in \mathbf{C} \text{ such that } \inf_{\|x\|=1, x \in \mathcal{D}(A)} \|(\lambda - A)x\| = 0\Big\}$$

Definition 2.1.9. (Schmoeger) Let X be a Banach space.let $A \in \mathcal{L}(X)$ them

$$\sigma_{e8}(A) := \bigcap_{K \in \mathcal{K}(X)} \sigma_{\delta}(A+K)$$

where

$$\sigma_{\delta}(A) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is not surjective } \}.$$

Definition 2.1.10. For $A \in L(X, Y)$, set

$$f_M(A) = \inf_{N \subset M} \bar{\gamma}\left(Ai_N\right) \tag{2.4}$$

and

$$f(A) = \sup_{M \subset X} f_M(A) \tag{2.5}$$

where M, N represent infinite dimensional subspaces of X and Ai_N denotes the restriction of A to N.

The semi-norm $f(\cdot)$ is a measure of non-strict-singularity, it was introduced by M. Schechter in [40].

Proposition 2.1.8. (M. Schechter [41, Theorem 5.4, p. 180]) Let X be a Banach space and let $A \in \mathcal{C}(X)$. Then, $\lambda \notin \sigma_{e5}(A)$ if, and only if, $\lambda \in \Phi_A^0$, where $\Phi_A^0 = \{\lambda \in \Phi_A$ such that $i(\lambda - A) = 0\}$. For $n \in \mathbb{N}^*$, let

$$\mathscr{I}_n(X) = \left\{K \in \mathscr{L}(X) \text{ satisfying } f\left((KB)^n\right) < 1 \text{ for all } B \in L(X)\right\},$$

where $f(\cdot)$ is a measure of non-strict-singularity given in (2.5).

Theorem 2.1.5. (N. Moalla [31]) LetA, $B \in \mathscr{C}(X)$ such that $\rho(A) \cap \rho(B) \neq$ D. If, for some $\lambda \in \rho(A) \cap \rho(B)$, the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathscr{I}_n(X)$, then

$$\sigma_{e5}(A) = \sigma_{e5}(B).$$

We recall some properties of $\sigma_{e7}(\cdot)$ (resp. $\sigma_{e8}(\cdot)$) due to V. Rakocević (resp. C. Schmoeger), given in [42].

Theorem 2.1.6. [2] Let $A \in \mathcal{C}(X)$ such that $\rho(A)$ is not empty. Then,

(i) If $\mathbb{C} \backslash \sigma_{e4}(A)$ is connected, then

$$\sigma_{e4}(A) = \sigma_{e5}(A).$$

(ii) If $\mathbb{C} \backslash \sigma_{e5}(A)$ is connected, then

$$\sigma_{e5}(A) = \sigma_{e6}(A).$$

Proposition 2.1.9. [36] Let $A \in \mathcal{C}(X)$.

(1) $\partial \sigma_{e6}(A) \subseteq \sigma_{\mathrm{e}7}(A)$ (where $\partial \sigma_{e}(A)$ denotes the boundary of $\sigma_{e}(A)$).

(2) $\sigma_{\mathrm{e}7}(A) \neq \emptyset$.

(3) $\lambda \notin \sigma_{\mathrm{e}7}(A) \Leftrightarrow \lambda I - A \in \Phi_{+}(X)$ and ind $(\lambda I - A) \leq 0$.

(4) $\sigma_{\mathrm{e}7}(A)$ is compact, $\sigma_{\mathrm{e}7}(A) \subseteq \sigma(A)$.

Proof. For (1), (2), see [35], Theorem 1. For (3), see [35], Lemma 1 and 2. $\square$

Proposition 2.1.10. [42] Let $A \in L(X)$ and let λ_0 be a boundary point of $\sigma(A)$. If $\lambda_0 \in \Sigma(A)$ then λ_0 is an isolated point of $\sigma(A)$.

Proof. Theorem 3 of [20] shows the existence of $\delta > 0$ such that $\lambda \in \Sigma(A)$ for $|\lambda - \lambda_0| < \delta, \alpha(\lambda I - A)$ is a constant for $0 < |\lambda - \lambda_0| < \delta$ and $\beta(\lambda I - A)$ is a constant for $0 < |\lambda - \lambda_0| < \delta$. Take $\mu_0 \in \varrho(A)$ with $0 < |\mu_0 - \lambda_0| < \delta$.
Then $\alpha(\mu_0 I - A) = \beta(\mu_0 I - A) = 0$, thus $\alpha(\lambda I - A) = \beta(\lambda I - A) = 0$ for $0 < |\lambda - \lambda_0| < \delta$. This shows that $\lambda \in \varrho(A)$ for $0 < |\lambda - \lambda_0| < \delta$. $\square$

Proposition 2.1.11. [35] Let $A \in L(X)$ and $h \in L(A)$. If h has no zeroes in $\sigma_{\mathrm{e}7}(A)$ then h has at most a finite number of zeroes in $\sigma(A)$.

Proof. Assume that the number of zeroes of h in $\sigma(A)$ is infinite.
Then there is $z_0 \in \sigma(A)$ such that z_0 is an accumulation point of the zeroes of h in $\sigma(A)$.
Denote by C the connected component of $\sigma(A)$ which contains z_0 and by K the connected component of $\Delta(h)$ which contains z_0 (where $\Delta(h)$ is the open set of the definition of h).
It follows that $C \subseteq K$ and $h \equiv 0$ on K.
Let $\lambda_0 \in \partial C$. Then $h(\lambda_0) = 0$.
Since h does not vanish on $\sigma_{\mathrm{e}7}(A)$, we have $\lambda_0 \notin \sigma_{\mathrm{e}7}(A)$ and therefore $\lambda_0 \in \Sigma(A)$. Since C is a connected component of $\sigma(A)$, we also have $\lambda_0 \in \partial\sigma(A)$. By Proposition 2.1.10 we see that λ_0 is an isolated point of $\sigma(A)$. Thus $C = \{\lambda_0\}$.
Hence we get $z_0 = \lambda_0$, a contradiction, since z_0 is an accumulation point of $\sigma(A)$. $\square$

Proposition 2.1.12. Let (A_n) be a sequence in $L(X)$ converging to $A \in L(X)$ in the operator norm. If $V \subseteq \mathbb{C}$ is open and $0 \in V$, then there exists $n_0 \in \mathbb{N}$ such that

$$\sigma_{\mathrm{e}7}(A_n) \subseteq \sigma_{\mathrm{e}7}(A) + V \quad \text{ for all } n \geq n_0.$$

Proof. Assume not. Then by passing to a subsequence (if necessary) it may be assumed that for each n there

$$\exists \lambda_n \in \sigma_{\mathrm{e}7}(A_n) \text{ such that } \lambda_n \notin \sigma_{\mathrm{e}7}(A) + V.$$

Since (λ_n) is bounded, we may assume (if necessary pass to a subsequence) that $\lim_{n\to\infty} \lambda_n = \lambda_0$. This gives $\lambda_0 \notin \sigma_{\mathrm{e}7}(A) + V$, hence $\lambda_0 \notin \sigma_{\mathrm{e}7}(A)$. Thus $\lambda_0 I - A \in \Phi_{+}^{-}(X)$.
Since $\Phi_{+}^{-}(X)$ is an open multiplicative semigroup (see [12], §2) and

$$\lambda_n I - A_n \to \lambda_0 I - A(n \to \infty),$$

we get some

$$N \in \mathbb{N} \text{ such that } \lambda_n I - A_n \in \Phi_+^-(X) \text{ for all } n \geq N.$$

Use again to derive $\lambda_n \notin \sigma_{e7}\ (A_n)$ for each $n \geq N$, a contradiction. □

Proposition 2.1.13. [2] Let $A \in \mathcal{C}(X)$, then

(i) $\lambda \notin \sigma_{e7}\ (A)$ if and only if $\lambda - A \in \Phi_+(X)$ and $i(\lambda - A) \leqslant 0$

(ii) $\lambda \notin \sigma_{e8}(A)$ if and only if $\lambda - A \in \Phi_-(X)$ and $i(\lambda - A) \geqslant 0$.

(iii) If A is a bounded linear operator, then $\sigma_{e8}(A) = \sigma_{e7}\ (A^*)$, where A^* stands for the adjoint operator.

Proof. (i) Let $\lambda \in \Phi_{+A}$ such that $i(\lambda - A) \leqslant 0$. Then by Lemma 2.1.1, $\lambda - A$ can be expressed in the form $\lambda - A = +K$ where $K \in \mathcal{K}(X)$ and $A \in \mathcal{C}(X)$ an operator with closed range and $\alpha(A) = 0$. Hence by [37, Theorem 5.1, p. 70] there exists a constant $c > 0$ such that $\|Ax\| \geqslant c\|x\|$, for all $x \in \mathcal{D}(A)$. Thus $\lambda \notin \sigma_{ap}(A + K)$ and therefore $\lambda \notin \sigma_{\mathrm{eap}}\ (A)$. Conversely, if $\lambda \notin \sigma_{\mathrm{ecp}}\ (A)$, then there exists $K \in \mathcal{K}(X)$ such that

$$\inf_{\|x\|=1, x \in \mathcal{D}(A)} \|(\lambda - A - K)x\| > 0.$$

The use of [37, Theorem 5.1, p. 70] leads to $\lambda - A - K \in \Phi_+(X)$ and $\alpha(\lambda - A - K) = 0$, hence it follows from Lemma 2.1 that $\lambda - A \in \Phi_+(X)$ and $i(\lambda - A) \leqslant 0$. This completes the proof of (i).

The proof of (ii) is a straightforward adoption of the proof of Proposition 7 in [42].

(iii) This assertion follows, immediately, from (i) and (ii). □

Proposition 2.1.14. [43] Let $A \in \mathcal{C}(X)$, then

(i) $\sigma_{e7}(A) \neq \emptyset$,

(ii) $\sigma_{e8}(A) \neq Q$, and

(iii) $\sigma_{e7}(A)$ and $\sigma_{e8}(A)$ are compact.

The following proposition gives a characterization of the essential approximate point spectrum and the essential defect spectrum by means of upper semi-Fredholm and lower semi-Fredholm operators, respectively.

2.1.6 Essential spectrum radius

Definition 2.1.11. Let X be a Banach space, $L(X)$ be the bounded linear operators on X and let $\mathscr{K}$ be the closed ideal of compact operators on X. The quotient algebra $L(X)/\mathscr{K}$ is a Banach algebra called the Calkin algebra. The essential spectrum $\sigma_e(A)$ of A is defined by:

$$\sigma_e(A) = \{\lambda \in C : \Pi_1(\lambda - A) \text{ is not invertible in } L(X)/\mathscr{K}\}$$

where Π_1 is the natural homomorphism from $L(X)$ onto $L(X)/\mathscr{K}$. An operator $T \in L(X)$ is called a Fredholm operator if (i) the range of A is closed and (ii) the kernel of A and the cokernel of A are of finite dimension. For such an operator A,
The connection between the class of Fredholm operators and the Calkin algebra is contained in the following theorem of Atkinson [4].

Proposition 2.1.15. [34] Let $A \in \mathcal{C}(X)$

Therefore an equivalent definition for the essential spectrum is:

$$\sigma_e(A) = \{\lambda \in C : \lambda - A \text{ is not Fredholm }\}.$$

The radius $r_e(A)$ of the essential spectrum $\sigma_e(A)$ is defined by

$$r_e(A) = \sup\{|\lambda| : \lambda \in \sigma_e(A)\}.$$

In this section, we shall consider bounded A and shall obtain some other equivalents for the spectral radius. Our discussion is motivated by the results of Nussbaum [34] and Lebow and Schechter [26].

Nussbaum uses the concept of the ball measure of noncompactness to obtain a formula for $r_e(A)$.

For a bounded set D in X, the ball measure of noncompactness of D, denoted by $\gamma(D)$, is defined as

$$\gamma(D) = \inf\left\{r > 0 : D \subset \bigcup_{i=1}^{k} B(x_i, r)\right\}.$$

Here $B(x_i, r)$ stands for the ball centered at $x_i \in X$ with radius r and k is arbitrary but finite. We set $\gamma(A) = \gamma(A(\mathscr{U}_X))$. Nussbaum proves that

$$r_e(A) = \lim_n (\gamma(A^n))^{1/4}$$

Lebow and Schechter [26] show

$$r_e(A) = \lim_n (\|A^n\|_M)^{1/n}, \text{ where } \|A\|_M = \inf\{\eta : \exists a$$

subspace M of finite codimension $\ni \|Ax\| < \eta\|x\|, x \in M\}$.

Lemma 2.1.2. [26] The following are equivalent:

(i) $K \in \mathscr{K}(\mathscr{H})$.

(ii) $r_e(A+K) = 0$ for all $A \in L(\mathscr{H})$ for which $r_e(A) = 0$.

Proof. Assume that $K \in \mathscr{K}(\mathscr{H})$, and let $A \in L(\mathscr{H})$ be an operator such that

$$r_e(A) = 0.$$

Denote by π the quotient map from $L(\mathscr{H})$ onto $\mathscr{C}(\mathscr{H})$, and note that $\pi(K) = 0$ and that

$$\begin{aligned} r_e(A+K) &= r(\pi(A+K)) \\ &= r(\pi(A) + \pi(K)) \\ &= r(\pi(A)) \\ &= r_e(A) = 0 \end{aligned}$$

This establishes the implication (i) $\Rightarrow$ (ii).

Conversely, assume that $K \in L(\mathscr{H})$ is an operator such that

$$r_e(A+K) = 0 \text{ for all } A \in L(\mathscr{H}) \text{ for which } r_e(A) = 0.$$

Equivalently,

$$r(\pi(A) + \pi(K)) = 0 \text{ for all } A \in \mathscr{H} \text{ for which } r(\pi(A)) = 0.$$

By the spectral Zemánek's characterization of the radical, we have $\pi(K) \in \mathrm{rad}(C(\mathscr{H})) = \{0\}$, and K is a compact operator. $\square$

2.1.7 Interaction between the essential Spectrum definition

Proposition 2.1.16. [9] Let $A \in \mathcal{C}(X)$, then we have:

- $\sigma_{\mathrm{e}1}(A) \cap \sigma_{\mathrm{e}2}(A) = \sigma_{\mathrm{e}3}(A) \subseteq \sigma_{\mathrm{e}4}(A) \subseteq \sigma_{\mathrm{e}5}(A) \subseteq \sigma_{\mathrm{e}6}(A)$.
- $\sigma_{\mathrm{e}5}(A) = \sigma_{\mathrm{e}7}(A) \cup \sigma_{\mathrm{e}8}(A)$.
- $\sigma_{\mathrm{e}1}(A) \subset \sigma_{\mathrm{e}7}(A)$.
- $\sigma_{\mathrm{e}\,2}(A) \subset \sigma_{\mathrm{e}8}(A)$.

Remark 2.1.5. Let $A \in \mathcal{C}(X)$

(i) $\sigma_{\mathrm{e}5}(A) = \sigma_{\mathrm{eap}}(A) \cup \sigma_{e\delta}(A)$.

(ii) $\sigma_{e1}(A) \subset \sigma_{eap}(A)$ and $\sigma_{e2}(A) \subset \sigma_{e\delta}(A)$.

Theorem 2.1.7. [2] Let $A \in \mathcal{C}(X)$ such that $\rho(A)$ is not empty.

(i) If $\mathbb{C}\backslash\sigma_{e4}(A)$ is connected, then $\sigma_{e4}(A) = \sigma_{e5}(A), \sigma_{e2}(A) = \sigma_{e8}(A)$,

(ii) If $\mathbb{C}\backslash\sigma_{e5}(A)$ is connected, then $\sigma_{e5}(A) = \sigma_{e6}(A)$.

Proof. Since the inclusion
$$\sigma_{e5}(A) \subset \sigma_{e6}(A)$$
thes is known, it is sufficient to show that
$$\sigma_{e6}(A) \subset \sigma_{e5}(A)$$
which is equivalent to
$$[\mathbb{C}\backslash\sigma_{e5}(A)] \bigcap \sigma_{e6}(A) = \emptyset.$$
In fact, let us suppose the following
$$[\mathbb{C}\backslash\sigma_{e5}(A)] \cap \sigma_{e6}(A) \neq \emptyset.$$
Then,
$$\exists \lambda_0 \in \mathbb{C}, \text{ such that } \lambda_0 \in \mathbb{C}\backslash\sigma_{e5}(A) \text{ and } \lambda_0 \in \sigma_{e6}(A).$$
Let $\lambda \in \mathbb{C}\backslash\sigma_{e5}(A)$.
Since $\mathbb{C}\backslash\sigma_{e5}(A)$ is connected, and using Proposition 2.1.4 (iii), it follows that
$$\alpha(\lambda - A) = \alpha(\lambda_0 - A) \text{ and } \beta(\lambda - A) = \beta(\lambda_0 - A).$$
We claim that $\lambda \in \sigma_{e6}(A)$.
In fact, if $\lambda \in \rho_6(A)$,
then there exists a neighborhood V_λ of λ such that $V_\lambda\backslash\{\lambda\} \subset \rho(A)$.
Since $\mathbb{C}\backslash\sigma_{e5}(A)$ is connected, there exists $\lambda_1 \in \mathbb{C}\backslash\sigma_{e5}(A)$ such that $\lambda_1 \in V_\lambda\backslash\{\lambda\} \subset \rho(A)$.
Hence,
$$\alpha(\lambda_1 - A) = \beta(\lambda_1 - A) = 0 \text{ and so, } \alpha(\lambda_0 - A) = \beta(\lambda_0 - A) = 0.$$
In this way, we see that
$$\lambda_0 \in \rho(A) \subset \rho_6(A),$$

which is contrary to the properties of λ_0. This proves the claim and shows that

$$\mathbb{C}\backslash\sigma_{e5}(A) \subset \sigma_{e6}(A).$$

Hence,

$$\emptyset \neq \rho(A) \subset \rho_5(A) \subset \sigma_{e6}(A) \subset \sigma(A)$$

which is a contradiction. This shows that

$$[\mathbb{C}\backslash\sigma_{e5}(A)] \bigcap \sigma_{e6}(A) = \emptyset$$

and completes the proof of the theorem. $\square$

Remark 2.1.6. (i) If $\lambda \in \sigma_c(A)$ (the continuous spectrum of A, then $R(\lambda - A)$ is not closed (otherwise $\lambda \in \rho(A)$ see [41, Lemma 5.1 p. 179]). Therefore, $\lambda \in \sigma_{ei}(A), i = 1, \cdots, 6$. Consequently, we have

$$\sigma_c(A) \subset \bigcap_{i=1}^{6} \sigma_{ei}(A).$$

If the spectrum of A is purely continuous, then

$$\sigma(A) = \sigma_c(A) = \sigma_{ei}(A) \quad i = 1, \cdots, 6$$

(ii) Let E_0 be a core of A, i.e., a linear subspace of $D(A)$ such that the closure $\overline{A_{|E_0}}$ of its restriction A_{E_0} equals A. Then,

$$\sigma_{ap}(A) := \left\{ \lambda \in \mathbb{C} \text{ such that } \inf_{\|x\|=1, x \in E_0} \|(\lambda - A)x\| = 0 \right\}.$$

(iii) Set $\widetilde{\alpha}(A) := \inf\{\|Ax\| \text{ such that } x \in D(A) \text{ and } \|x\| = 1\}$. Whenever E_0 is a core of A, then

$$\widetilde{\alpha}(A) := \inf \left\{ \|Ax\| \text{ such that } x \in E_0 \text{ and } \|x\| = 1 \right\},$$

holds. Using this quantity, we obtain

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \text{ such that } \widetilde{\alpha}(\lambda - A) = 0\}.$$

Theorem 2.1.8. [39] Let $A \in \mathcal{C}(X)$. If $0 \in \rho(A)$, then for all $\lambda \in \mathbb{C}, \lambda \neq 0$ we have

(i) $\lambda \in \sigma_{ei}(A)$ if, and only if, $\frac{1}{\lambda} \in \sigma_{ei}\left(A^{-1}\right)$, for $i = 1, 2, 3, 4, 5, 7, 8$.

(ii) If $\mathbb{C}\backslash\sigma_{e5}(A)$ and $\mathbb{C}\backslash\sigma_{e5}\left(A^{-1}\right)$ are connected, then $\lambda \in \sigma_{e6}(A)$ if, and only if, $\frac{1}{\lambda} \in \sigma_{e6}\left(A^{-1}\right)$.

Proof. (i) Using [39], we deduce that

$$\lambda \in \sigma_{ei}(A) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ei}\left(A^{-1}\right) \text{ for } i = 4, 5.$$

Moreover, for $\lambda \neq 0$, we can write

$$A - \lambda = -\lambda\left(A^{-1} - \lambda^{-1}\right) A.$$

Since A is one-to-one and onto, then

$$\alpha(A-\lambda)=\alpha\left(A^{-1}-\lambda^{-1}\right) \text{ and } R(A-\lambda)=R\left(A^{-1}-\lambda^{-1}\right).$$

This shows that $\lambda \in \Phi_{+A}$ (resp. Φ_{-A}) if, and only if, $\frac{1}{\lambda} \in \Phi_{+A^{-1}}$ (resp. $\Phi_{-A^{-1}}$) and we have $i(A-\lambda)=i\left(A^{-1}-\lambda^{-1}\right)$. Therefore, we infer that

$$\lambda \in \sigma_{ei}(A) \text{ if, and only if, } \frac{1}{\lambda} \in \sigma_{ei}\left(A^{-1}\right) \text{ for } i=1,2,3,4,5,7,8.$$

(ii) The sets $\mathbb{C}\backslash\sigma_{e5}(A)$ and $\mathbb{C}\backslash\sigma_{e5}\left(A^{-1}\right)$ are connected. $\rho(A)$ and $\rho\left(A^{-1}\right)$ are not empty sets. So, applying Theorem 2.1.7 (ii), we deduce that $\lambda \in \sigma_{e6}(A)$ if, and only if, $\frac{1}{\lambda} \in \sigma_{e6}\left(A^{-1}\right)$.

□

Theorem 2.1.9. Let X be one of the spaces $L_p(\mu), 1 \leq p \leq \infty$, or $C(\wp)$ and assume that $A \in \mathcal{C}(X)$. If $S \in A\mathscr{S}(X)$, then

$$\sigma_{ei}(A)=\sigma_{ei}(A+S), \quad i=1,4,5.$$

Moreover, if $C\sigma_{e5}(A)$ t the complement of $\sigma_{e5}(A)$] is connected and neither $\rho(A)$ nor $\rho(A+S)$ is empty, then

$$\sigma_{e6}(A)=\sigma_{e6}(A+S)$$

In [23], it is shown (Theorem 3.2) that

$$\sigma_{e5}(A)=\bigcap_{S\in\mathscr{S}(L_p(\mu))}\sigma(A+S). \tag{2.6}$$

The next result shows that this result is not optimal and (2.6) may be expressed in terms of A-strictly singular perturbations. More precisely, we have:

Remark 2.1.7. 1. For $p \in (1,2]$, Proposition 2.1.1 was already established in [23, Theorem 3.1(b)]. The proof relies on the properties of superprojective Banach spaces [38]. Our proof here uses the identity $\mathscr{S}\left(L_p(\mu)\right)=C\mathscr{S}\left(L_p(\mu)\right)$ (valid for $p \in [1,\infty)$, see [24, 36]) and works for $p \in \{1\} \cup (2,\infty)$ as well as for $p \in (1,2]$.

2. Since $\mathscr{S}(X) \subseteq A\mathscr{S}(X)$, it follows from Theorem 2.1.9 that the result of Proposition 2.1.1 holds also true for $\sigma_{e1}(\cdot), \sigma_{e4}(\cdot), \sigma_{e5}(\cdot)$, and $\sigma_{e6}(\cdot)$ (see also [23, Theorem 3.1]).

The next result gives a convenient criterion for the invariance of essential spectra of perturbed linear operators.

Proposition 2.1.17. Let X be one of the spaces $L_p(\mu), p \in \{1\} \cup (2,\infty)$, or $C(\wp)$ and let $A, B \in \mathscr{C}(X)$. If, for some $\lambda \in \rho(A) \cap \rho(B)$, we have $(\lambda-A)^{-1}-(\lambda-B)^{-1} \in \mathscr{S}(X)$, then

$$\sigma_{ei}(A)=\sigma_{ei}(B), \quad i=1,2,3,4, \text{ and } 5.$$

Remark 2.1.8. For $X=L_p(\mu)$ with $p \in [1,\infty)$, the conclusion of Proposition 2.1.17 concerning $\sigma_{e1}(\cdot), \sigma_{e4}(\cdot)$, and $\sigma_{e5}(\cdot)$ was already established in [23]. The following lemma is essential in proving Propositions 2.1.1 and 2.1.17.

Theorem 2.1.10. [48] Let X be a complex Banach space. Let A be a member of $B(X)$. Let f be a complex-valued function which is analytic in a neighborhood of $\sigma(A)$, the spectrum of A. Then the operator $f(A)$ in $B(X)$ is well defined by the complex functional calculus . Suppose now that $e_3(A) = \sigma(A)$. Then

1. $f\left(e_3(A)\right) = e_3(f(A))$,
2. $e_3(f(A)) = \sigma(f(A))$,
3. $f\left(e_4(A)\right) = e_4(f(A))$,
4. $e_4(f(A)) = \sigma(f(A))$.

Proof. Let d be an arbitrary complex number. Suppose that for each c in $e_3(A)$, $f(c) \neq d$. Then $f(z) - d$ is an analytic function in a neighborhood of $\sigma(A)$ which does not vanish on $\sigma(A)$. Thus by the complex functional calculus $f(A) - dI$ has a bounded inverse on X. Thus d is in the resolvent set of A. In particular, d is not a member of $e_3(f(A))$. Thus $e_3(f(A)) \subset f\left(e_9(A)\right)$. Suppose now that $d = f(c)$, where c is a member of $e_3(A)$. Set

$$g(z) = \begin{cases} (f(z) - d)/(z - c), & \text{if } z \neq c \\ f'(c), & \text{if } z \approx c \end{cases}$$

Then $g(z)$ is analytic in a neighborhood of $\sigma(A)$ and $g(z)(z-c) = f(z) - d$. Thus by the functional calculus, $g(A)(A - cI) = f(A) - dI = (A - cI)g(A)$. Suppose that d is not a member of $e_3(f(A))$. Then $f(A) - d$ is a member of $\Phi(X)$. Observe that as a consequence of the equalities above, $N(A - c) \subset N(f(A) - d)$ and $R(f(A) - d) \subset R(A - c)$. Thus, $a(A-c) \leqslant a(f(A)-d) < \infty$ and $b(A-c) \leqslant b(f(A)-d) < \infty . R(f(A)-d)$ is closed in X and $X = R(f(A-d) \oplus N$, where N is a finite dimensional subspace of X. Hence $R(A-c)$ is closed in view of the above inclusion. Thus $A - c$ is a member of $\Phi(X)$. Therefore c is not a member of $e_3(A)$. Hence $f\left(e_3(A) \subset e_3(f(A)\right.$, and the proof of (i) is completed.

By the standard spectral mapping theorem, $f(\sigma(A)) = \sigma(f(A))$. Observing further that for an arbitrary operator $A, e_3(A) \subset e_4(A) \subset \sigma(A)$, and using (i), we obtain (ii), (iii), and (iv) immediately. $\square$

Theorem 2.1.11. [42] Let $A, B \in \mathcal{C}(X)$ such that $\rho(A) \cap \rho(B) \neq \emptyset$.

(i) If for some $\lambda \in \rho(A) \cap \rho(B)$ the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{F}_+^b(X)$, then

$$\sigma_{\text{eap}}(A) = \sigma_{\text{e7}}(B) \text{ and } \sigma_{\text{e1}}(A) = \sigma_{e1}(B).$$

(ii) If for some $\lambda \in \rho(A) \cap \rho(B)$ the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{F}_-^b(X)$, then

$$\sigma_{e8}(A) = \sigma_{e8}(B) \quad \text{ and } \quad \sigma_{e2}(A) = \sigma_{e2}(B).$$

Proof. Without loss of generality, we suppose that $\lambda = 0$. Hence $0 \in \rho(A) \cap \rho(B)$. Therefore, we can write for $\mu \neq 0$

$$\mu - A = -\mu\left(\mu^{-1} - A^{-1}\right) A.$$

Since, A is one to one and onto, then

$$\alpha(\mu - A) = \alpha\left(\mu^{-1} - A^{-1}\right) \text{ and } R(\mu - A) = R\left(\mu^{-1} - A^{-1}\right).$$

This shows that $\mu \in \Phi_{+A}$ (resp. Φ_{-A}) if and only if $\mu^{-1} \in \Phi_{+A^{-1}}$ (resp. $\Phi_{-A^{-1}}$), in this case we have $i(\mu - A\ ) = i\left(\mu^{-1} - A^{-1}\right)$. Therefore, it follows from Lemma 1.3.2 that $\Phi_{+A} = \Phi_{+B}$ (resp. $\Phi_{-A} = \Phi_{-B}$) and $i(\mu - A) = i(\mu - B)$, for each $\mu \in \Phi_{+A}$ (resp. Φ_{-A}), since $A^{-1} - B^{-1} \in \mathcal{F}_{+}^{b}(X)$ (resp. $\in \mathcal{F}_{-}^{b}(X)$). Hence the use of Proposition 2.1.13 makes us conclude that $\sigma_{\mathrm{e}7}\ (A) = \sigma_{\mathrm{e}7}\ (B)$ (resp. $\sigma_{e8}(A) = \sigma_{e8}(B)$). This proves the first part of (i) and (ii). The second part of these assertions follows from [24, Theorem 3.2] . □

Theorem 2.1.12. [35] Let $A \in \mathcal{C}(X)$ with a non-empty resolvent set, and let f be a complex-valued function that is holomorphic on an open set containing $\sigma(A) \cup \{\infty\}$. Then

$$\sigma_{e7}(f(A)) \subseteq f\left(\sigma_{ec7}(A)\right).$$

Proposition 2.1.18. Let A_1 and A_2 be two elements of $\mathcal{C}(X)$ such that $\left(\lambda - A_1\right)^{-1} - \left(\lambda - A_2\right)^{-1} \in \mathcal{F}_{+}^{D}(X)\big($ resp. $\in \mathcal{F}_{-}^{D}(X)\big)$ for some $\lambda \in \rho\left(A_1\right) \cap \rho\left(A_2\right)$. If $\sigma_{\mathrm{e}7}\ \left(A_1\right) = \sigma_{\mathrm{e}1}\ \left(A_1\right)$ (resp. $\sigma_{e8}\left(A_1\right) = \sigma_{\mathrm{e}2}\ \left(A_1\right)$) and if f is a complex-valued function holomorphic on a neighborhood of $\sigma\left(A_1\right) \cup \sigma\left(A_2\right) \cup \{\infty\}$, then

$$\sigma_{\mathrm{e}7}\ \left(f\left(A_k\right)\right) = f\left(\sigma_{\mathrm{eap}}\ \left(A_k\right)\right), \quad k = 1, 2$$

(resp.

$$\sigma_{e8}\left(f\left(A_k\right)\right) = f\left(\sigma_{e8}\left(A_k\right)\right), \quad k = 1, 2).$$

Proof. For $k = 1$ the result follows from the hypothesis $\sigma_{\mathrm{e}7}\ \left(A_1\right) = \sigma_{\mathrm{e}1}\ \left(A_1\right)$ and [6, Theorem 7]. For the case $k = 2$, the inclusion $\sigma_{\mathrm{e}7}\ \left(f\left(A_2\right)\right) \subset f\left(\sigma_{\mathrm{e}7}\ \left(A_2\right)\right)$ follows from Theorem 2.1.12 It remains to show that $f\left(\sigma_{\mathrm{e}7}\ \left(A_2\right)\right) \subset \sigma_{\mathrm{e}7}\ \left(f\left(A_2\right)\right)$. To do so, we consider $\lambda \in f\left(\sigma_{\mathrm{e}7}\ \left(A_2\right)\right)$, then there exists $\mu \in \sigma_{\mathrm{e}7}\ \left(A_2\right)$ such that $\lambda = f(\mu)$. Hence it follows, from Theorem 2.1.11 and the hypothesis $\sigma_{\mathrm{e}7}\ \left(A_1\right) = \sigma_{\mathrm{e}1}\ \left(A_1\right)$, that $\mu \in \sigma_{\mathrm{e}1}\ \left(A_2\right)$. Now, applying the spectral mapping theorem for $\sigma_{\mathrm{e}1}\ (\cdot)$[6, Theorem 7(a)], we obtain $f(\mu) \in \sigma_{\mathrm{e}2}\ \left(f\left(A_2\right)\right) \subseteq \sigma_{\mathrm{e}7}\ \left(f\left(A_2\right)\right)$. Thus, $\lambda \in \sigma_{\mathrm{e}7}\ \left(f\left(A_2\right)\right)$. This proves the result for $\sigma_{\mathrm{e}7}\ (\cdot)$. The result for $\sigma_{e8}(\cdot)$ can be proved in the same way. □

Lemma 2.1.3. (See [28].) Let D be a bounded subset of $\mathbb{R}^n$ and $1 < p < \infty$. If the hypothesis (5.1) and (5.2) are satisfied, the meosure $d\mu$ satisfies

$$\left\{\begin{array}{l}\text{the hyperplanes have zero } d\mu \text{ measure, i.e.,} \\ \text{for each } e \in S^{n-1}, d\mu\left\{v \in \mathbb{R}^n, v \cdot e = 0\right\} = 0,\end{array}\right.$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ and the collision operotor $K : L_p^{\sigma}\left(\mathbb{R}^n\right) \longrightarrow L_p\left(\mathbb{R}^n\right)$ is compact. Then for any $\lambda \in \mathbb{C}$ sudh that $\operatorname{Re}\lambda > -\sigma_0$, the operotor $K(\lambda - T)^{-1}$ is compact on X_p.

Theorem 2.1.13. Assume thot the hypotheses of Lemme 2.1.3 are satisfied. Then

$$\sigma_{\mathrm{e}7}\ (A) = \sigma_{e8}(A) = \left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\sigma_0\right\}.$$

Proof. It follows from the proof of Lemma 2.1.3 that for any $\lambda \geqslant 1$, $\left.\|(\lambda - T)^{-1}\|_{\mathcal{L}(X_p} \cdot X_p^{\sigma}\right) \leqslant \sup_{v \in \mathbb{R}^n} \frac{\sigma(v)}{(K \in \lambda + \sigma(v))^p}$. Therefore, since X_p^{σ} is continuously embedded in X_p, we infer that $\lim_{\lambda \to +\infty} \|K(\lambda - T)^{-1}\|\, \mathcal{L}\left(X_p\right) = 0$. So, there exists $\lambda \in \rho(A)$ such that $r_{\sigma}\left(K(\lambda - T)^{-1}\right) < 1$. For such λ we have

$$(\lambda - A)^{-1} - (\lambda - T)^{-1} = \sum_{n \geqslant 1} (\lambda - T)^{-1}\left[K(\lambda - T)^{-1}\right]^n.$$

Now, the result follows from Theorem 2.1.11 and Lemma 2.1.3. □

Remark 2.1.9. (i) If $\lambda \in \sigma C(T)$ then $\mathcal{R}(\lambda - T)$ is not closed (otherwise $\lambda \in \rho(T)$. Therefore $\lambda \in \sigma_{ei}(T), i = 1, \ldots, 6$. Consequently we have $\sigma C(T) \subset \bigcap_{i=1}^{6} \sigma_{ei}(T)$. If the spectrum of T is purely continuous then $\sigma(T) = \sigma C(T) = \sigma_{ei}(T) i = 1, \ldots, 6$.

(ii) Let $T \in \mathcal{C}(X)$, then it follows from the closedness of T that $\mathcal{D}(T)$ endowed with the graph norm $\|.\|_T$ (i.e., $\|x\|_T := \|x\| + \|Tx\|, x \in \mathcal{D}(T)$) is a Banach space. Let X_T denotes $(\mathcal{D}(T), \ \| \cdot \|_T)$. In this new space the operator T satisfies $\|Tx\| \leq \|x\|_T$ and consequently T is a bounded operator from X_T into X.

Theorem 2.1.14. [47] Let $A \in \mathcal{C}(X)$ such that $\rho(A)$ is not empty.

(i) (i) If $\mathbb{C} \backslash \sigma_{e4}(A)$ is connected, then $\sigma_{e4}(A) = \sigma_{e5}(A), \sigma_{e2}(A) = \sigma_{e8}(A),$.

(ii) If $\mathbb{C} \backslash \sigma_{e5}(A)$ is connected, then $\sigma_{e5}(A) = \sigma_{e6}(A)$.

Example 2.1.4. consider the operator $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), T : e_j \mapsto e_{j-1}$ for $j \neq 0, T : e_0 \mapsto 0$. Since $\|T\| = 1$, one has $\sigma(T) \subset \overline{\mathbb{D}_1}$.
For any $z \in \mathbb{C}$ with $|z| = 1$, the range of $T - zI$ is dense but not closed, hence the boundary of the unit disc is in the first type of the essential spectrum: $\partial \mathbb{D}_1 \subset \sigma_{\mathrm{e}\ 1}(T)$.
For any $z \in \mathbb{C}$ with $|z| < 1, T - zI$ has a closed range, onedimensional kernel, and one-dimensional cokernel, so $z \in \sigma(T)$ although $z \notin \sigma_{\mathrm{e}\ k}(T)$ for $1 \leq k \leq 4$; thus, $\sigma_{\mathrm{e}\ k}(T) = \partial \mathbb{D}_1$ for $1 \leq k \leq 4$.
There are two components of $\mathbb{C} \backslash \sigma_{\mathrm{e}\ 1}(T) : \{z \in \mathbb{C} : |z| > 1\}$ and $\{z \in \mathbb{C} : |z| < 1\}$.
The component $\{|z| < 1\}$ has no intersection with the resolvent set; by definition,

$$\sigma_{\mathrm{e}\ 5}(T) = \sigma_{\mathrm{e}1}(T) \cup \{z \in \mathbb{C} : |z| < 1\} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

2.2 Essential Spectra for the Sum of Closed and Bounded Linear Operators

Theorem 2.2.1. ([45, Theorem 3.3]). Let F and Δ be point sets in the plane. Let F be closed, Δ be open and $F \subset \Delta$. Suppose that the boundary $B(\Delta)$ of Δ is nonempty and bounded. Then, there exists a Cauchy domain D, such that:

(i) $F \subset D$,

(ii) $\bar{D} \subset \Delta$,

(iii) the curves forming $B(D)$ are polygons, and

(iv) D is unbounded if Δ is unbounded.

Theorem 2.2.2. ([44]). Let $A \in \mathcal{L}(X)$, and let $f(\lambda) = 1$. Then, $f^*(A) = I + K, K \in \mathcal{K}(X)$.

Lemma 2.2.1. ([44, Lemma 7.4]). Let $\mu_i \in \Phi_i^0(A)$ ($\Phi_i^0(A)$ being the set of all $\lambda \in \Phi_i(A)$ such that $\frac{-1}{\lambda - \lambda_i} \in \sigma(T_i)$ and T_i is defined in Eq. (2.2.6)). Let D be a bounded Cauchy domain with the following: $\bar{D} \subset \Phi_i(A), \mu_i \in D$, and no other points of $\Phi^0(A)$ are contained in $\bar{D}$. Then, $\frac{1}{2\pi i} \int_{+B(\mathcal{D})} R'_\lambda(A) d\lambda = K \in \mathcal{K}(X)$.

Lemma 2.2.2. Let $A \in \mathcal{C}(X), B \in \mathcal{L}(X), \lambda \in \Phi_A \backslash \Phi^0(A)$ and $\mu \in \Phi_B \backslash \Phi^0(B)$. If there exist a positive integer n and a Fredholm perturbation F_1, such that $B : \mathcal{D}(A^n) \longrightarrow \mathcal{D}(A)$ and $ABx = BAx + F_1 x$, for all $x \in \mathcal{D}(A^n)$, then there exists a Fredholm perturbation F depending analytically on λ and μ, such that $R'_\lambda(A) R'_\mu(B) = R'_\mu(B) R'_\lambda(A) + F$.

Theorem 2.2.3. [10] Let U and V be two bounded linear operators on a Banach space X. If $UV \in \mathcal{F}(X)$ then

$$\sigma_{ei}(U+V) \backslash \{0\} \subset [\sigma_{ei}(U) \cup \sigma_{ei}(V)] \backslash \{0\}, i = 4, 5.$$

If, further, $VU \in \mathcal{F}(X)$, then

$$\sigma_{e4}(U+V) \backslash \{0\} = [\sigma_{e4}(U) \cup \sigma_{e4}(V)] \backslash \{0\}.$$

Moreover, if the complement of $\sigma_{e4}(U)$ is connected, then

$$\sigma_{e5}(U+V) \backslash \{0\} = [\sigma_{e5}(U) \cup \sigma_{e5}(V)] \backslash \{0\}.$$

Theorem 2.2.4. [2] Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{L}(X)$. Suppose that there exist a positive integer n and $F \in \mathcal{F}(X)$, such that $B : \mathcal{D}(A^n) \longrightarrow \mathcal{D}(A)$ and $ABx = BAx + Fx$, for all $x \in \mathcal{D}(A^n)$. Then,

(i) $\sigma_{e4}(A+B) \subseteq \sigma_{e4}(A) + \sigma_{e4}(B)$. If $\sigma_{e4}(A)$ is empty, then $\sigma_{e4}(A) + \sigma_{e4}(B)$ is also an empty set.

(ii) If, in addition $\mathbb{C} \backslash \sigma_{e4}(A), \mathbb{C} \backslash \sigma_{e4}(B)$, and $\mathbb{C} \backslash \sigma_{e4}(A+B)$ are connected and if $\rho(A)$ and $\rho(A+B)$ are nonempty sets, then $\sigma_{e5}(A+B) \subseteq \sigma_{e5}(A) + \sigma_{e5}(B)$.

(iii) Moreover, if $\mathbb{C} \backslash \sigma_{e5}(A), \mathbb{C} \backslash \sigma_{e5}(B)$, and $\mathbb{C} \backslash \sigma_{e5}(A+B)$ are connected and if $\rho(A)$ and $\rho(A+B)$ are nonempty sets, then $\sigma_{e6}(A+B) \subseteq \sigma_{e6}(A) + \sigma_{e6}(B)$.

Proof. (i) First, it is clear that the theorem is trivially true if we suppose that $\sigma_{e4}(A) + \sigma_{e4}(B)$ constitutes the entire complex plane. Hence, let us assume

that $\sigma_{e4}(A) + \sigma_{e4}(B)$ is not the entire plane. Second, we fix a point γ such that $\gamma \notin \sigma_{e4}(A) + \sigma_{e4}(B)$ and we define the operator A_1 as $A_1 := \gamma - A$. Hence, it is easy to verify that if $\lambda \in \sigma_{e4}(B)$, the element $\gamma - \lambda$ will be in Φ_A which is equivalent to say that $\lambda \in \Phi_{A_1}$. In the following, we will find a Cauchy domain $\mathcal{D}$ such that $R'_\lambda(A_1)$ and $R'_\lambda(B)$ are analytic on $B(\mathcal{D})$, the boundary of $\mathcal{D}$ where $R'_\lambda(.)$ is defined in Hence, T_i is a quasi-inverse of $(\lambda_i - A)$. Moreover, when $\lambda \in \Phi_i(A)$ and $\frac{-1}{\lambda - \lambda_i} \in \rho(T_i)$, the operator

$$R'_\lambda(A) := T_i \left[(\lambda - \lambda_i) T_i + I \right]^{-1}.$$

In fact,$\sigma_{e4}(A)$ is closed and $\sigma_{e4}(B)$ is compact. Then, there exists an open set $U \supset \sigma_{e4}(B)$ such that $B(U)$, the boundary of U, is bounded and when $\lambda \in U, (\gamma - \lambda) \in \Phi_A$.
Therefore, $\sigma_{e4}(B) \subset U \subset \Phi_{A_1}$. Using Theorem 2.2.1, we infer that there exists a bounded Cauchy domain $\mathcal{D}$, such that $\sigma_{e4}(B) \subset \mathcal{D} \subset U$.
Note that $\Phi^0(A_1)$ [resp. $\Phi^0(B)$] does not accumulate in Φ_{A_1} [resp. Φ_B]. So, we can choose $\mathcal{D}$ such that $R'_\lambda(A_1)$ and $R'_\lambda(B)$ are analytic on $B(\mathcal{D})$. We also claim that $R'_\lambda(A_1)$ is of the form $TC(\lambda)$ where $C(\lambda)$ is a bounded operator-valued

analytic function of λ and T is a fixed bounded operator, such that $T : X \longrightarrow \mathcal{D}(A_1) = \mathcal{D}(A)$.
Now, let us define the following operators M_1 and M_2 as follows

$$M_1 = -\frac{1}{2\pi i}\int_{+B(\mathcal{D})} R'_\lambda(A_1)\, R'_\lambda(B)d\lambda$$

and

$$M_2 = -\frac{1}{2\pi i}\int_{+B(\mathcal{D})} R'_\lambda(B)R'_\lambda(A_1)\, d\lambda.$$

In order to prove this assertion, we will show that $\gamma \in \Phi_{A+B}$. Hence, it is sufficient to find two Fredholm perturbations F_1 and F_2, such that

$$(\gamma - B - A)M_1 = I + F_1$$

and

$$M_2(\gamma - B - A) = I + F_2 \text{ on } \mathcal{D}(A).$$

Now, writing the operator $\gamma - B - A$ as follows

$$(\gamma - B - A) = (\gamma - \lambda - A) + (\lambda - B) = -(\lambda - A_1) + (\lambda - B),$$

we get

$$\begin{aligned}(\gamma - B - A)M_1 = &- \frac{1}{2\pi i}\int_{+B(\mathcal{D})} -(\lambda - A_1)\, R'_\lambda(A_1)\, R'_\lambda(B)d\lambda \\ &- \frac{1}{2\pi i}\int_{+B(\mathcal{D})} -(\lambda - B)R'_\lambda(A_1)\, R'_\lambda(B)d\lambda.\end{aligned} \tag{2.7}$$

Obviously, $(\lambda - A_1)\, R'_\lambda(A_1) = I + \mathfrak{F}$, where $\mathfrak{F}$ is a bounded finite rank operator depending analytically on λ. Then, the first integral of the above equality is of the following form $-\frac{1}{2\pi i}\int_{+B(\mathcal{D})} -(I+\mathfrak{F})R'_\lambda(B)d\lambda$. Using Theorem 2.2.2, we deduce that

$$\frac{1}{2\pi i}\int_{+B(\mathcal{D})} R'_\lambda(B)d\lambda = I + K_1, \text{ where } K_1 \in \mathcal{K}(X).$$

Moreover, we also mention that

$$\int_{+B(\mathcal{D})} -(I + \mathfrak{F})R'_\lambda(B)d\lambda$$

is a compact operator. Hence, we infer that the first integral of (2.7) is of the form $I + K_2$, where $K_2 \in \mathcal{K}(X)$. Applying Lemma 2.2.1, we get $R'_\lambda(A_1)\, R'_\lambda(B) = R'_\lambda(B)R'_\lambda(A_1) + F$, where F is a Fredholm perturbation. Then, the second integral of the same equality is equal to

$$-\frac{1}{2\pi i}\int_{+B(\mathcal{D})} -(\lambda - B)R'_\lambda(B)R'_\lambda(A_1)\, d\lambda - \frac{1}{2\pi i}\int_{+B(\mathcal{D})} (\lambda - B)Fd\lambda.$$

Since $\int_{+B(\mathcal{D})} R'_\lambda(A_1)\, d\lambda$ is compact (see Lemma 2.2.1), then a same reasoning as in the first part allows us to write $-\frac{1}{2\pi i}\int_{+B(\mathcal{D})} -(\lambda - B)R'_\lambda(B)R'_\lambda(A_1)\, d\lambda = I + K_3$, where $K_3 \in \mathcal{K}(X)$. Using the fact that $\frac{1}{2\pi i}\int_{+B(\mathcal{D})}(\lambda - B)Fd\lambda$ is also a Fredholm perturbation, we have $(\gamma - B - A)M_1 = I + F_1, F_1 \in \mathcal{F}(X)$. By a similar argument, we obtain $M_2(\gamma - B - A) = I + F_2$, where $F_2 \in \mathcal{F}(X)$. Therefore, $(\gamma - B - A) \in \Phi(X)$, and we deduce that $\sigma_{e4}(A + B) \subseteq \sigma_{e4}(A) + \sigma_{e4}(B)$.

(ii) This assertion follows immediately from Theorem 2.2.3 (i).

(iii) The proof of this assertion holds from Theorem 2.2.3 (ii).

□

2.3 Essential Spectra for the Product of Closed and Bounded Linear Operators

Theorem 2.3.1. ([44, Theorem 9]). Let $f(\lambda)$ and $g(\lambda)$ be in $\mathcal{R}'_\infty(A)$. Then $f(A).g(A) = (f.g)(A) + K, K \in \mathcal{K}(X)$.

Theorem 2.3.2. ([44, Theorem 14]). Let $A \in \mathcal{L}(X)$, and let $f(\lambda) = \lambda$. Then, $f^*(A) = A + K, K \in \mathcal{K}(X)$.

Theorem 2.3.3. Let $A \in \mathcal{C}(X)$ and $B \in \Phi^b(X)$. Let $B : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ and suppose that there exists $F \in \mathcal{F}(X)$ such that $ABx = BAx + Fx$, for all $x \in \mathcal{D}(A)$. Then, BA is closable and

1. $\sigma_{e4}(\overline{BA}) \subseteq \sigma_{e4}(A)\sigma_{e4}(B)$ and $\sigma_{e4}(AB) \subseteq \sigma_{e4}(A)\sigma_{e4}(B)$.
2. If, in addition, $C\backslash\sigma_{e4}(\overline{BA}), \mathbb{C}\backslash\sigma_{e4}(AB), \mathbb{C}\backslash\sigma_{e4}(A), \mathbb{C}\backslash\sigma_{e4}(B)$ are connected, and if $\rho(A), \rho(\overline{BA})$, and $\rho(AB)$ are nonempty sets, then $\sigma_{e5}(\overline{BA}) \subseteq \sigma_{es}(A)\sigma_{es}(B)$ and $\sigma_{es}(AB) \subseteq \sigma_{e5}(A)\sigma_{es}(B)$.
3. Moreover, if $C\backslash\sigma_{es}(BA), \mathbb{C}\backslash\sigma_{e4}(AB), \mathbb{C}\backslash\sigma_{es}(A), \mathbb{C}\backslash\sigma_{es}(B)$ are connected, and if $\rho(A), \rho(\overline{BA})$, and $\rho(AB)$ are nonempty sets, then $\sigma_{e6}(\overline{BA}) \subseteq \sigma_{e6}(A)\sigma_{e6}(B)$ and $\sigma_{e6}(AB) \subseteq \sigma_{e6}(A)\sigma_{e6}(B)$.

Proof. 1. Since the operator F is bounded and the restriction of the operator AB on $\mathcal{D}(A)$ is closable, then BA is closable. Furthermore, it is clear that $0 \notin \sigma_{e4}(B)$ and $\sigma_{e4}(B)$ is not empty. So, the theorem is trivially true if $\sigma_{e4}(A) = \mathbb{C}$, and we will assume, in the following, that $\sigma_{e4}(A) \neq \mathbb{C}$. Now, let γ be a fixed point not in $\sigma_{e4}(B)\sigma_{e4}(A)$. In what follows, we will show that $\gamma \in \Phi_{BA}$. Observing that $\sigma_{e4}(A)$ is closed, $\sigma_{e4}(B)$ is compact and $0 \notin \sigma_{e4}(B)$, we infer that there exists an open set U, with a bounded boundary $B(U)$, containing $\sigma_{E4}(B)$ and satisfying that $0 \notin U$ and $(\gamma - \mu A) \in \Phi(X), \forall \mu \in U$. Let $\mathcal{D}$ be a bounded Cauchy domain, such that $\sigma_{e4}(B) \subset \mathcal{D} \subseteq U$. Writing $(\gamma - \mu A)$ as follows:

$$(\gamma - \mu A) = \mu\gamma\left(\frac{1}{\mu} - \frac{1}{\gamma}A\right) = \frac{\gamma}{\lambda}\left(\lambda - \frac{1}{\gamma}A\right), \lambda = \frac{1}{\mu}$$

and taking $\mathcal{D}'$ as the image of $\mathcal{D}$ under the map $\lambda = \frac{1}{\mu}$, we can assume that $R'_\lambda(A_1)$ is analytic in λ on $B(\mathcal{D}')$, where $A_1 := \frac{1}{\gamma}A$. This assumption holds true, thanks to the fact that $\forall\mu \in \overline{\mathcal{D}}, \frac{1}{\mu} \in \Phi_{A_1}$ and that the operator $R'_\lambda(A_1)$ is analytic in λ throughout Φ_{A_1} except for, at most, an isolated set having no accumulation in Φ_{A_1}. Let us define the following operators M_1 and M_2 as follows: $M_1 = -\frac{1}{2\pi i}\int_{+B(D)} \frac{1}{\gamma\lambda} R'_\lambda(A_1) R'_\perp(B) d\lambda$ and $M_2 = -\frac{1}{2\pi i}\int_{+B(D)} \frac{1}{p} R'_\perp(B) R'_\lambda(A_1) d\lambda$.

Since $R(M_1) \subset \mathcal{D}(A)$, the operator $(\gamma - \overline{BA})M_1$ is well defined, and we have $(\gamma - \overline{BA})M_1 = (\gamma - BA)M_1$. Moreover,

$$\begin{aligned}(\gamma - BA) &= (\gamma - B\gamma A_1)\\ &= \gamma B(\lambda - A_1) - \gamma\lambda B + \gamma I\\ &= \gamma B(\lambda - A_1) + \gamma(I - \lambda B).\end{aligned}$$

Then,

$$\begin{aligned}(\gamma - \overline{BA})M_1 = -\frac{1}{2\pi i}\int_{+B(D)} &\left(\frac{1}{\lambda}B(\lambda - A_1)R'_\lambda(A_1)R'_\perp(B)\right.\\ &\left.+\left(\frac{1}{\lambda} - B\right)R_\lambda(A_1)R'_\perp(B)\right)d\lambda\end{aligned}$$

On the one hand, the first part of the integrand can be written as follows:

$$\begin{aligned}\int_{+B(D')}\frac{1}{\lambda}B(\lambda - A_1)R'_\lambda(A_1)R'_{\frac{1}{\lambda}}(B)d\lambda &= \int_{+B(D')}\frac{1}{\lambda}B(I + K_1(\lambda))R'_{\frac{1}{\lambda}}(B)d\lambda\\ &= \int_{+B(\mathcal{D})}\frac{1}{\lambda}BR'_{\pm}(B)d\lambda + K_2\\ &= \int_{+B(\mathcal{D})}\frac{1}{\mu}BR'_\mu(B)d\mu + K_2\end{aligned}$$

where $K_l \in K(X), i = 1, 2$. On the other hand, since $0 \notin D$ hence, using Theorems 2.3.1,2.2.2, and 2.3.1 , we get $\frac{1}{2\pi i}\int_{+R(D)}\frac{1}{\mu}BR'_\mu(B)d\mu = I + K_3$, where $K_3 \in \mathcal{K}(X)$. Note that the second part of the integrand can also be written as:

$$\begin{aligned}&\int_{+B(\mathcal{D})}\left(\frac{1}{\lambda} - B\right)R'_\lambda(A_1)R'(B)d\lambda\\ &= \int_{+B(D)}\left(\frac{1}{\lambda} - B\right)\left[R'_\perp(B)R'_\lambda(A_1) + F_1\right]d\lambda\\ &= \int_{+B(\mathcal{D})}\left[I + K_4(\lambda)\right]R'_\lambda(A_1)d\lambda + F_2\end{aligned}$$

$$= \int_{+E(D)} R'_\lambda(A_1)d\lambda + F_3.$$

where $K_4 \in K(X)$ and $F_i \in \mathcal{F}(X)$, with $i = 1, 2, 3$. We claim that $R'_\lambda(A_1)$ is analytic in $\mathcal{D}'$ except for, at most, a finite number of points. Then, by using Lemma 2.2.1, we deduce that $\frac{1}{2\pi l}\int_{+B(D)}R'_\lambda(A_1)d\lambda = K_5 \in \mathcal{K}(X)$.

Therefore, $(\gamma - \overline{BA})M_1 = I + F_4$, where $F_4 \in \mathcal{F}(X)$. Now, we can easily check that $\mathcal{D}(\overline{BA}) \subseteq \mathcal{D}(AB)$ and $\overline{BA}x = ABx + Fx\forall x \in \mathcal{D}(\overline{BA})$. Hence,

$$\begin{aligned}(\gamma - \overline{BA}) &= \gamma\overline{B(\lambda - A_1)} + \gamma(I - \lambda B)\\ &= \gamma(\lambda - A_1)B + \gamma(I - \lambda B) + F_5.\end{aligned}$$

where $F_5 \in \mathcal{F}(X)$. Then,

$$
\begin{aligned}
& M_2(\gamma - \overline{BA}) \\
& = \frac{1}{2\pi i} \int_{+B(\mathcal{D})} \frac{1}{\gamma\lambda} R'_{\perp}(B) R'_{\lambda}(A_1) \left[\gamma(\lambda - A_1) B + \gamma(I - \lambda B) + F_5\right] d\lambda \\
& = -\frac{1}{2\pi i} \int_{+B(\mathcal{D})} \frac{1}{\lambda} R'_{\frac{1}{\lambda}}(B)(I + K_6) B d\lambda - \\
& \frac{1}{2\pi i} \int_{+B(\mathcal{D})} \frac{1}{\gamma\lambda} \left(R'_{\lambda}(A_1) R'_{\perp}(B) - F_1\right)\left(\gamma(I - \lambda B) + F_5\right) d\lambda \\
& = \left[\frac{1}{2\pi i} \int_{+B(\mathcal{D})} \frac{1}{\mu} R'_{\mu}(B) d\mu\right] B + K_7 - \\
& \frac{1}{2\pi i} \int_{+B(\mathcal{D})} R'_{\lambda}(A_1) R'_{\perp}(B) \left(\frac{1}{\lambda} - B\right) d\lambda + F_6 \\
& \qquad = I + F_7 - \frac{1}{2\pi i} \int_{+B(\mathcal{D}')} R'_{\lambda}(A_1)(I + K_8) d\lambda \\
& \qquad = I + F_8,
\end{aligned}
$$

where $K_i \in \mathcal{K}(X)$, with $i = 6, 7$ and $F_i \in \mathcal{F}(X)$, with $i = 6,\ 7,\ 8$. Therefore, we conclude that $(\gamma - \overline{BA}) \in \Phi(X)$, and the proof of the first inclusion is completed. Now, in order to show that $\sigma_{e4}(AB) \subseteq \sigma_{e4}(B)\sigma_{e4}(A)$, we will prove that $(\gamma - AB) \in \Phi(X)$. Since $R(M_1) \subset \mathcal{D}(A)$ and $ABx = BAx - Fx$ for all $x \in \mathcal{D}(A)$, we obtain $(\gamma - AB)M_1 = (\gamma - BA + F)M_1 = (\gamma -$ BA$)\ M_1 + FM_1 = I + F_4 + F_9 = I + F_{10}$, where $F_i \in \mathcal{F}(X)$, with $i = 9, 10$. Furthermore, we have $M_2(\gamma - \overline{AB}) = M_2\left[\gamma(\lambda - A_1) B + \gamma(I - \lambda B)\right] = I + F_{11}$, where $F_{11} \in \mathcal{F}(X)$. Hence, $(\gamma - \overline{AB}) \in \Phi(X)$ and, we deduce that $\sigma_{e4}(AB) \subseteq \sigma_{e4}(A)\sigma_{e4}(B)$.

2. The proof of this assertion holds from Theorem 2.2.3 (i).

3. This assertion follows immediately from Theorem 2.2.3 (ii).

$\square$

Chapter 3

Applications

3.1 Application to a transport operator

The purpose of this section is to apply Theorems 2.1.8 and 2.2.3 to study the essential spectra of the following onespeed neutron transport operator $A_0 = T_0 + K$ with vacuum boundary conditions in slab geometry (cf. [8] [9] , [11] [28] .)

We define the advection operator T_0 by

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset X_1 \longrightarrow X_1 \\ \qquad\qquad \psi \longrightarrow T_0\psi(x,\xi) = -\xi\frac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi), \\ \mathcal{D}(T_0) = \{\psi \in W_P \text{ such that } \psi(a,-\xi) = \psi(-a,\xi) = 0, \xi \in (0,1)\} \end{cases}$$

where $X_1 = L_1[(-a,a) \times (-1,1);$ dxdk$]$ $(0 < a < \infty), x \in [-a,a]$ for a parameter, $\xi \in [-1,1]$, $\sigma(.) \in L^\infty(-1,1)$ and $W_p = \left\{\psi \in X_1 \text{ such that } \xi\frac{\partial_v}{\sigma x} \in X_1\right\}$ is the partial Sobolev space. Next, we consider the collision operator K defined by

$$\begin{cases} K : X_1 \longrightarrow X_1 \\ \qquad \psi \longrightarrow \displaystyle\int_{-1}^{1} \kappa(x,\xi,\xi')\,\psi(x,\xi')\,d\xi', \end{cases}$$

where $\kappa(.,\ldots)$ is a measurable function from $[-a,a] \times [-1,1] \times [-1,1]$ to $\mathbf{R}$. Observe that the operator K acts only on the variable ξ, so x may be viewed merely as a parameter in $[-a,a]$. Hence we may consider K as a function $K : x \in [-a,a] \longrightarrow K(x) \in \mathcal{Z}$ where $\mathcal{Z} := \mathcal{L}(L_1([-1,1],d\xi))$.
In the following we will make the assumptions:

$$(H1)\begin{cases} K \text{ is a measurable, i.e.,} \\ \{x \in [-a,a] \text{ such that } K(x) \in \mathcal{O}\} \text{ is measurable if } \mathcal{O} \subset \mathcal{Z} \text{ is open,} \\ \text{there exists a compact subset } T \subset \mathcal{Z} \text{ such that } K(x) \in T \text{ a.e.,} \\ \text{and finally } K(x) \in \mathcal{K}(L_1([-1,1],d\xi)) \text{ a.e.,} \end{cases}$$

where $\mathcal{K}(L_1([-1,1],d\xi))$ denotes the set of all compact operators on $L_1([-1,1],d\xi)$.

Definition 3.1.1. A collision operator K is said to be regular if it satisfies the assumptions (Hl).

Proposition 3.1.1. ([28]) If the collision operator K is regular. Then, for any $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > -\lambda^*$, the operator $(\lambda - T_0)^{-1} K$ is weakly compact on X_1.

Note that, the essential spectra of the operator T_0 was analyzed in detail in ([23]). In particular we have

$$\sigma_{ei}\left(T_0\right)=\sigma\left(T_0\right)=\sigma C\left(T_0\right)=\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda^*\}, \quad \text{for} \quad i=1,2,\ldots,8,$$

where $\sigma C\left(T_0\right)$ is the continuous spectrum of T_0 and $\lambda^* := \liminf_{|\xi|\to 0} \sigma(\xi)$, (for more detail see [28]).

In the following we impose the condition

$$(H2) \quad \lambda^* > \|K\|. \tag{3.1}$$

Let λ the leading eigenvalue of A_0. In [46], the strip $-\lambda^* < \bar{\lambda} \leq -\lambda^* + \|K\|$ contains at most isolated points of $\sigma\left(A_0\right)$. Applying Eq. (3.15) and the condition (H2), we have

$$0 \in \rho\left(T_0\right) \cap \rho\left(A_0\right). \tag{3.2}$$

Using Theorem 2.1.8 , we have the following lemma

Lemma 3.1.1. [11] If the condition (H2) is satisfied. Then

$$\sigma_{ei}\left(T_0^{-1}\right)=\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\frac{1}{\lambda} \leq -\lambda^*\right\}, \text{ for } \quad i=1,2,\ldots,8.$$

Lemma 3.1.2. Let U is bounded weakly compact operator on X_1. If the condition (H2) is satisfied. Then

$$\sigma_{\varepsilon i}\left(T_0^{-1}+U\right)\backslash\{0\}=\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\frac{1}{\lambda} \leq -\lambda^*\right\}, \text{ for } i=1,2,\ldots,8.$$

Proof. Since U is bounded weakly compact operator on X_1, so U is a Riesz operator on X_1, therefore

$$\sigma_{ei}(U)=\{0\} \text{ for } i=1,2,\ldots,8.$$

Then $T_0^{-1}U$ and UT_0^{-1} are weakly compacts on X_1.
Applying Theorem 2.2.3 and Lemma 3.1.1, we obtain

$$\begin{aligned}
\sigma_{ei}\left(T_0^{-1}+U\right)\backslash\{0\} &= \left[\sigma_{ei}\left(T_0^{-1}\right) \cup \sigma_{ei}(U)\right]\backslash\{0\} \\
&= \left[\sigma_{ei}\left(T_0^{-1}\right) \cup \{0\}\right]\backslash\{0\} \\
&= \sigma_{ei}\left(T_0^{-1}\right)\backslash\{0\} \\
&= \left\{\lambda \in \mathbb{C} \text{ such that } Re\frac{1}{\lambda} \leq -\lambda^*\right\}, \quad \text{for} \quad i=1,2,4,\ldots,8.
\end{aligned}$$

In the same way, using Theorem 2.1.8 (v) and Lemma 3.1.1 we have

$$\begin{aligned}
\sigma_{e3}\left(T_0^{-1}+U\right)\backslash\{0\} &= \left[(\sigma_{e3}(U)\cup\sigma_{e3}(V))\cup(\sigma_{e1}(U)\cap\sigma_{e2}(V))\cup(\sigma_{e2}(U)\cap\sigma_{e1}(V))\right]\backslash\{0\} \\
&= \left[\left(\sigma_{e3}\left(T_0^{-1}\right)\cup\{0\}\right)\cup\left(\sigma_{e1}\left(T_0^{-1}\right)\cap\{0\}\right)\cup\left(\sigma_{e2}\left(T_0^{-1}\right)\cap\{0\}\right)\right]\backslash\{0\} \\
&= \sigma_{e3}\left(T_0^{-1}\right) \\
&= \left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\frac{1}{\lambda} \leq -\lambda^*\right\}.
\end{aligned}$$

Consequently, we get

$$\sigma_{ei}\left(T_0^{-1}+U\right)\backslash\{0\}=\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\frac{1}{\lambda} \leq -\lambda^*\right\}, \quad \text{for} \quad i=1,2,\ldots,8.$$

Now, we give the following theorem. $\square$

Theorem 3.1.1. [11] If the condition (H2) is satisfied and K is regular on X_1. Then

$$\sigma_{ei}\left(A_0\right)=\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq-\lambda^*\right\}, \quad \text{for} \quad i=1,2, \ldots, 8.$$

Proof. Since $\lambda^*>\|K\|$. By [23], we can write $\left\|T_0^{-1} K\right\| \leq \frac{\|K\|}{\lambda^*}$, hence $\left\|T_0^{-1} K\right\|<1$. Therefore, we have

$$A_0^{-1}=T_0^{-1}+U$$

where

$$U=\sum_{n \geq 1}\left[T_0^{-1} K\right]^n T_0^{-1}$$

We have U is weakly compact on X_1. Therefore, using Lemma 3.1.2 , we obtain

$$\begin{aligned}
\sigma_{ei}\left(A_0^{-1}\right) \backslash\{0\} &=\sigma_{ei}\left(T_0^{-1}+U\right) \backslash\{0\} \\
&=\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \frac{1}{\lambda} \leq-\lambda^*\right\}, \text{ for } i=1,2, \ldots, 8.
\end{aligned}$$

Consequently, applying Theorem 2.1.8 and Eq. (3.2) we obtain

$$\sigma_{ei}\left(A_0\right)=\left\{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq-\lambda^*\right\}, \quad \text{for} \quad i=1,2, \ldots, 8.$$

□

3.2 Application to two-group transport equation

The goal of this part is to establish the results concerning the essential spectra of the one-speed neutron transport operator represented in [5] in slab geometry on L_1 -spaces.

Let

$$X:=L_1(D ; d x d v)$$

where $D=(0,1) \times K$ with K is the unit sphere of $\mathbf{R}^3, x \in(0,1)$ and $v=\left(v_1, v_2, v_3\right) \in K$. Define the following sets representing the incoming D^i and the outgoing D^0 boundary of the phase space D :

$$\begin{aligned}
& D^t=D_1^i \cup D_2^i=\{0\} \times K^1 \cup\{1\} \times K^0, \\
& D^0=D_1^0 \cup D_2^0=\{0\} \times K^0 \cup\{1\} \times K^1,
\end{aligned}$$

for

$$K^0=K \cap\left\{v_3<0\right\} \text{ and } K^1=K \cap\left\{v_3>0\right\}$$

Moreover, we shall denote the boundary spaces X^i and X^0 by the following ways:

$$\begin{aligned}
X^i & :=L_1\left(D^i,\left|v_3\right| d v\right) \\
& :=L_1\left(D_1^i,\left|v_3\right| d v\right) \oplus L_1\left(D_2^i,\left|v_3\right| d v\right) \\
& :=X_1^i \oplus X_2^i
\end{aligned}$$

endowed with the norm

$$\begin{aligned}
\left\|\psi^i, X^i\right\| & =\left\|\psi_1^i, X_1^i\right\|+\left\|\psi_2^i, X_2^i\right\| \\
& =\int_{K^1}\left|\psi(0, v)\right|\left|v_3\right| d v+\int_{K^0}\left|\psi(1, v)\right|\left|v_3\right| d v,
\end{aligned}$$

and

$$\begin{aligned} X^0 &:= L_1\left(D^0, |v_3|\,dv\right) \\ &:= L_1\left(D_1^0, |v_3|\,dv\right) \oplus L_1\left(D_2^0, |v_3|\,dv\right) \\ &:= X_1^0 \oplus X_2^0 \end{aligned}$$

equipped with the norm

$$\begin{aligned} \left\|\psi^0, X^0\right\| &= \left\|\psi_1^0, X_1^0\right\| + \left\|\psi_2^0, X_2^0\right\| \\ &= \int_{K^0} \left|\psi(0,v)\right\| v_3 \Big| dv + \int_{K^1} \left| \psi(1,v) \right\| v_3 \Big|\, dv. \end{aligned}$$

Denote by $\mathcal{W}$ the space defined by

$$\mathcal{W} = \left\{ \psi \in X \text{ such that } v_3 \frac{\partial \psi}{\partial x} \in X \right\}.$$

It is well-known that any function $\psi \in \mathcal{W}$ possesses traces (see [6]) on the spatial boundary denoted by $\psi^0 = (\psi_1^0, \psi_2^0)^T$, and $\psi^i = (\psi_1^i, \psi_2^i)^T$ given by

$$\begin{cases} \psi_1^i(v) = \psi(0,v), & v \in K^1, \\ \psi_2^i(v) = \psi(1,v), & v \in K^0, \\ \psi_1^0(v) = \psi(0,v), & v \in K^0, \\ \psi_2^0(v) = \psi(1,v), & v \in K^1. \end{cases}$$

We consider the following two-group transport operators

$$\mathcal{A} = \mathcal{T} + \mathcal{K},$$

which

$$\begin{aligned} \mathcal{T}_\psi &:= \begin{pmatrix} -v_3 \frac{\partial \psi_1}{\partial x}(x,v) - \sigma_1(x,v)\psi_1(x,v) & 0 \\ 0 & -v_2 \frac{\partial \psi_2}{\partial x}(x,v) - \sigma_2(x,v)\psi_2(x,v) \end{pmatrix} \\ &:= \begin{pmatrix} T_1 & 0 \\ 0 & T_2^H \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{aligned}$$

and

$$\mathcal{K} = \begin{pmatrix} 0 & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

where $x \in [0,1], v = (v_1, v_2, v_3) \in K$ and $K_{ij}, (i,j) \in \{(1,2),(2,1),(2,2)\}$ are bounded linear operators on X by

$$\begin{cases} K_{\text{ij}} : X \quad X \\ \qquad \psi \quad \longrightarrow K_{\text{ij}}\psi(x,v) = \int_K \kappa_{ij}\left(x,v,v'\right)\psi\left(x,v'\right)dv', \end{cases} \tag{3.3}$$

and the kernels $\kappa_{ij} : (0,1) \times K \times K \longrightarrow \mathbf{R}$ are assumed to be measurable. Let us introduce the boundary operator H,

$$\begin{cases} H : X^0 \longrightarrow X^i \\ H \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{cases}$$

with for $j,k \in \{1,2\}, H_{jk} : X_k^0 \longrightarrow X_j^i, H_{jk} \in \mathcal{L}\left(X_k^0, X_j^i\right)$, defined such that, on natural identification, the boundary conditions can be written as $\psi^H = H\left(\psi^0\right)$. The operators T_1 and T_2^H are defined by

$$\begin{cases} T_1 : \mathcal{D}\left(T_1\right) \subseteq X \longrightarrow X \\ \mathcal{D}\left(T_1\right) = \mathcal{W} \end{cases}$$

and

$$\begin{cases} T_2^H : \mathcal{D}\left(T_2^H\right) \subseteq X \longrightarrow X \\ \psi \longrightarrow T_2^H \psi(x,v) = -v_3 \dfrac{\partial \psi}{\partial x}(x,v) - \sigma_2(x,v)\psi(x,v) \\ \mathcal{D}\left(T_2^H\right) = \left\{\psi \in \mathcal{W} \text{ such that } \psi_{|D^i} := \psi^i \in X^i, \psi_{|D^v} := \psi^0 \in X^0 \text{ and } \psi^i = H\left(\psi^0\right)\right\}, \end{cases}$$

where $\sigma_j(\ldots),, j = 1,2$, is a positive bounded function and the boundary operator H. It is clear that the operator $\mathcal{T}$ is defined on $\mathcal{W} \times \mathcal{D}\left(T_2^H\right)$.
Next, we will define $\mathcal{A}$ by

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{W} \times \mathcal{D}\left(T_2^H\right) \text{ such that } \psi_1^i = \psi_2^i \right\}$$

and

$$\mathcal{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$\begin{cases} A = T_1, \\ B = K_{12}, \\ C = K_{21}, \\ D = T_2^H + K_{22}, \end{cases}$$

Now, it is easy to check that this operator is of the matrix form of Section 3 , with $Z := Z_1 := X^i, X = Y := L_1(D, dxdv)$

$$\begin{cases} \Gamma_X : \mathcal{W} \longrightarrow X^i \\ \varphi \longrightarrow \Gamma_X \varphi = \varphi^i, \\ \text{with } \mathcal{D}\left(\Gamma_X\right) = \mathcal{D}\left(T_1\right) \end{cases}$$

and

$$\begin{cases} \Gamma_Y : X \longrightarrow X^i \\ \qquad \psi \longrightarrow \Gamma_Y \psi = H\psi^0 \end{cases}$$

We define the operator A_1 by

$$\begin{cases} A_1\psi(x,v) := T_1\psi(x,v) := -v_3 \frac{\partial \psi}{\partial x}(x,v) - \sigma_1(x,v)\psi(x,v), \\ \mathcal{D}\left(A_1\right) = \left\{\psi \in \mathcal{D}\left(T_1\right) \text{ such that } \psi^i = 0\right\}. \end{cases}$$

Remark 3.2.1. It is well-known that the operators T_1 and T_2^H are closed linear operators. Moreover, the derivative of ψ in the definition of T_1 and T_2^H is meant in distributional sense. Note that if $\psi \in \mathcal{D}\left(T_1\right)$ (resp. $\psi \in \mathcal{D}\left(T_2^H\right)$) then it is absolutely continuous with respect to x. Hence the restriction of ψ to D^i and D^0 are meaningful. Note also that $\mathcal{D}\left(T_1\right)$ (resp. $\mathcal{D}\left(T_2^H\right)$) is dense in X because it contains $C_0^\infty\left(D^0\right)$. Now, we will determine the expression of the resolvent of the operator T_2^H. Let $\varphi \in X, \lambda \in \mathbf{C}$ and consider the problem

$$\left(\lambda - T_2^H\right)\psi = \varphi \tag{3.4}$$

where the unknown ψ must be sought in $\mathcal{D}\left(T_2^H\right)$. Let $\lambda_j^*, j=1,2$ denote the real number defined by

$$\lambda_j^* := \operatorname{ess}-\inf\left\{\sigma_j(x,v), \quad (x,v)\in D\right\}, \quad j=1,2$$

and

$$\lambda_0 := \begin{cases} -\lambda_2^* & \text{if } \|H\|\leq 1, \\ -\lambda_2^* + \log(\|H\|) & \text{if } \quad \|H\|>1. \end{cases}$$

Thus, for $\operatorname{Re}\lambda > -\lambda_2^*$, the solution of (3.4) is formally given by

$$\begin{aligned} &\psi(x,v) \\ &= \begin{cases} \psi(0,v)e^{-\int_0^x \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds} + \frac{1}{|v_3|}\int_0^x e^{-\int_{x'}^x \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi\left(x',v\right)dx', & v\in K^1, \\ \psi(1,v)e^{-\int_x^1 \frac{\sigma_2(s,v)+\lambda}{|v|}ds} + \frac{1}{|v_3|}\int_x^1 e^{-\int_x^{x'} \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi\left(x',v\right)dx', & v\in K^0, \end{cases} \end{aligned} \tag{3.5}$$

whereas $\psi(1,v)$ and $\psi(0,v)$ are given by

$$\psi(1,v) = \psi(0,v)e^{-\int_0^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds} + \frac{1}{|v_3|}\int_0^1 e^{-\int_{x'}^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi\left(x',v\right)dx', \quad v\in K^1, \tag{3.6}$$

$$\psi(0,v) = \psi(1,v)e^{-\int_0^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds} + \frac{1}{|v_3|}\int_0^1 e^{-\int_0^{x'} \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi\left(x',v\right)dx', \quad v\in K^0. \tag{3.7}$$

For the clarity of our subsequent analysis, we introduce the following bounded operators depending on the parameter λ,

$$\begin{cases} N_\lambda : X^i \longrightarrow X^0, N_\lambda u := \left(N_\lambda^+ u, N_\lambda^- u\right) \quad \text{with} \\ \left(N_\lambda^+ u\right)(0,v) := u(1,v)e^{-\int_0^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}, \quad v\in K^0; \\ \left(N_\lambda^- u\right)(1,v) := u(0,v)e^{-\int_0^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}, \quad v\in K^1; \end{cases}$$

$$\begin{cases} B_\lambda : X^i \longrightarrow X, B_\lambda u := \chi_{K^0}(v)B_\lambda^+ u + \chi_{K^1}(v)B_\lambda^- u \quad \text{with} \\ \left(B_\lambda^- u\right)(x,v) := u(0,v)e^{-\int_0^x \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}, \quad v\in K^1; \\ \left(B_\lambda^+ u\right)(x,v) := u(1,v)e^{-\int_x^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}, \quad v\in K^0; \end{cases}$$

$$\begin{cases} G_\lambda : X \longrightarrow X^0, G_\lambda\varphi := \left(G_\lambda^+\varphi, G_\lambda^-\varphi\right) \quad \text{with} \\ G_\lambda^-\varphi := \frac{1}{|v_3|}\int_0^1 e^{-\int_x^1 \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi(x,v)dx, \quad v\in K^1; \\ G_\lambda^+\varphi := \frac{1}{|v_3|}\int_0^1 e^{-\int_0^x \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi(x,v)dx, \quad v\in K^0; \end{cases}$$

and

$$\begin{cases} F_\lambda : X \longrightarrow X, F_\lambda\varphi := \chi_{K^0}(v)F_\lambda^+\varphi + \chi_{K^1}(v)F_\lambda^-\varphi \quad \text{with} \\ F_\lambda^-\varphi := \frac{1}{|v_3|}\int_0^x e^{-\int_{x'}^x \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi\left(x',v\right)dx', \quad v\in K^1; \\ F_\lambda^+\varphi := \frac{1}{|v_3|}\int_x^1 e^{-\int_x^{x'} \frac{\sigma_2(s,v)+\lambda}{|v_3|}ds}\varphi\left(x',v\right)dx', \quad v\in K^0, \end{cases}$$

where $\chi_{K^0}(.)$ and $\chi_{K^1}(.)$ denote, respectively, the characteristic functions of the sets K^0 and K^1. The operators $N_\lambda, B_\lambda, G_\lambda$ and F_λ are bounded on their respective spaces.

Their norms are bounded above, respectively by $e^{-\left(\operatorname{Re}\lambda+\lambda_2^*\right)}$, $(\lambda_2^*+\operatorname{Re}\lambda)^{-1}$, $(\lambda_2^*+\operatorname{Re}\lambda)^{-1}$ and $(\operatorname{Re}\lambda+\lambda_2^*)^{-1}$.

The fact that ψ must satisfy the boundary conditions, Eqs. (3.6) and (3.7) can be written in the space X^0 in the operator form

$$\psi^0 = N_\lambda H\psi^0 + G_\lambda\varphi,$$
$$(I - N_\lambda H)\,\psi^0 = G_\lambda\varphi.$$

If $\operatorname{Re}\lambda > \lambda_0$, the solution of the last equation reduces to the following form

$$\psi^0 = \sum_{n\geq 0} (N_\lambda H)^n\, G_\lambda\varphi. \tag{3.8}$$

On the other hand, Eqs. (3.5) can be rewritten as

$$\psi = B_\lambda H\psi^0 + F_\lambda\varphi \tag{3.9}$$

Substituting (3.8) into (3.9) we get

$$\psi = \sum_{n\geq 0} B_\lambda H\,(N_\lambda H)^n\, G_\lambda\varphi + F_\lambda\varphi.$$

Thus

$$\left(\lambda - T_2^H\right)^{-1} = \sum_{n\geq 0} B_\lambda H\,(N_\lambda H)^n\, G_\lambda + F_\lambda. \tag{3.10}$$

On the other hand, observe that the operator F_λ is nothing else but $(\lambda - T_2^0)^{-1}$ where T_2^0 designate the operator T_2^H with boundary condition $H = 0$. Let $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > \lambda_0$, then $\lambda \in \rho\left(T_2^H\right) \cap \rho\,(T_2^0)$. From Eq. (3.10) we have,

$$\left(\lambda - T_2^H\right)^{-1} - \left(\lambda - T_2^0\right)^{-1} = \sum_{n\geq 0} B_\lambda H\,(N_\lambda H)^n\, G_\lambda. \tag{3.11}$$

It is easy to see that

$$\sigma\left(T_2^0\right) = \sigma C\left(T_2^0\right) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda_2^*\}\,. \tag{3.12}$$

According to the Remark 2.1.9 -(i) with Eq. (3.12) we get the following:

$$\sigma_{ei}\left(T_2^0\right) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda_2^*\}\,, \quad i = 1, \ldots, 6.$$

In addition, it is well-known that the spectrum of the operator A_1 is reduced to the continuous spectrum. More precisely, we have

$$\sigma_{ei}\,(A_1) = \sigma C\,(A_1) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda_1^*\}\,, \quad i = 1, \ldots, 6 \tag{3.13}$$

(see [21]). Since H is weakly compact, then it follows from [10] together with Eq. (3.11) that the operator $\left(\lambda - T_2^H\right)^{-1} - (\lambda - T_2^0)^{-1}$ is weakly compact. Therefore, using [24] we get

$$\sigma_{ei}\left(T_2^H\right) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda_2^*\}\,, \quad i = 1, \ldots, 5. \tag{3.14}$$

Also, the fact that $C\sigma_{e5}\left(T_2^H\right)$ is connected and $\rho\left(T_2^H\right) \neq \emptyset$ yields $\sigma_{e5}\left(T_2^H\right) = \sigma_{e6}\left(T_2^H\right)$.
Verification of assumptions $(\mathrm{H1}) - (\mathrm{H8})$: The conditions (H1)-(H2) are satisfied since A is closed with $\mathcal{D}(A) = \mathcal{D}\left(\Gamma_X\right)$. It follows from the general theory of $C_0^\infty\left(D^0\right)$ that the operator A_1 satisfies (H3). The result defined by assumption (H4) follows from the fact that K_{21} is bounded. In this aim to verify the hypothesis (H5), we will determine the expression of the solution of the equation

$$(\lambda - A)\psi = 0$$

where the unknown ψ must be sought in $\mathcal{D}(A)$. Thus, for each $\operatorname{Re}\lambda > -\lambda_1^*$, the solution is given by

$$\psi(x,v) = \begin{cases} \psi(0,v)e^{-\int_0^x \frac{\sigma_1(s,v)+\lambda}{|v_3|}ds}, & v \in K^1 \\ \psi(1,v)e^{-\int_x^1 \frac{\sigma_1(s,v)+\lambda}{|v_3|}ds}, & v \in K^0 \end{cases}$$

Hence, the operator K_λ is defined on X^i by

$$\begin{cases} K_\lambda : X^i \longrightarrow X, K_\lambda u := \chi_{K^0} K_\lambda^+ u + \chi_{K^1} K_\lambda^- u \quad \text{with} \\ \left(K_\lambda^- u\right)(x,v) := u(0,v)e^{-\int_0^x \frac{\sigma_1(s,v)+\lambda}{|v|}ds}, \quad v \in K^1 \\ \left(K_\lambda^+ u\right)(x,v) := u(1,v)e^{-\int_x^1 \frac{\sigma_1(s,v)+\lambda}{|v_3|}ds}, \quad v \in K^0, \end{cases}$$

which is bounded by $\left(\operatorname{Re}\lambda + \lambda_1^*\right)^{-1}$. Finally, (H6), (H7) and (H8) can be easily verified. Observe that the collision operators $K_{ij}, (i,j) \in \{(1,2),(2,1),(2,2)\}$ acts only on the variables v', so x may be viewed merely as a parameter in $[0,1]$. Then, we will consider K_{ij} as a function

$$K_{ij}(.) : x \in [0,1] \to K_{ij}(x) \in \mathcal{L}\left(L_1(K,dv)\right)$$

Remark 3.2.2. In a L^1 -space setting, M.Mokhtar-Kharroubi [22] noticed that the compactness assumption is too restrictive and that a non-negative collision operator Kij can be assumed to be only dominated by a compact operator. Unfortunately, this class of operators dominated by a compact operator is not yet optimal as explained by B. Lods [27] who use the class of operators which are weakly compact. However, from the definition about the regular operator introduced in [27] we can give the diverse essential spectra in the case of transport equations. This weak compactness assumption is more general than the ones we usually find in the literature in particular used by [32]

In the sequel, we will make the following definition introduced in [27].

Definition 3.2.1. Let K_{ij} be the collision operator defined by (3.3). Then, K_{ij} is said to be regular if

$$\left\{\kappa_{ij}\left(x,.,v'\right);\left\{\left(x,v'\right) \in (0,1) \times K\right\}\right.$$

is a relatively weak compact subset of $L_1(K,dv)$.

Remark 3.2.3. This definition asserts that for every $x \in [0,1]$

$$f \in L^1(K) \longrightarrow \int_K \kappa\left(x,v,v'\right) f\left(v'\right) dv' \in L^1(K)$$

is a weakly compact operator and this weak compactness holds collectively in $x \in (0,1)$. The class of regular operators enjoys the following approximate property:

Theorem 3.2.1. ([27]) Let $K_{ij} \in \mathcal{L}(X)$ be a regular and nonnegative operator. Then, there exist $(K_m)_m \subset \mathcal{L}(X)$ such that:

(i) $0 \leq K_m \leq K_{ij}$ defined for any $m \in \mathbf{N}$.

(ii) For any $m \in \mathbf{N}, K_m$ is dominated by a rank-one operator in $\mathcal{L}\left(L_1(K, ds)\right)$.

(iii) $\lim_{m \to +\infty} \|K_{ij} - K_m\| = 0$.

Proposition 3.2.1. [29] Let $(\Omega, \sum, \mu)$ be a σ-finite, positive measure space and let S, T be two bounded linear operator on $L_1(\Omega, d\mu)$. Then the following assertions holds:

(i) The set of all weakly compact operators is a norm-closed subset of $\mathcal{L}\left(L_1(\Omega, d\mu)\right)$.

(ii) If T is weakly compact on X and $0 \leq S \leq T$, then S is also weakly compact.

Lemma 3.2.1. [11] We assume that the collision operator K_{21} is non-negative, with its kernel $\kappa_{21}(.,.,.) satisfying$ that $\left\{\frac{\kappa_{21}(x,.,v')}{|v_3'|}, (x, v') \in (0,1) \times K\right\}$ is a relatively weak compact subset of $L_1(K, dv)$, then for any $\lambda \in \mathbf{C}$ satisfying $\operatorname{Re}\lambda > -\lambda_1^*, K_{21}\left(\lambda - A_1\right)^{-1}$ is weakly compact operator on X.

Proof. Let $\lambda \in \mathbf{C}$ be such that $\operatorname{Re}\lambda > -\lambda_1^*$. We have $K_{21}\left(\lambda - A_1\right)^{-1} = K_{21}F_\lambda$. It suffices to show that $K_{21}F_\lambda$ is weakly compact on X.
Let $\varphi \in X$.

$$\begin{aligned}
\left(K_{21}F_\lambda\varphi\right)(x,v) &= \int_K K_{21}\left(x,v,v'\right)\left(F_\lambda\varphi\right)\left(x,v'\right)dv' \\
&= \int_{K^0} K_{21}\left(x,v,v'\right)F_\lambda^+\varphi\left(x,v'\right)dv' + \int_{K^1} K_{21}\left(x,v,v'\right)F_\lambda^-\varphi\left(x,v'\right)dv' \\
&= \int_{K^0} \frac{\kappa_{21}\left(x,v,v'\right)}{|v_3'|}\int_x^1 e^{-\int_x^{x'}\frac{\sigma_1(s,v')+\lambda}{|v_3'|}ds}\varphi\left(x',v'\right)dx'dv' \\
&\quad + \int_{K^1} \frac{\kappa_{21}\left(x,v,v'\right)}{|v_3'|}\int_0^x e^{-\int_{x'}^{x}\frac{\sigma_1(s,v')+\lambda}{|v_3'|}ds}\varphi\left(x',v'\right)dx'dv' \\
&= K_{21}'\widetilde{F}_\lambda\varphi,
\end{aligned}$$

where K_{21}' and $\widetilde{F}_\lambda$ denote the following bounded operators:

$$\begin{cases}
K_{21}' : X \longrightarrow X \\
\qquad \varphi \longrightarrow \displaystyle\int_K \frac{\kappa_{21}\left(x,v,v'\right)}{|v_3'|}\varphi\left(x,v'\right)dv',
\end{cases}$$

and

$$\begin{cases}
\widetilde{F}_\lambda : X \longrightarrow X \\
\qquad \psi \longrightarrow \chi_{K^0}\left(v'\right)\displaystyle\int_0^x e^{-\int_{x'}^{x}\frac{\sigma_1(s,,')+\lambda}{|v_3'|}ds}\psi\left(x',v'\right)dx' \\
\qquad\qquad + \chi_{K^1}\left(v'\right)\displaystyle\int_x^1 e^{-\int_x^{x'}\frac{\sigma_1(s,v')+\lambda}{|v_3'|}ds}\psi\left(x',v'\right)dx'.
\end{cases}$$

We claim that $K'_{21}\tilde{F}_\lambda$ depend continuously on K'_{21}. Let $\varphi \in X$

$$
\begin{aligned}
\left\|\widetilde{F}_\lambda\varphi\right\| &\leq \int_0^1 \left[\int_{K^0} dv' \int_0^x e^{-\int_{x'}^{z} \frac{\sigma_1(s,v')+\lambda}{|v'_3|}ds} \left|\varphi\left(x',v'\right)\right| dx'\right] dx \\
&+ \int_0^1 \left[\int_{K^1} dv' \int_x^1 e^{-\int_{x}^{x'} \frac{\sigma_1(s,v')+\lambda}{|v'_3|}ds} \left|\varphi\left(x',v'\right)\right| dx'\right] dx \\
&\leq \int_0^1 \left[\int_{K^0} dv' \int_0^x e^{-\int_{x}^{x} \frac{Re\lambda+\lambda_i^*}{|v'_3|}ds} \left|\varphi\left(x',v'\right)\right| dx'\right] dx \\
&+ \int_0^1 \left[\int_{K^1} dv' \int_x^1 e^{-\int_{x}^{x'} \frac{Rc\lambda+\lambda_i^*}{|v'_3|}ds} \left|\varphi\left(x',v'\right)\right| dx'\right] dx \\
&\leq \int_0^1 \left[\int_{K^0} dv' \int_0^x \left|\varphi\left(x',v'\right)\right| dx'\right] dx + \int_0^1 \left[\int_{K^1} dv' \int_x^1 \left|\varphi\left(x',v'\right)\right| dx'\right] dx \\
&\leq \left[\int_{K^0}\int_0^x \left|\varphi\left(x',v'\right)\right| dx'dv' + \int_{K^1}\int_x^1 \left|\varphi\left(x',v'\right)\right| dx'dv'\right] \\
&\leq \|\varphi\|.
\end{aligned}
$$

Then,

$$\left\|K'_{21}\widetilde{F}_\lambda\right\| \leq \|K'_{21}\|$$

According to Theorem 3.2.1 and Proposition 3.2.1-(i), it suffices to proves the result when K'_{21} is dominated by a rank-one operator in $\mathcal{L}\left(L_1(K,dv)\right)$. This asserts that K'_{21} have a kernel

$$\kappa'_{21}(v,v) = \frac{\kappa_{21}\left(x,v,v'\right)}{|v'_3|} = \kappa'_1(v)\kappa'_2\left(v'\right); \quad \kappa'_1(.) \in L_1(K), \kappa'_2(.) \in L_\infty(K)$$

Let $\mathcal{O}$ be a bounded set of X, and let $\psi \in \mathcal{O}$. We have for all measurable subset E of D

$$
\begin{aligned}
\int_E \left|K'_{21}\widetilde{F}_\lambda\psi(x,v)\right| dxdv &\leq \int_E\int_{K^0}\int_0^x \kappa'_1(v)\kappa'_2\left(v'\right) e^{-\int_{x}^{x} \frac{\sigma_1(s,v')+\lambda}{|v'_3|}ds} \left|\varphi\left(x',v'\right) dx'dv'\right| dxdv \\
&+ \int_E\left|\int_{K^1}\int_x^1 \kappa'_1(v)\kappa'_2\left(v'\right) e^{-\int_{x}^{x'} \frac{\sigma_1(s,v')+\lambda}{|v'_3|}ds}\right| \varphi\left(x',v'\right) dx'dv' \mid dxdv \\
&\leq \|\kappa'_2\|_\infty \int_E \left|\kappa'_1(v)\right| dxdv \\
&\times \left[\int_{K^1}\int_x^1 \left|\psi\left(x',v'\right)\right| dx'dv' + \int_{K^0}\int_0^x \left|\psi\left(x',v'\right)\right| dx'dv'\right] \\
&\leq \|\kappa'_2\|_\infty \|\psi\|_1 \int_E \left|\kappa'_1(v)\right| dxdv
\end{aligned}
$$

Since

$$\lim_{|E|\longrightarrow 0} \int_E \left|\kappa'_1(v)\right| dxdv = 0, \quad \kappa'_1 \subseteq L_1(K,dv)$$

and according to [7] , [17] we infer that the set $K'_{21}\widetilde{F}_\lambda(\mathcal{O})$ is weakly compact. Hence the weak compactness of $K_{21}\left(\lambda - A_1\right)^{-1}$ is proved. $\square$

Lemma 3.2.2. [5] We assume that the collision operator K_{12} is non-negative, regular in the sense above, then for $\lambda \in \mathbf{C}$ satisfying $\operatorname{Re}\lambda > -\lambda_1^*$, the operator $(\lambda - A_1)^{-1} K_{12}$ is weakly compact on X. Proof. Let $\lambda \in \mathbf{C}$ be such that $\operatorname{Re}\lambda > -\lambda_1^*$. It follows that

$$\left\|(\lambda - A_1)^{-1}\right\| \leq \|F_\lambda\| \leq \frac{1}{(\operatorname{Re}\lambda + \lambda_1^*)}.$$

Let $\varepsilon > 0$, for $\operatorname{Re}\lambda > -\lambda_1^* + \varepsilon$, we get

$$\left\|(\lambda - A_1)^{-1} K_{12}\right\| \leq \frac{1}{\varepsilon} \|K_{12}\|.$$

Then, $(\lambda - A_1)^{-1} K_{12}$ depends continuously on K_{12}, unfortunately on $\{\operatorname{Re}\lambda > -\lambda_1^* + \varepsilon\}$. From Theorem 3.2.1 and Proposition 3.2.1-(i), it suffices to prove the result when K_{12} is dominated by a rank-one operator in $\mathcal{L}(L_1(K, dv))$. This asserts that the kernel of K_{12} can be rewritten as follows:

$$\kappa_{12}(v, v') = \kappa_1(v)\kappa_2(v'), \quad \kappa_1(.) \in L_1(K), \quad \kappa_2(.) \in L^\infty(K).$$

Now, we will shows that $F_\lambda K_{12}$ is weakly compact. Let $\varphi \in X$

$$\begin{aligned}
\left(F_\lambda^+ K_{12}\varphi\right)(x, v) &= \frac{1}{|v_3|} \int_x^1 e^{-\int_x^{x'} \frac{\sigma_1(s,v)+\lambda}{|v_3|} ds} K_{12}\varphi(x', v)\, dx' \\
&= \frac{1}{|v_3|} \int_x^1 \int_K e^{-\int_x^{x'} \frac{\sigma_1(s,v)+\lambda}{|v_3|} ds} \kappa_{12}(x', v, v')\, \varphi(x', v')\, dx' dv', \quad v \in K^0 \\
&= J_\lambda U_\lambda \varphi
\end{aligned}$$

where J_λ and U_λ denote the following bounded operators:

$$\begin{cases} U_\lambda : X \longrightarrow L_1((0,1), dx) \\ \qquad \varphi \longrightarrow \displaystyle\int_K \kappa_2(v')\, \varphi(x', v')\, dv' \end{cases}$$

and

$$\begin{cases} J_\lambda : L_1((0,1), dx) \longrightarrow X_1^0 \\ \qquad \varphi \longrightarrow \displaystyle\frac{1}{|v_3|} \int_x^1 e^{-\int_x^{x'} \frac{\sigma_1(s,v)+\lambda}{|v_3|} ds} \kappa_1(v)\varphi(x')\, dx'. \end{cases}$$

Now, it sufficient to show that J_λ is weakly compact. To do so, let $\mathcal{O}$ be a bounded set of $L_1((0,1), dx)$, and let $\psi \in \mathcal{O}$. It follows that for all measurable subset E of K^0

$$\int_E \left| J_\lambda \psi(v) \right| \left| v_3 \right| dv \leq \|\psi\| \int_E \left| \kappa_1(v) \right| dv$$

Since

$$\lim_{|E| \to 0} \int_E |\kappa_1(v)|\, dv = 0, \quad \kappa_1(.) \in L_1(K, dv),$$

where $|E|$ is the measure of E, applying [7] we deduce that the set $J_\lambda(\mathcal{O})$ is weakly compact. Hence the weak compactness of $F_\lambda^+ K_{12}$ is checked. Reasoning analogously shows that the operator $F_\lambda^- K_{12}$ is weakly compact.

We are now in the position to express the main result of this section in the following theorem

Theorem 3.2.2. [34] If the operator H is weakly compact, positive, and the operators K_{12}, K_{22} are non-negative, regular and if in addition $\left\{\frac{\kappa_{21}(x,,,v')}{|\Gamma_s'|},(x,v') \in (0,1)\times K\right\}$ is relatively weakly compact then

$$\sigma_{ei}(\mathcal{A}) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\min(\lambda_1^*, \lambda_2^*)\}, \quad i = 1, \dots, 6$$

Proof. Let $\mu \in \rho(A_1)$, the operator M_μ is given by

$$M_\mu = D + K_{21}B_\mu H\psi^0 - K_{21}(\mu - A_1)^{-1} K_{12}$$

Using Lemma 3.2.2 and the weak compactness of H we conclude that

$$K_{21}B_\mu H\psi^0 - K_{21}(\mu - A_1)^{-1} K_{12}$$

is weakly compact, then it follows from [24] that

$$\sigma_{ei}\left(\bar{M}_\mu\right) = \sigma_{ei}(D), \qquad \text{for } i = 1, \dots, 6$$

For $\mu \in \rho\left(T_2^H\right)$ such that $r_\sigma\left(\left(\mu - T_2^H\right)^{-1} K_{22}\right) < 1$, then $\mu \in \rho\left(T_2^H\right) \cap \rho\left(T_2^H + K_{22}\right)$. Hence, we have

$$(\mu - D)^{-1} - \left(\mu - T_2^H\right)^{-1} = \sum_{n \geq 1}\left(\left(\mu - T_2^H\right)^{-1} K_{22}\right)^n \left(\mu - T_2^H\right)^{-1} \tag{3.15}$$

Since K_{22} is non-negative, regular operator, implies that for all $n \geq 1$, $\left(\left(\mu - T_2^H\right)^{-1} K_{22}\right)^n \left(\mu - T_2^H\right)^{-1}$ is weakly compact on X.

So, $(\mu - D)^{-1} - \left(\mu - T_2^H\right)^{-1}$ is weakly compact. The use of Eq. (3.15) and [20, Theorem 3.3] leads to

$$\sigma_{ei}(D) = \sigma_{ei}\left(T_2^H\right) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda_2^*\}, \quad i = 1, \dots, 6$$

Then from Eq. (3.14)

$$\sigma_{ei}\left(\bar{M}_\mu\right) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda_2^*\}, \quad i = 1, \dots, 6.$$

Now, applying Theorem 3.9, by using Eqs. (3.13) and (28), we get

$$\sigma_{ei}(\mathcal{A}) = \{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\min(\lambda_1^*, \lambda_2^*)\}, \quad i = 1, \dots, 6$$

□

Remark 3.2.4. In this application, we have determined the essential spectra of the two-group transport operator $\mathcal{A}$ on L_1-space without knowing that of the operator A. But we know only the restriction of the operator A on the intersection densely domain, $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$. So, the result obtained in [32] is a special case of this work.

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