Ministry of Higher Education and Scientific Research
Echahid Hamma Lakhdar University – El Oued
Faculty of Technology
Department of Electrical Engineering

Lecture Notes of

Control systems

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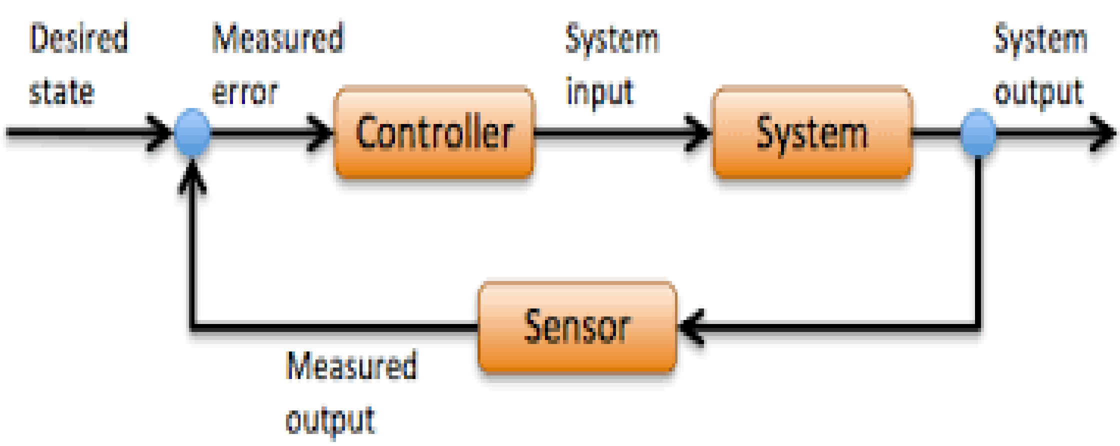
Electrotechnical Licence (3-ELT)



Contents overview

Control systems are an integral component of any industrial society and are necessary to provide useful economic products for society. Control engineering is based on the foundation of feedback theory and linear system analysis, and integrates the concepts of network theory and communication theory. Therefore, control engineering is not limited to any engineering discipline but is equally applicable for aeronautical, chemical, mechanical, and electrical engineering.

This document has been developed of the control Systems course for the Electrotechnical license at the University of El-Oued and is mainly intended for third-year students. In particular, it details the modeling and analysis of continuous and time-invariant linear systems. This document was not designed to be self-sufficient. Presence in class and additional work are essential to achieve mastery of the objectives of the course. However, it will allow you to deepen the aspects that can only be dealt with quickly in class.

The Servo Systems course (control system) aims to familiarize the student with the properties of control structures of continuous linear systems and the time and frequency analysis of basic systems (1st and 2nd order system). At the end of this course, the student should be able to:

- the student know the definitions and concepts specific to continuous linear systems.
- become familiar with the structure of control systems (open loop and closed loop) and the role of each component in these systems.
- the student be able to determine the static and dynamic performances of a continuous linear system.
- the student know the classical techniques of representation of continuous linear systems: differential equation, transfer function, functional diagram and fluence graph.
- the student be able to analyze basic systems in the time and frequency domain.

- the student analyze the stability of a controlled system.

The prior knowledge required for taking this support is: basic mathematics (algebra, integral and differential calculus, complex numbers, etc.), fundamental notions of signal processing and basic electronics (linear circuits).

General introduction chapter provides an introduces the key concepts related to history of automatic control systems, terminology and definition, concept of systems, dynamic behavior, static behavior, static systems, dynamic systems, linear systems, introductory examples, open loop systems and closed loop systems with performance of servo systems.

Systems modeling chapter focuses on the representation of systems by their differential equations, Laplace transform, the differential equation to the transfer function, functional blocks and subsystems, rules of simplification, representation of dynamical systems by fluence graphs, Mason's rule, calculation of transfer functions of looped systems.

Temporal responses of linear systems chapter: in this chapter, the focus is on the definition of the response of a system, transient regime, permanent regime, notions of stability, static speed and accuracy impulse response (1st and 2nd order). temporal characteristics. step response (1st and 2nd order), identification of first and second order systems at from the time response.

Frequency responses of linear systems chapter provides a definition of frequency responses of linear systems, Bode and Nyquist diagram, Frequency characteristics of systems basic dynamics (1st and 2nd order), phase and gain margins.

Stability and Accuracy of Servo Systems chapter provides a definition of conditions of stability, algebraic criterion of Routh-Herwitz, Criteria of the reverse in the Nyquist and Bode plans, Stability margins, accuracy of servo systems, static accuracy, calculation of the static difference, dynamic precision.

Abstract

Control systems are an integral component of any industrial society and are necessary to provide useful economic products for society. Control engineering is based on the foundation of feedback theory and linear system analysis, and integrates the concepts of network theory and communication theory. Therefore, control engineering is not limited to any engineering discipline but is equally applicable for aeronautical, chemical, mechanical, and electrical engineering.

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Key word : Control systems , dynamic performances , diagram and fluence graph , transfer function ,Stability

ملخص

تعد أنظمة التحكم جزءًا لا يتجزأ من أي مجتمع صناعي وهي ضرورية لتوفير منتجات اقتصادية مفيدة للمجتمع. تعتمد هندسة التحكم على أساس نظرية التغذية المرتدة وتحليل النظام الخطي، وتدمج مفاهيم نظرية الشبكة ونظرية الاتصالات. لذلك، لا تقتصر هندسة التحكم على أي تخصص هندسي ولكنها تنطبق أيضًا على هندسة الطيران والكيميائية والميكانيكية والكهربائية.

تم تطوير هذه الوثيقة من دورة أنظمة التحكم للرخصة الكهروتقنية بجامعة الوادي وهي مخصصة بشكل أساسي لطلاب السنة الثالثة. وعلى وجه الخصوص، فإنه يعرض تفاصيل نمذجة وتحليل الأنظمة الخطية المستمرة والثابتة مع الزمن. لم يتم تصميم هذه الوثيقة لتكون مكتفية ذاتيا. يعد التواجد في الفصل والعمل الإضافي ضروريين لتحقيق التمكن من أهداف الدورة. ومع ذلك، سيسمح لك ذلك بتعميق الجوانب التي لا يمكن التعامل معها إلا بسرعة في الفصل.

يهدف مقرر الأنظمة المؤازرة (نظام التحكم) إلى تعريف الطالب بخصائص هياكل التحكم للأنظمة الخطية المستمرة وتحليل الوقت والتردد للأنظمة الأساسية (نظام الرتبة الأولى والثانية)

يعرف الطالب التعاريف والمفاهيم الخاصة بالأنظمة الخطية المستمرة.

 التعرف على بنية أنظمة التحكم (الحلقة المفتوحة والحلقة المغلقة) ودور كل مكون في هذه الأنظمة.

 أن يكون الطالب قادراً على تحديد الأداء الساكن والديناميكي لنظام خطي مستمر.

 يتعرف الطالب على التقنيات الكلاسيكية لتمثيل الأنظمة الخطية المستمرة: المعادلة التفاضلية، دالة النقل، المخطط الوظيفي، ورسم فلوينس.

 أن يكون الطالب قادراً على تحليل الأنظمة الأساسية في مجال الزمن والتردد.

الكلمات المفتاحية: أنظمة التحكم، الأداء الديناميكي، الرسم البياني والمخططات الانسيابية، التردد للأنظمة الأساسية

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Chapter 1

Introduction to servo systems

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Introduction to servo systems

This chapter introduces the main definitions and notions necessary for the study of linear servo systems (object of this course). We will illustrate the difference between the notion of open loop control and closed loop control. We will also establish the distinction between operation in regulation and operation in tracking. Finally, we will expose the procedure to be followed for the realization of a control system.

1. Definitions :

Definition 1.1: The system is an assembly of components or elements so as to produce a given function. It can be classified as SISO systems and MIMO systems based on the **number of inputs and outputs** present. **SISO** (Single Input and Single Output) systems have one input and one output. Whereas, **MIMO** (Multiple Inputs and Multiple Outputs) systems have more than one input and more than one output. It is common to represent a system or one of its component elements using a diagram, known as a block diagram. Figure 1.1 provides an example of such a representation.

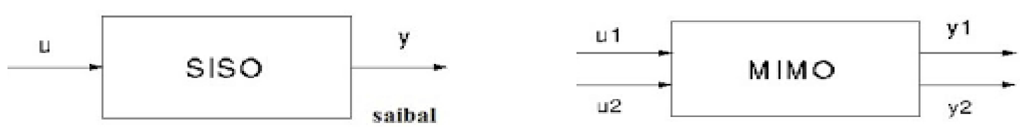


Figure 1.1: block diagram of a system

The inputs affecting a system are generally denoted by the letter *u* and outputs by the letter *y*. (command) to get the desired output. There may also be inputs which are called disturbances. The latter disturb the desired functioning at the output (Figure 1.2) [17].

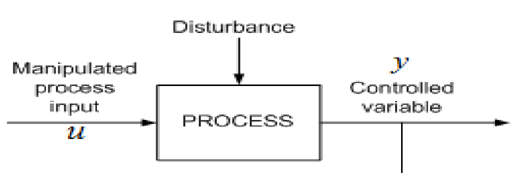


Figure 1.2: Various signal of System

Definition 1.2: In a system when the output quantity is controlled by varying the input quantity the system is called ***control system*** (***Example:*** a lamp controlled by a switch)

The output quantity is called controlled variable or response. The input quantity is called command signal or excitation

Definition 1.3: automatic control is the science that deals with the modelling, analysis, identification and control of dynamic systems (objects studied).

Definition 1.4: The model is a mathematical object (differential equations) linking between inputs and outputs signals.

1.2 Notion of Open/Closed Loop:

There are generally two control structures: open-loop control and closed-loop control, also called feedback control.

1.2.1 Open loop control:

A control system is in open loop when no output measurement y(t) is used (does not include feedback) to develop the command u(t) (Figure 1.3). This requires knowledge of an operating model of the system to be controlled,

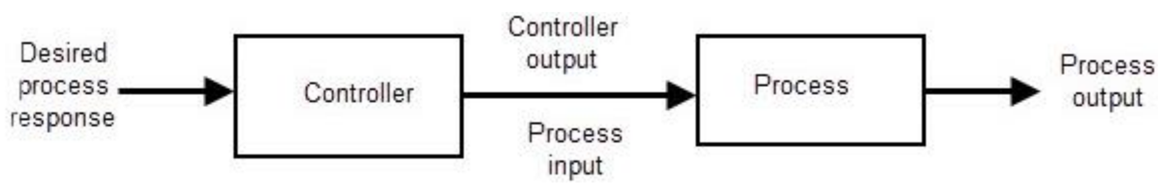


Figure 1.3: Open loop control

Example 1: knowledge of an operating model of a DC motor will make it possible to know the input voltage that will have to be applied to it to obtain the speed desired rotation.

Open-loop control has the following disadvantages:

- you cannot control a system that suffers from instability problems (we will come back to the concept of stability in more detail in a later chapter),

- difficult correction: having no information on the output, the operator cannot develop any adjustment strategy to obtain an output with the desired value with precision and speed.

1.2.2 Closed loop control:

In order to solve the problems of open loop control and to automate the system (remove human action) a feedback loop (or feedback) is introduced. In this control strategy, a measurement of the output is used and compared with the desired signal by the controller (Figure 1.4).

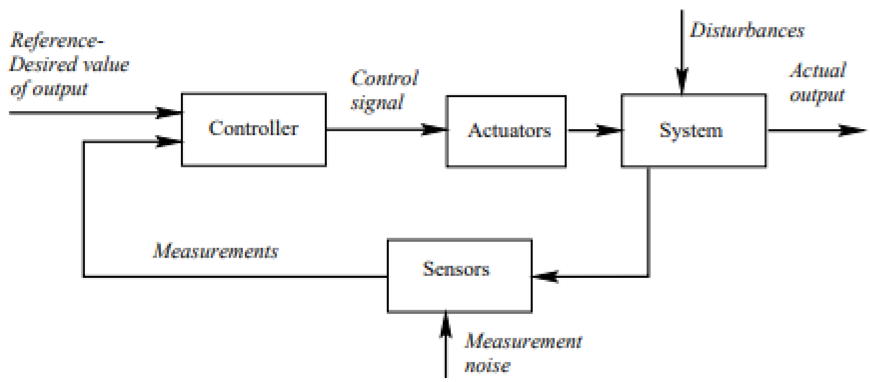


Figure 1.4: Closed loop control

- **Desired Signal** (yr(t) or yc(t)): it corresponds to the value desired at the output,
- **Output Signal** (y(t)): represents the physical phenomenon that the control system must enslave .
- **Error** (ε(t)): represents the difference between the setpoint and the output,
- **Comparator**: constantly compares what you get with what you want to get at the output (elaborates the difference ε(t)),
- **Corrector**: (controller) processes the deviation signal and determines the control signal,
- **Actuator**: it receives the control signal from the corrector and acts directly on the system to be controlled (for example: power amplifier, motor, cylinder, valve, etc.),

- **Sensor**: measuring device which gives an image as faithful as possible of the output y(t) and transforms it into a signal understandable by the comparator (most often electrical) (for example: accelerometer, thermometer, gyroscopes, etc.).

We can therefore define servo system as a looped system (closed loop) whose operation tends to cancel the difference between a controlled signals (the output y(t) and the reference yr(t)).

Thus, in example 1 An automatic speed control system, also called cruise control, works by using the difference, or error, between the actual and desired speeds and knowledge of the car’s response to fuel increases and decreases to calculate via some algorithm an appropriate gas pedal position, so to drive the speed error to zero. This decision process is called a control law and it is implemented in the controller. The system configuration is shown in Fig. 1.5 The car dynamics of interest are captured in the plant. Information about the actual speed is fed back to the controller by sensors, and the control decisions are implemented via a device, the actuator, that changes the position of the gas pedal. The knowledge of the car’s response to fuel increases and decreases is most often captured in a mathematical model.

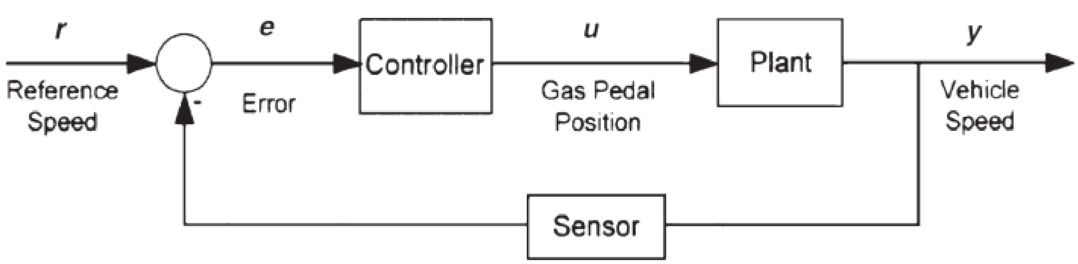


Figure 1.5: Feedback control configuration with automatic speed control system

1.3 Regulation and pursuit:

A servo (slaved system, closed loop system, control system) can be used in regulation or tracking. A servo is used in regulation (regulated system) when the setpoint is constant or changing in stages). The output must remain constant despite the presence of disturbances. If, on the other hand, the setpoint constantly evolves or follows a physical quantity independent of the system itself (radar of tracking, position servoing, etc...) the servo operates in tracking (following system). The objective is then that the response of the system follows the instruction [14].

1.4 Methodology:

In automation, the methodology used to design a control system can operate as follows [8]:

- **Specifications:** the starting point of any project is the specifications. The automation engineer must be aware of the problem and the various specifications. He must, on this occasion, clearly define the system (with its inputs and outputs) and the expected performance,

- **Modeling:** often the automation engineer must describe the behavior of the system studied using the laws of physics (modeling) or from input/output measurements to obtain a set of algebraic and differential equations. Then, he must reformulate these so as to obtain a classic automatic model (a transfer function for example),

- **Analysis:** in this phase, automatic control techniques should be used to judge the performance of the system (stability, response time, oscillations, precision...) from the model,

- Synthesis: the last phase consists in designing a corrector, i.e. a loop which gives the system studied the desired performance if possible.

1.5 Performances of a servo:

1.5.1 Stability: A general notion of stability says that a system is stable if after being removed from a stable state, the system returns to the original state provided no external input is applied.

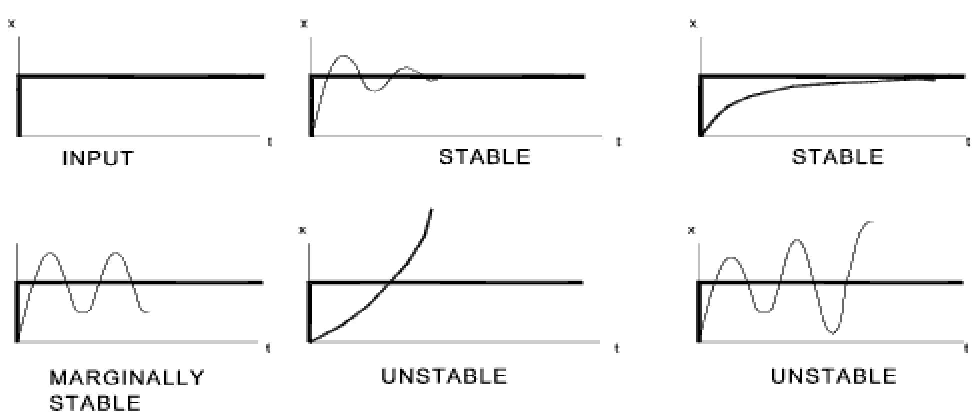


Figure 1.6: Stability of system

1.5.2 Accuracy: By calling y(t) the output corresponding to the setpoint y_c (t), we can characterize the precision of an error control at each instant: $\varepsilon(t) = y_c(t) - y(t)$. There are two types of accuracy:

- **Dynamic accuracy:** as long as the transient part of the response has not become negligible, ε(t) characterizes the dynamic error,

- **Static accuracy:** the static error is the error that remains in steady state, i.e. lim ε(t) when t tends to infinity.

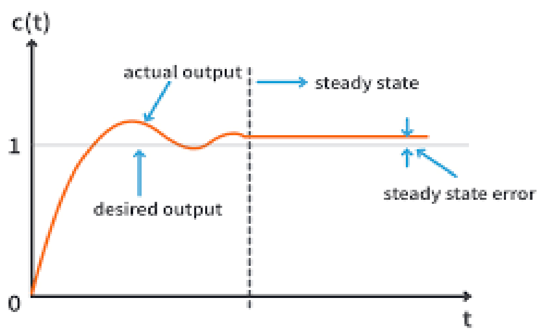


Figure 1.6: Static error

1.5.3 Speed: The speed of a system can be measured by the response time, at a given percentage, to a step input. The 5% response time is frequently used, i.e. the time from which the response is between 95% and 105% of its final value.

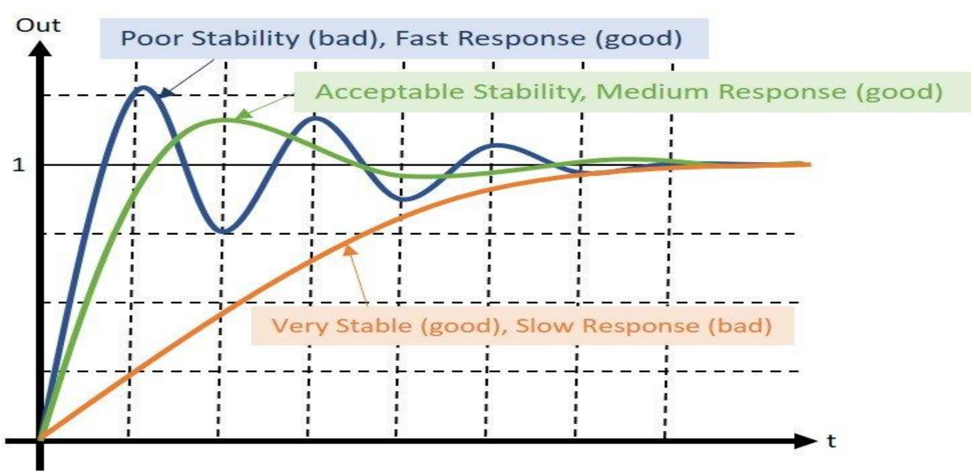


Figure 1.6: Response of system

Chapter 2

Systems modeling

Chapter 2

Systems modeling

The first step in the analysis and design of control systems is mathematical modeling of the system. The two most common methods are the transfer function representation and the state equation approach. The dynamic performance of the physical systems is obtained by utilizing the physical laws of electrical, mechanical, fluid and thermodynamic systems. We generally model physical systems with linear differential equations with constant coefficients when possible.

2.1. Representation by a differential equation

The input-output relations of various physical components of a system are governed by differential equation. The differential equations of a linear time invariant system with constant coefficients as follows:

$$a_n \frac{d^n c}{dt^n} + a_{n-1} \frac{d^{n-1} c}{dt^{n-1}} + \cdots .. + a_1 \frac{dc}{dt} + a_0 c = b_m \frac{d^m r}{dt^m} + b_{m-1} \frac{d^{m-1} r}{dt^{m-1}} + \cdots + b_1 \frac{dr}{dt} + b_0 r \qquad (2.1)$$

Where is the output, and is the input of the system. The solution of this equation represents the evolution of the system output y(t) over time as a function of the input u(t) and initial conditions.

Exemple 2.1 : Consider the RLC circuit below:

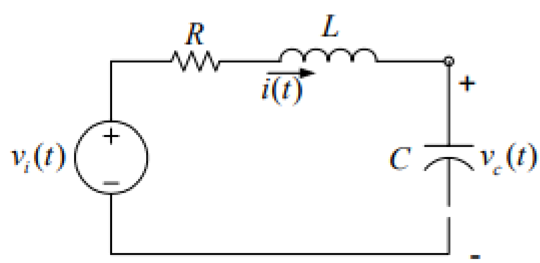


Figure 2.1 RLC Circuit

We want to determine the relationship between $u(t)$ (supply voltage) and y(t) as $v_c(t)$ (capacitor voltage), The mesh equation gives:

$$R * i(t) + L * \frac{di(t)}{dt} + v_c(t) = u(t) \qquad (2.2)$$

$$R * C\frac{dv_c(t)}{dt} + L * C\frac{d^2v_c(t)}{dt} + v_c(t) = u(t) \quad (2.3)$$

2.2 Transfer function

Transfer function is the ratio of Laplace transform of outputs of the system to the Laplace transform of the inputs under the assumption that all initial conditions are zero

- **Laplace transform**

The Laplace transform of function *f (t)* with zero initial condition (causal) is defined by:

$$L\big(f(t)\big) = F(S) = \int_0^{\infty} f(t)e^{-st}dt \quad (2.4)$$

- Initial Value Theorem: If $f(0^+)$ exists, then

$$f(0^+) = \lim_{t\to o+} f(t) = \lim_{s\to\infty} s\,F(S) \quad (2.5)$$

- Final value theorem: let *F(s)* be a function which has no poles in the half-plane $\Re(s) \geq 0$, except at most one simple pole at the origin. So :

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(S) \quad (2.6)$$

The main properties of the Laplace transform are shown in Table 2.1and Table 2.2. .

The system transfer function is : $G(S) = \frac{Y(S)}{U(S)}$. We can then represent the system in the graphical form below:

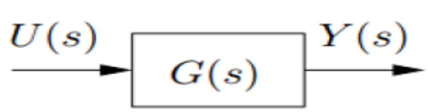


Figure 2.2 Representation of system transfer function

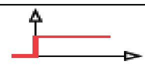
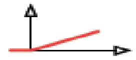
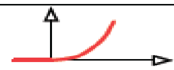
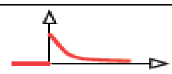
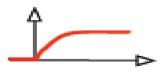
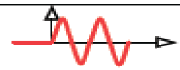
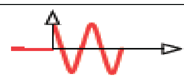
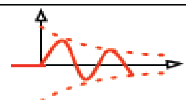
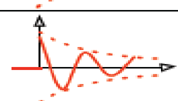
	$b \times \delta(t)$	b
	$b \times u(t)$	$\frac{b}{p}$
	$b \times t \times u(t)$	$\frac{b}{p^2}$
	$b \times t^n \times u(t)$	$b \times \frac{n!}{p^{n+1}}$
	$b \times e^{-at} \times u(t)$	$\frac{b}{p+a}$
	$b \times (1 - e^{-at}) \times u(t)$	$\frac{b}{p} \times \frac{a}{p+a}$
	$b \times sin(\omega t).u(t)$	$b \times \frac{\omega}{p^2+\omega^2}$
	$b \times cos(\omega t).u(t)$	$b \times \frac{p}{p^2+\omega^2}$
	$e^{-at} sin(\omega t).u(t)$	$\frac{\omega}{(p+a)^2+\omega^2}$
	$e^{-at} cos(\omega t).u(t)$	$\frac{p+a}{(p+a)^2+\omega^2}$

Table 2.1: Laplace transforms of usual excitation signals

Property	***f (t)***	***F (s)***
Linearity	$k_1 f_1(t)+ k_2 f_2(t)+ k_3 f_3(t)+\dots$	$k_1 F_1(s)+ k_2 F_2(s)+ k_3 F_3(s)+\dots$
Time scale	$f(at)$, $a>0$	$1/a\ F(s/a)$
First derivative	$\frac{df(t)}{dt}$	$sF(s)-f(0)$
Second derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2F(s)-sf(0)-f'(0)$
Derivative of order n	$\frac{d^n f(t)}{dt^n}$	$s^nF(s)-s^{n-1}f(0)- s^{n-2}f'(0)- s^{n-3}f''(0)-\dots-f^{n-1}(0)$
Integral	$\int_0^t f(x)dx$	$\frac{F(s)}{s}$
Time shift	$f(t-a)$	$e^{as}F(s)$
Multiply by an exponential	$e^{-as}f(t)$	$F(s+a)$
Multiply by t	$t\,f(t)$	$\frac{dF(s)}{ds}$

Table 2.2: Properties of the Laplace transform

Remarks :

- ✓ The transfer function is also the Laplace transform of the impulse response (the impulse response to an input in the form of a Dirac impulse, with null initial conditions).
- ✓ The denominator of the transfer function is said to be the characteristic polynomial of the system.
- ✓ The order of the system is the degree of this polynomial.

Example 2.2 Find the transfer function relating the capacitor voltage to the input voltage of RLC circuit in example 2.1. so : Applying KVL

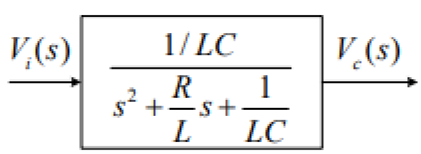
$R * i(t) + L * \frac{di(t)}{dt} + v_c(t) = v_i(t)$ with $i(t) = c * \frac{dv(t)}{dt}$

Taking TL with zero initial conditions, we obtain : $V_c(S)\left[LCS^2 + RC.S + 1\right] = V_i(S)$

the system transfer function is :

$$\frac{V_c(S)}{V_i(S)} = G(s) = \frac{1}{LCS^2 + RC.S + 1} = \frac{1/LC}{S^2 + \frac{R}{L}S + \frac{1}{Lc}}$$

Therefor :



- ➢ ***Advantages of Transfer function:*** *1*. The response of the system to any input can be determined very easily. 2. It gives the gain of the system. 3. It help in the study of stability of the system 4. Since Laplace transform is used it converts time domain equations to simple algebraic equations 5. Poles and zeroes of a system can be determined from the knowledge of the transfer function of the system.

2.2.2 Transfer Function with Multivariable System

The definition of transfer function can be extended to a system with a multiple number of inputs and outputs (multivariable system). Consider a system with two inputs and two outputs as shown in Figure 2.5.

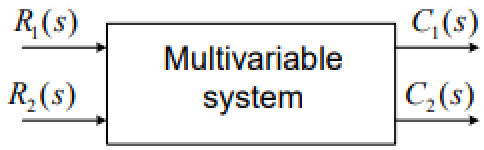


Figure 2.1 Representation of Multivariable System transfer function

When dealing with the relationship between one input and one output, it is assumed that all other inputs are set to zero, e.g.,

$$G_{11} = \left.\frac{C_1(s)}{R_1(s)}\right|_{R_2=0} \qquad G_{12} = \left.\frac{C_1(s)}{R_2(s)}\right|_{R_1=0}$$
$$G_{21} = \left.\frac{C_2(s)}{R_1(s)}\right|_{R_2=0} \qquad G_{22} = \left.\frac{C_2(s)}{R_2(s)}\right|_{R_1=0} \tag{2.7}$$

Since the principle of superposition is valid for linear system, the total effect on any output variable due to all the inputs acting simultaneously is obtained by adding up the outputs due to each input acting alone.

$$C_1(s) = G_{11}R_1(s) + G_{12}R_2(s)$$
$$C_2(s) = G_{21}R_1(s) + G_{22}R_2(s) \tag{2.8}$$

Representing the above by matrix equation we have

$$C(s) = G(s)R(s) \qquad \text{where} \qquad G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{2.9}$$

2.2.3 State Variable

Alternatively, the behavior of an invariant linear system with input *u(t)* and output *y(t)* can be described by a finite number of quantities called state variables. A state model is a finite set of first-order differential equations. The differential equations of a lumped linear network can be written in the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad (2.10)$$

his system of first-order differential equations is known as the state equation of the system and is the state vector and is the input vector. The second equation is referred to as the output equation. A is called the state matrix, B the input matrix, C the output matrix, D the direct transition matrix. the block diagram given in Figure 2.3 to illustrate this representation.

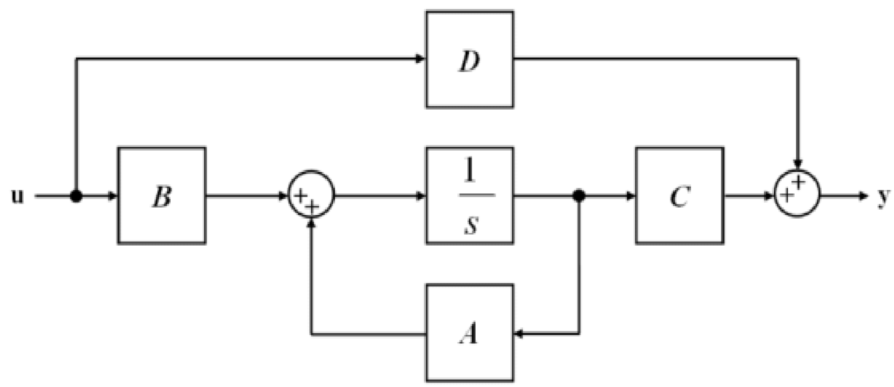


Figure 2.1 block diagram of State Variable

Example 2.2 circuit ***RLC*** , with the output of the system is the capacitor voltage. We want to determine the relationship between *u(t)* (supply voltage) and *y(t)* (current i(t)):

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dT = u(t)$$

$$R\frac{dy(t)}{dt} + L\frac{d^2i(t)}{dt^2} + \frac{1}{C}y(t) = \frac{du(t)}{dt}$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = u(t) \qquad (2.11)$$

We choose two state variables (n = 2): $x_1(t)= y(t)$ and $x_2(t)= dy(t)/dt$. Then, equation (2.11) :

$$\frac{d^2x_2(t)}{dt^2} = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{LC}u(t) \qquad (2.12)$$

We deduce the state representation :

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{LC}\end{bmatrix}u(t),\ y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + 0u(t) \qquad (2.13)$$

Example 2.3 : Obtain the state equation in phase variable form for the following differential equation.

$$2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 10u(t)$$

The differential equation is third order, thus there are three state variables as follows

$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}$

and the derivatives are

$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3,$ and $\dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + 5u(t)$

Or in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}5u(t) \qquad , \qquad y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

2.2.3 Transition from state representation to transfer function

Applying the Laplace transform to the state model (2.10) gives us:

$$\begin{matrix}\dot{x}(t) = Ax(t) + Bu(t)\\ y(t) = Cx(t) + Du(t)\end{matrix} \quad TL: \quad \begin{matrix}X(P) = (P.I - A)^{-1}BU(P)\\ Y(P) = CX(P) + DU(P)\end{matrix} \qquad (2.14)$$

Therefor : $\quad G(P) = \frac{Y(P)}{X(P)} = C(P.I - A)^{-1}B + D \qquad (2.15)$

2.3 Representation of dynamical systems by fluence graphs, Mason's rule,

2.3.1 2.5 Representation by fluence graph

This representation, very close to the representation by functional diagram, uses the properties of the linear systems. A variable is represented by a node and a link between variables by an arc, i.e.

an oriented branch (Figure 2.11). the orientation of a branch represents a single relationship in the indicated direction (Figure 2.12).

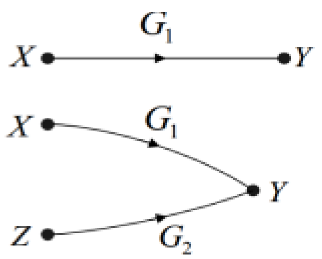


Figure 2.11: Nodes and branches

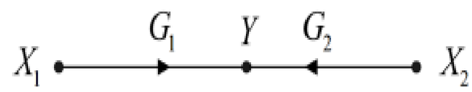


Figure 2.12: Orientation of a branch

$$Y = G_1 X$$
$$Y = G_1 X + G_2 Z$$

$$Y = G_1 X_1 + G_2 X_2 \qquad (2.16)$$

The value of the variable represented by a node is equal to the weighted summation of the variables which are linked to it by a branch directed towards the node (adder), the coefficient of weighting or gain being the value written near the branch (which can be a number or a transfer function). The main elements we will use are:

- ***source node (input):*** is a node associated with an input variable (external variable). It only admits divergent arcs,

- ***sink node (output):*** it represents a variable chosen as output. This knot only admits converging arcs,

- ***string***: is a link between two variables made by following the direction of the arrows and not passing through the same point twice,

- ***Chain of action (CA):*** is a chain starting at an entry node and ending at an exit node that does not pass through the same point twice,

- ***Loop:*** chain starting and ending at the same node,

- ***Chain gain***: is the product of the gains in the chain,

- ***Loop gain:*** is the product of the gains in the loop.

Remark 2.3: a node can always be transformed into an output node (Figure 2.13)

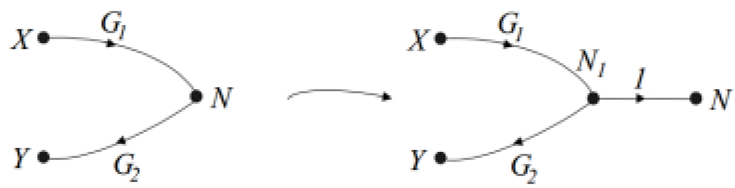


Figure 2.13: Transformed into an output node

Example 2.9: Consider a linear system described by the fluence graph in Figure 2.14.

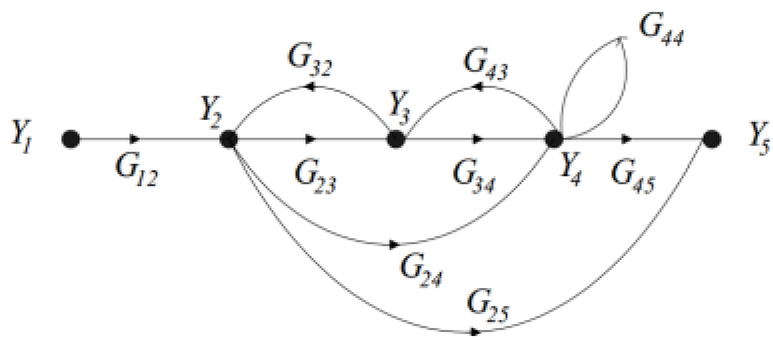


Figure 2.13: Fluence graph of a linear system

We can distinguish the following elements: sources: Y_1, sinks: Y_5, loops: $B_1 = G_{23} G_{32}$, $B_2 = G_{34} G_{43}$, $B_3 = G_{44}$, $B_4 = G_{24} G_{43} G_{32}$ the loops B_1 and B_3 are separate (do not touch each other), chain of action (CA) between Y_1 and Y_5: 3 CA: $G_{12} G_{23} G_{34} G_{45}$; $G_{12} G_{24} G_{45}$; $G_{12} G_{25}$

2.9 Mason's Gain Rul

It is possible to simplify signal flow graph in a manner similar to that of block diagram reduction. But it is also possible, and much less time-consuming, to write down the input-output relationship by inspection from the original signal flow graph. This can be accomplished by the use of general gain formula, known as Mason's Gain Formula.

$$G(P) = \frac{Y(P)}{U(P)} = \frac{1}{\Delta} \sum F_i \Delta_i \qquad (2.17)$$

where :

- F_i = Gain of the *k* th forward path

- Δ : it is the determinant of the graph., $\Delta = 1 - \sum$ (sum of all individual loop gains) +$\sum$ (sum of gain products of the gains of two loops that do not touch each other) -$\sum$ (all products of three-loop gains that do not touch each other) + . . .

- Δ_i: it corresponds to the value of Δ for the part of the graph not touching the ith chain of action.

Example 2.7 : For example 2.9, we have the following transfer function:

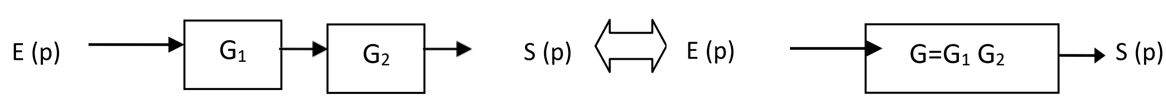
$$G(P) = \frac{Y(P)}{U(P)} = \frac{G_{12}G_{23}G_{34}G_{45} + G_{12}G_{25} + G_{12}G_{24}G_{45}G_{44}G_{34}G_{43}}{1 - G_{23}G_{32} - G_{34}G_{43} - G_{44} - G_{43}G_{32}G_{24} + G_{23}G_{32}G_{44}} \quad (2.18)$$

2.2 Block Diagrams

In general, the block diagram (functional diagram) is constituted of an assembly of rectangles, comparators, derivation points and arrows. The block is a short hand, pictorial representation of the cause-and-effect relationship between the input and output of a physical system.

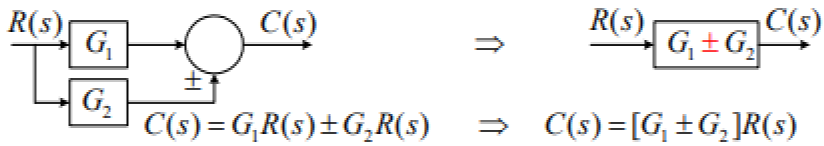
- **Serial systems (cascade)**

Two elements are cascaded when the output of the first is the input of the second. the multiplication of all elements.

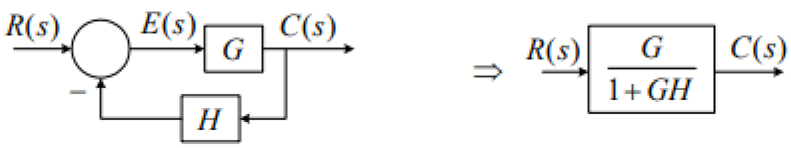


- **Parallel systems**

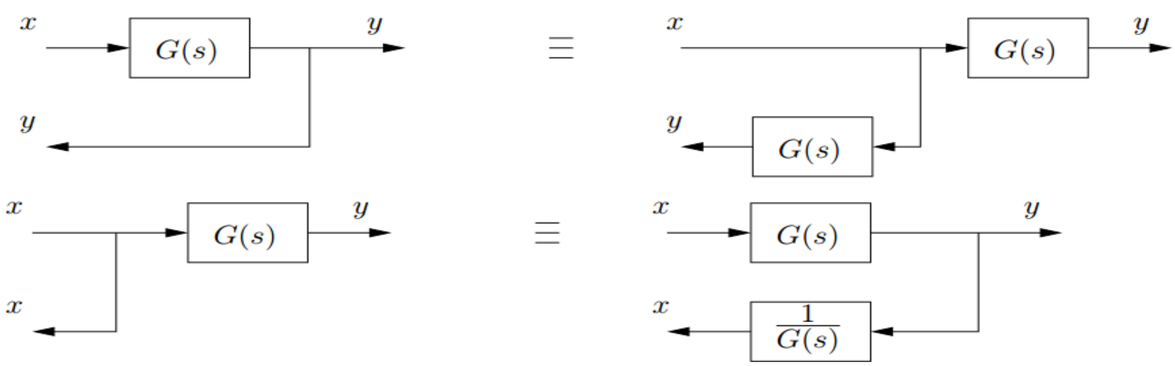
Two elements are in parallel when their inputs are common and their outputs are summed to give a single output



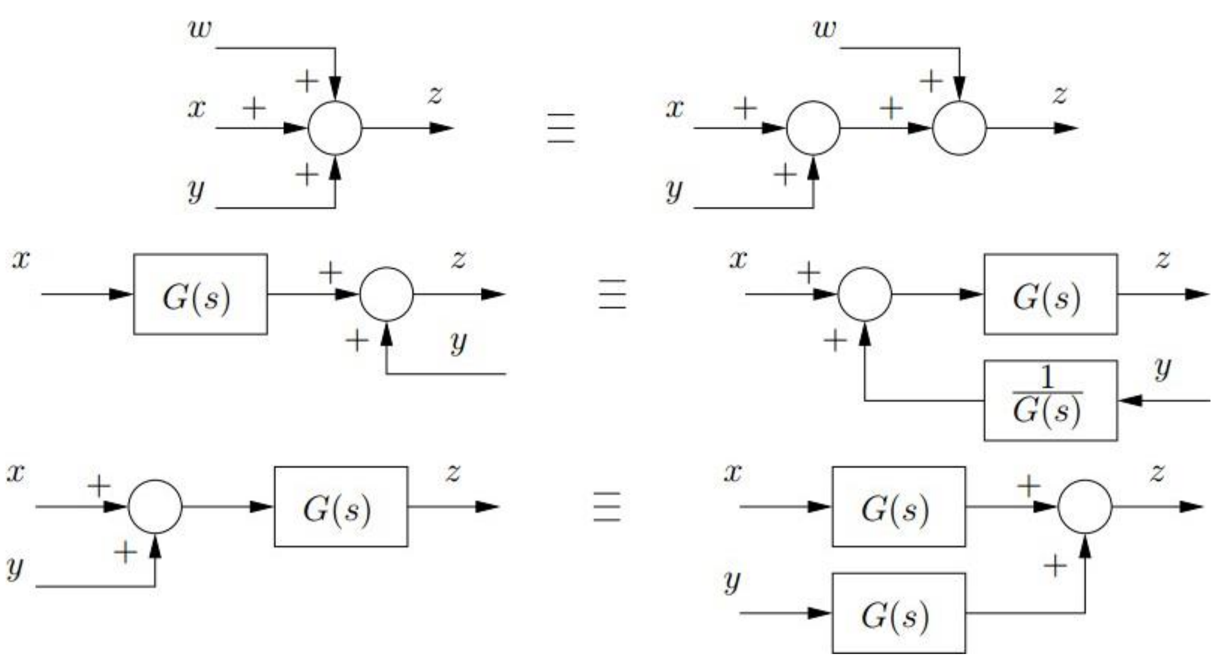
- **Eliminating single feedback loop**



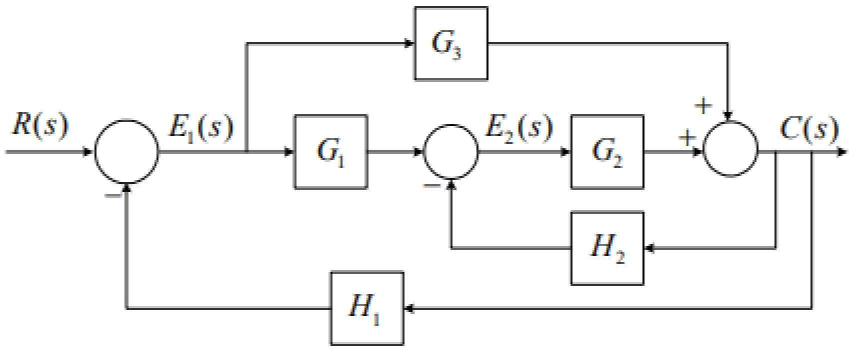
- **Moving a pickoff point**

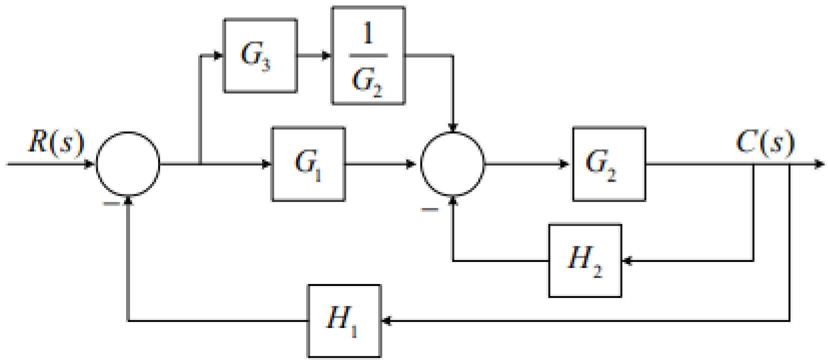


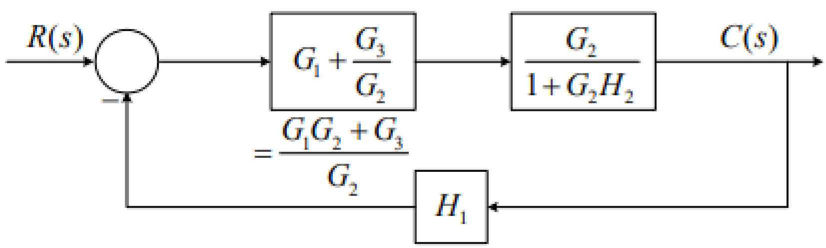
- **Moving a summing point**

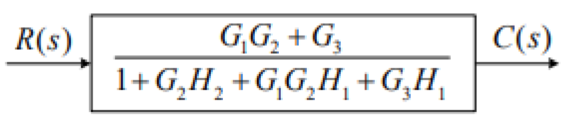


Example Consider the block diagram shown in Figure. Apply the block diagram transformation rules to determine the system transfer function.









Exercices on chapter 2

Laplace transf./ Block Diagrams /Transf. Rules

Exercice 1 : Using the Laplace transform form and properties, determine the Laplace transform for the following functions:

1) $f(t) = 1$; 2) $s(t) = t$, 3) $g(t) = e^{(-3t)}$;

4) $d_1(t) = 2.e^{(-0.5t)}$, 5) $d_2(t) = 5.(1 - e^{(-0.5t)})$, 6) $d_3(t) = 7.t^2$

Exercise 02: Find the inverse Laplace transform of the following expressions:

$D_1(p) = \frac{5}{p}$, $D_2(p) = \frac{5}{p+7}$, $D_3(p) = \frac{5}{p^2} + \frac{9}{p^2+9}$

$F_4(p) = \frac{p+3}{(p+1)(p+2)}$, $F_5(p) = \frac{p+3}{p^2+2p+4}$, $F_6(p) = \frac{p+3}{(p+2)^2}$

Exercise 3: Consider a system of input (t) and output $s(t)$ governed by the following differential equation:

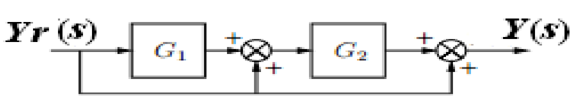
$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = e(t)$$

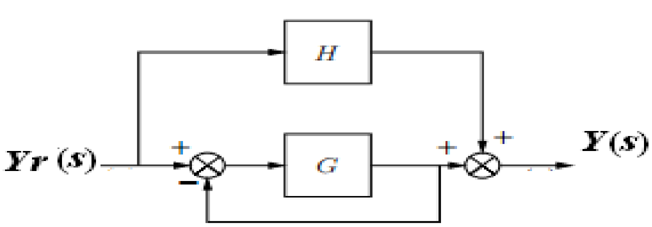
1) Determine the system transfer function: $F(p) = {}^{Y(p)}\!/_{E(p)}$.

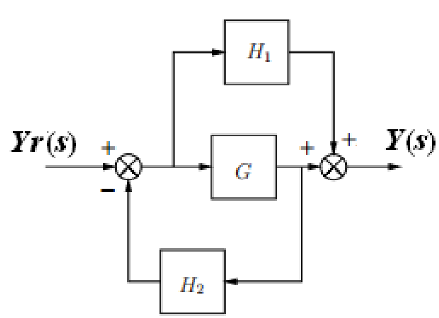
2) Calculate the poles and zeros of this system. with zero initial conditions.

3) Calculate the response of this system $y(t)$ with impulse signal.

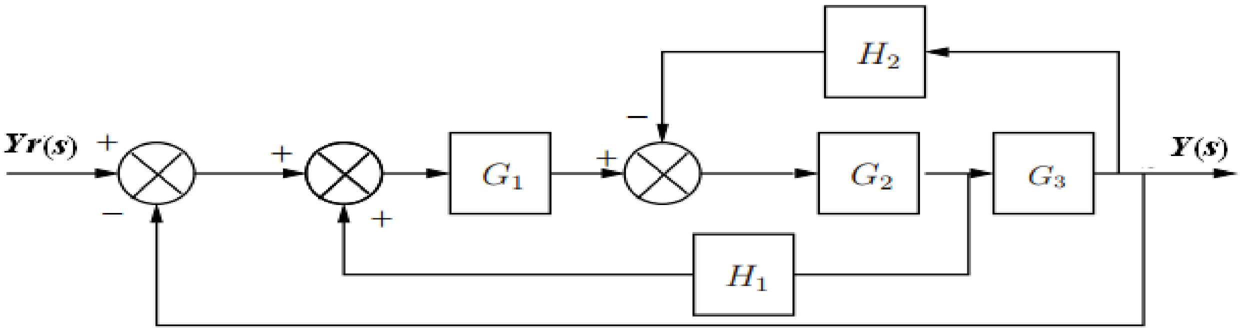
Exercise 4: Simplify the following functional diagrams (calculate the transfer function):

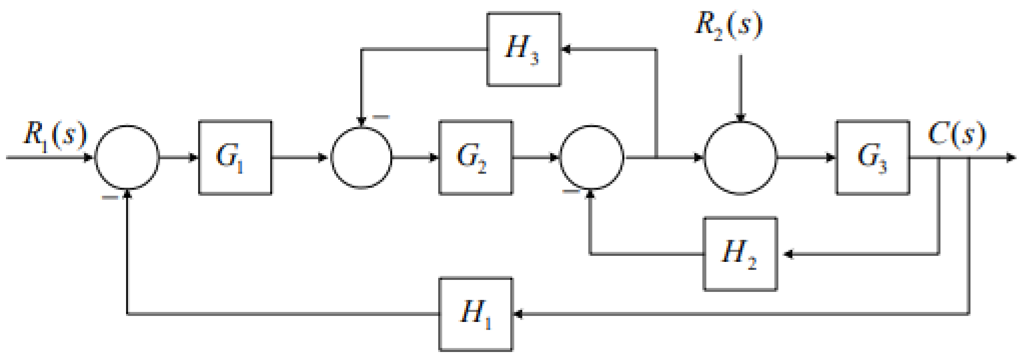


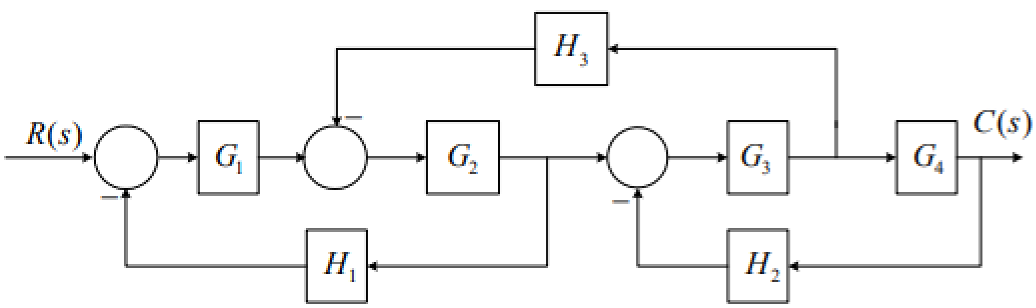




Exercise 5: Determine the global transfer function

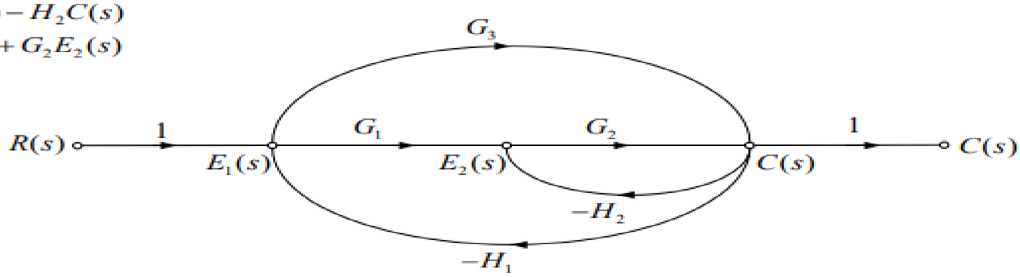






Exercice 06 : Determine the close loop transfer function by using Mason's gain formula. The following signal flow graph is obtained the following equations describing the block diagram.

$$E_1(s) = R(s) - H_1C(s)$$
$$E_2(s) = G_1E_1(s) - H_2C(s)$$
$$C(s) = G_3E_1(s) + G_2E_2(s)$$



Solution: $G(P) = \frac{C(P)}{R(P)} = \frac{G_1G_2+G_3}{1+G_2H_2+G_1G_2H_1+G_3H_1}$

Chapter 3

Temporal responses of linear systems

Chapter 3

Temporal responses of linear systems

The temporal response study of a system consists in determining its response (output) y(t) to an input signal u(t) which varies as a function of time. The performance of dynamic systems in the time domain can be defined in terms of the time response to standard test inputs (Dirac pulse, step input, sinusoidal input..). The purpose of this chapter is to study the temporal responses of first and second order systems.

3.1 Temporal response of First-order Systems

A linear system of first-order invariant with continuous time is governed by a differential equation of the first degree with constant coefficients. Its transfer function therefore has at most one zero and one pole.

$$\frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t) \tag{3.1}$$

Taking the Laplace transform, for the zero initial condition

$$PY(p) - y(0) + a_0 Y(p) = b_0 U(p) \tag{3.2}$$

$$Y(p) = \frac{b_0}{p+a_0} U(p),\ So:\ \ G(p) = \frac{K}{\tau p+1} \tag{3.3}$$

Where : $\tau = \frac{1}{a_0}$: it is the time constant of the system, $K = \frac{b_0}{a_0}$: it is the static gain.

3.1.2 impulse response

the system response to an input *u(t) = δ(t).* The Laplace transform of the input is then *U(p) = 1*. The expression for the output of the system is:

$$Y(p) = G(p) \overset{TL^{-1}}{\Longrightarrow} y(t) = TL^{-1}\left(\frac{K}{\tau p+1}\right) = \frac{K}{\tau}\ e^{-t/\tau}.u(t) \tag{3.4}$$

Figure 3.1 shows the form of this response. It is possible to determine the two parameters of the system from the intersection of the slope of the response at the origin with the abscissa axis, and the value of the output signal at the origin.

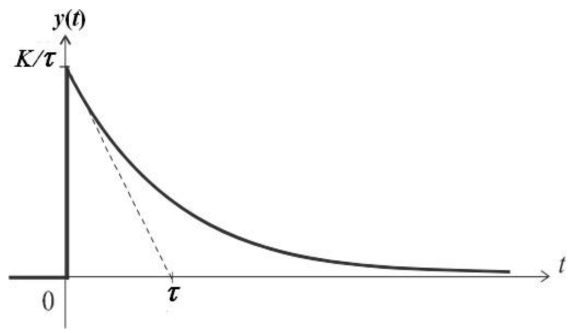


Figure 3.1: Impulse response of a first-order system

3.1.2 Unit Step response

If the input is a unit step function ***u (t) = 1.u(t)*** , i.e., ***U(p) =1/P*** , the response is

$$Y(p) = \frac{b_0}{p+a_0} . \frac{1}{p} = \frac{\frac{b_0}{a_0}}{p} - \frac{\frac{b_0}{a_0}}{p+a_0} \tag{3.5}$$

Taking the inverse Laplace transform :

$$\xRightarrow{TL^{-1}} y(t) = TL^{-1}\left(\frac{\frac{b_0}{a_0}}{p} - \frac{\frac{b_0}{a_0}}{p+a_0}\right) = \frac{b_0}{a_0}(1 - e^{-a_0 t}).u(t) = K\left(1 - e^{-t/\tau}\right).u(t) \tag{3.6}$$

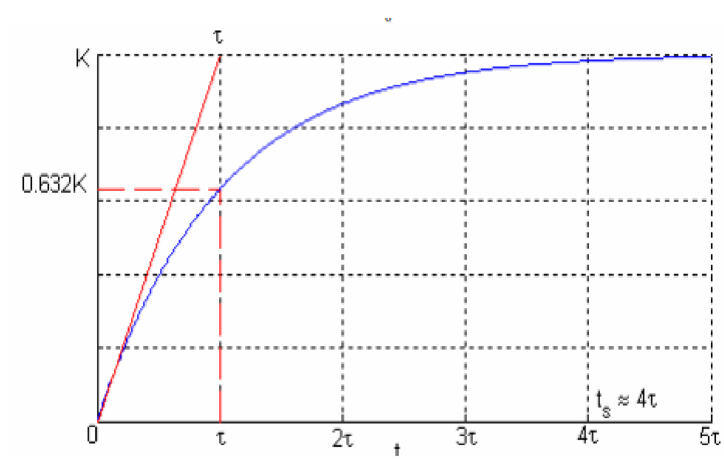


Figure 3.2: Step response of first-order system

- for $t = \tau$, $y(\tau) = K(1 - e^{-1}) \approx 0.63K$.

- The value of τ is then determined for a variation of the output of 63% of its total variation.

3.2 Temporal response of second-order systems

The standard form of the second-order transfer function is :

$$G(p) = \frac{Kw_n^2}{p^2+2\xi w_n p+w_n^2} \tag{3.7}$$

The characteristic equation is given by :

$$p^2 + 2\xi w_n s + w_n^2 = 0 \tag{3.8}$$

- For ***ζ >1,*** the poles of the transfer function are :

$$p_1 = -\xi w_n + w_n\sqrt{\xi^2 - 1}\,, p_2 = -\xi w_n - w_n\sqrt{\xi^2 - 1} \tag{3.9}$$

For the unit step response, and the p-domain step response can be written as

$$Y(p) = \frac{w_n^2}{(p-p_1)(p-p_2)}.\frac{1}{p} = \frac{1}{p} + \frac{A}{(p-p_1)} + \frac{B}{(p-p_2)} \tag{3.10}$$

Thus the inverse Laplace transform is

$$\xrightarrow{TL^{-1}} y(t) = TL^{-1}\left(\frac{1}{p} + \frac{A}{(p-p_1)} + \frac{B}{(p-p_2)}\right) = (1 - Ae^{-t/\tau_1} + Be^{-t/\tau_2}).u(t) \tag{3.11}$$

- The underdamped response (ζ <1) to a unit step input is given by :

$$Y(p) = \frac{w_n^2}{(p-\xi w_n - jw_d)(p - \xi w_n + jw_d)}.\frac{1}{p} = \frac{1}{p} + \frac{A}{(p-\xi w_n - jw_d)} + \frac{B}{(p-\xi w_n + jw_d)}$$

$$y(t) = TL^{-1}\left(\frac{1}{p} + \frac{A}{(p-\xi w_n - jw_d)} + \frac{B}{(p-\xi w_n + jw_d)}\right) = 1 - \frac{1}{\sqrt{1-\xi^2}}e^{-\xi w_n t}.sin(w_d t + \theta)$$

(3.12)

The responses is underdamped as shown in Figure 3.3.

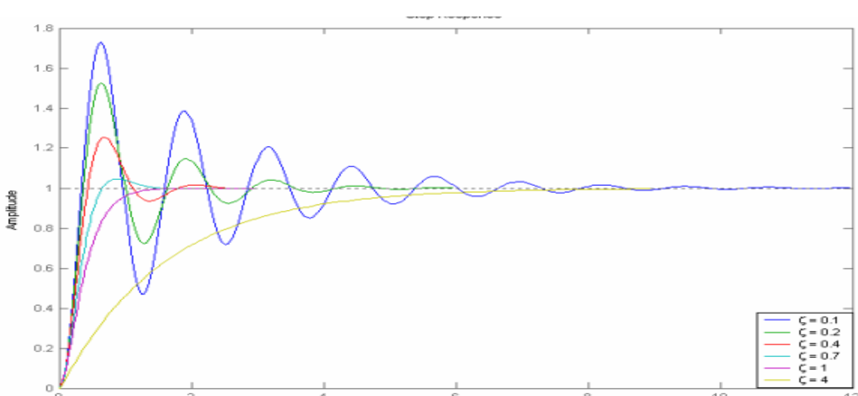


Figure 3.3: Response of a second-order system to a step input for different values of ζ

3.2.1 Time -domain performance specification of second-order systems

The performance criteria that are used to characterize the transient response to a unit step input include rise time, peak time, overshoot and settling time. The underdamped step response for a second order system is shown in Figure 3.4.

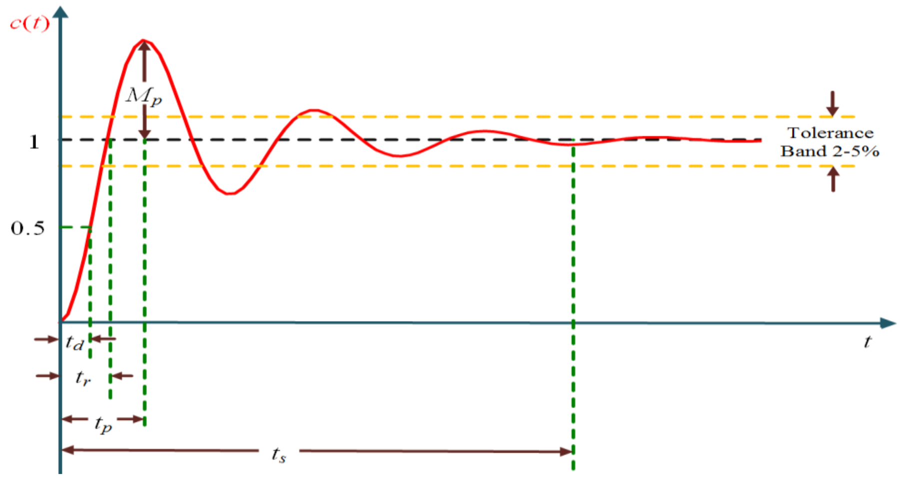


Figure 3.4: Step response (underdamped case) of a second order control system

- The peak time is obtained by setting the derivative of *y(t)* to zero:

$$t_p = \frac{\pi}{w_n\sqrt{\xi^2-1}} \tag{3.13}$$

- the rise time defined when the output signal reaches its final value for the first time,

$$t_m = \frac{\pi}{2w_n\sqrt{\xi^2-1}} \tag{3.14}$$

- For underdamped systems the percent overshoot P O or (M_p) . is defined as

$$P.O. = \frac{\text{Maximum value - Final value}}{\text{Final value}} \times 100 = \frac{M_{pt} - C_{ss}}{C_{ss}} \times 100$$

Therefore, the percent overshoot is : $P.O(\%) = 100 * \ e^{-\pi\xi\sqrt{1-\xi^2}}$ (3.15)

- settling time is the time required for the step response to reach within 2% of its final value:

$$t_s \approx \frac{4}{\xi w_n} \tag{3.16}$$

- The pseudo-period defined by the period of the damped sinusoid :

$$T_p = \frac{2\pi}{w_n\sqrt{\xi^2-1}} \quad (3.17)$$

Example 3.3:

Consider the following transfer function:

$\mathrm{G(s)} = \frac{9}{S^2+3S+9}$, we can deduce: $w^2=9$, $w=3$, $2\xi w=3$, $\xi=0.5$

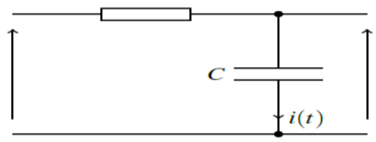
We then obtain: $P.O\%=16.4\%$, $T_p=2.4s$, $t_p=1.2S$, $t_m=0.6S$

,

Exercises of chapter 3

Temporal responses of linear systems

Exercise 01: Consider the following RC Circuit:

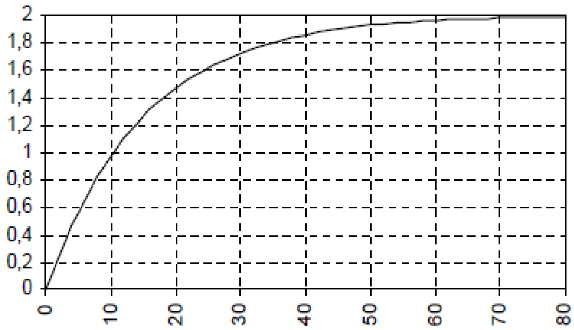


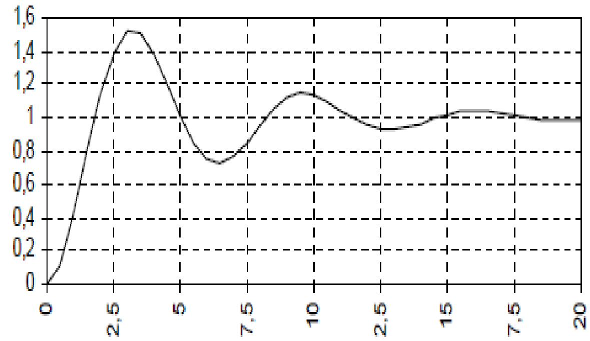
1. Determine the transfer function of this system. deduce the static gain K as well as the time constant.

Exercise 02: A linear system is characterized by the equation: $0.5\frac{dy(t)}{dt} + 1y(t) = 15u(t)$

Give the expression of the transfer function of the system, what are the type, the order, the gain and the time constant(s) of this system and the response time**.**

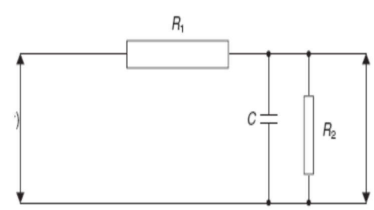
Exercise 03: Identify the parameters of the systems whose responses at a unit level are





Exercise 04:

1) Determine the transfer function linking the output $v_2(t)$ to the input $v_1(t)$ of the electric circuit given by the figure



2) For C=2µF and R1=R2=1MΩ, calculate the static gain K as well as the time constant .

3) Give the step response of this system.

Exercise 5: onsider a system of input x(t) and output y(t) governed by the differential equation:

$Y''(t) + 4y'(t) + 8y(t) = 2x(t)$ *with* $y'(0) = 0;\ y(0) = 0$

1. Determine the transfer function G(s) of the system.
2. Determine the order of the system, the static gain K, the damping coefficient z, the proper pulsation
3. Deduce the regime type for the step response. Calculate the value P.O% of the first overshoot, the time t_P of the first overshoot, the rise time T_m , the response time T_r *and* T_p.

Chapter 4.

Frequency responses of linear systems

Chapter 4.

Frequency responses of linear systems

The frequency analysis is interested in the behavior of the system considered in steady state, in response to a sinusoidal input, i.e. its harmonic response, according to the amplitude and the frequency of the input signal. This chapter presents methods for studying the frequency response of a First-Order System and of second-order systems whose transfer function is known.

4.1 Definition:

In general, the harmonic response of a system is determined by the complex number $G(j\omega)$ which is called the harmonic transfer function (or complex transfer function). The complex transfer function $G(j\omega)$ is obtained simply by replacing the Laplace variable ***P*** by ***jω*** in the expression of the transfer function $G(p)$. The frequency response of a linear system with transfer function $G(p)$ to a sinusoid input $u(t) = X\sin(\omega t)$ (X amplitude and pulsation ω) to sinusoid output $y(t) = Y\sin(\omega t+ \varphi)$ with the same pulsation ω and whose amplitude Y and phase φ with: $Y = X\,|G(j\omega)|$ ($|G(j\omega)|$ is the system gain) and $\varphi = arg(G(j\omega))$ (phase between output and input).

The magnitude $G(j\omega)$ and its angle θ for all ω constitute the system frequency response, and they provide a significant insight for the analysis and design of control systems. In the following, we study the Bode and Nyquist diagrams.

4.2 Bode diagram

This representation consists of two plots. The first represents the gain of the system in decibels (dB), when the pulsation ω (rad/s) varies:

$$G_{dB} = 20log|G(jw)| \tag{4.1}$$

The second shows the phase φ in degrees or rad, when the pulsation ω (rad/s) varies:

$$\varphi_t = arg\{G(jw)\} \tag{4.2}$$

The pulsation is on a logarithmic scale on the abscissa. It is common to place the phase plot below the amplitude plot as shown in Figure 4.1

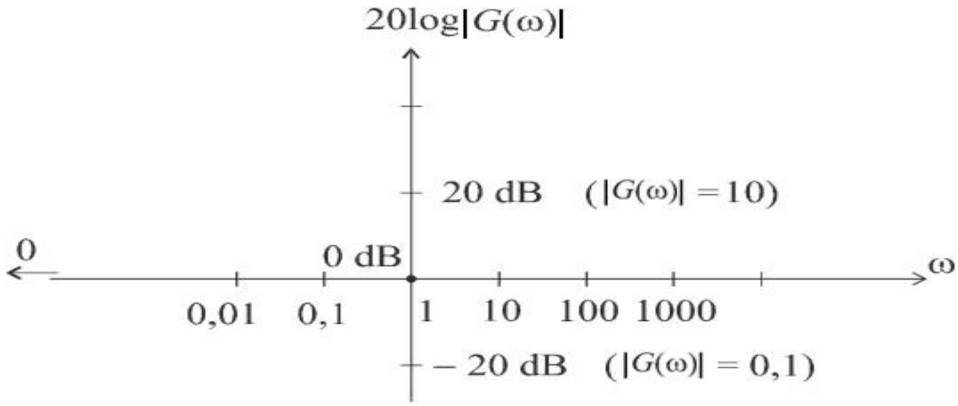


Figure 4.1: Logarithmic scale of the Bode diagram

In general, we plot the values of ω directly on the abscissa axis, and placing the pulsation ω = 1 at the origin of this axis since it corresponds to log10 ω = 0 as shown in Figure 4.2.

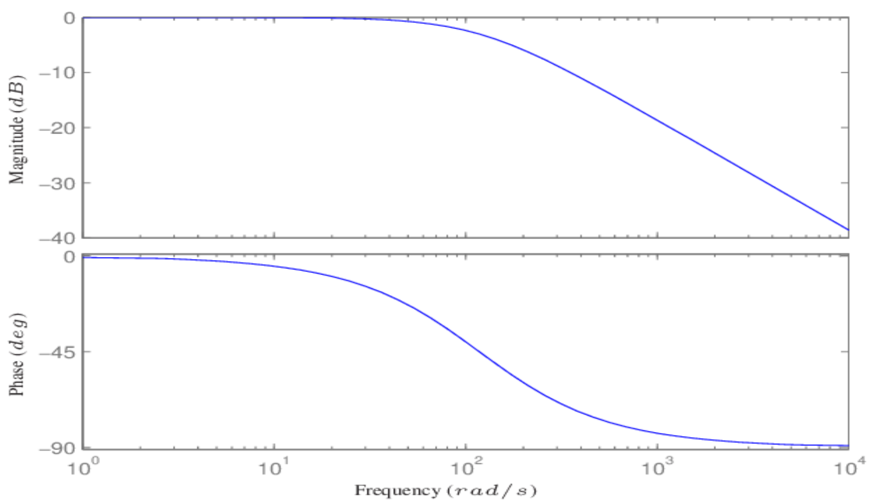


Figure 4.2: Bode diagram

the Magnitude in dB of the system considered is the sum of the elementary moduli in dB. On the other hand, the phase of the system is the sum of the arguments.

4.2.1 Bode diagram of First-Order System

Consider first the frequency response of a first-order system with the following transfer function

$$G(jw) = \frac{K}{1+j\tau w} \tag{4.3}$$

$$|G(jw)| = \frac{K}{\sqrt{1+(\tau w)^2}}, \quad \varphi(w) = actg(\tau w) \tag{4.4}$$

The design of the asymptotic diagram consists in separating the pulsation space into two domains, depending on whether *w* is very large or very small in front of 1.

- If $\tau w \ll 1$, therefore $\omega <<1/\tau$: $|G(jw)| = \frac{K}{\sqrt{1+(\tau w)^2}} \approx K$

  Therefore $|G(jw)|_{dB} = 20log(K)$.

  The gain curve follows a horizontal asymptote with an ordinate of 20log K,

- If τω $\tau w \gg 1$, therefore $w \gg 1/\tau$: $|G(jw)| \approx \frac{K}{\sqrt{(\tau w)^2}}$

  Therefore $|G(jw)|_{dB} = 20log(K) - 20log(\tau w)$

This corresponds to is a straight line with slope -20 dB/decade, which means that the gain decreases by 20 dB when the pulsation is multiplied by 10.

- At the pulsation $w_c = 1/\tau$, $|G(jw)|_{dB} = -3$dB which means that the real trace is therefore 3 dB below the asymptotes (see Figure 4.3). This pulse is called cutoff pulse.

Concerning the phase shift, we have:

- for $\tau w \ll 1$, therefore $\omega <<1/\tau$: $\varphi(w) \approx 0$, the phase curve is a horizontal asymptote

at the ordinate 0 rad,

- for $\tau w \ll 1$, therefore $\omega <<1/\tau$: $\varphi(w) \approx -\pi/_2$ then the phase curve is an asymptote

horizontal to the ordinate -π/2 rad.

At the pulsation $w_c = \frac{1}{\tau}$, we obtain a phase shift of -π/4 rad.

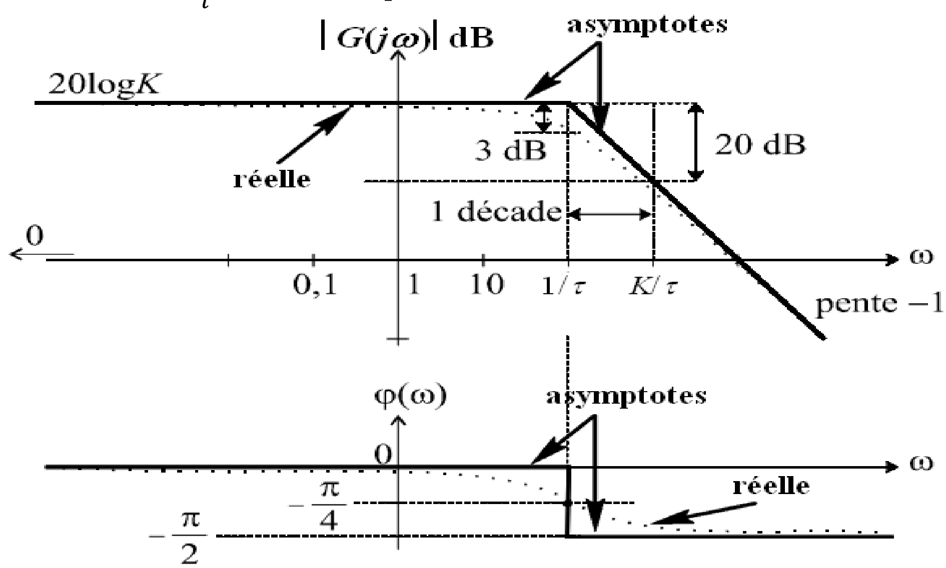


Figure 4.3: Bode diagram of a first-order system

4.2.2 Frequency response of second-order systems

Now consider the standard second-order transfer function :

$$G(s) = \frac{Kw_n^2}{s^2+2\xi w_n s+w_n^2} \tag{4.5}$$

$$G(jw) = \frac{K}{1-\frac{w^2}{w_n^2}+j\frac{2\xi w}{w_n}} \tag{4.6}$$

So : $|G(jw)| = \frac{K}{\sqrt{\left(1-\frac{w^2}{w_n^2}\right)^2+\left(\frac{2\xi w}{w_n}\right)^2}}, \quad \varphi(w) = -arct\left(\frac{\frac{2\xi w}{w_n}}{1-\frac{w^2}{w_n^2}}\right)$ (4.7)

The asymptotes are obtained in an identical way to that which was made for the canonical system of the first order.

- If $w \to 0$, therefore : $|G(jw)| \approx K$,so $|G(jw)|_{dB} = 20log(K)$, $\varphi(w) \approx 0$.
- If $w \to +\infty$, therefore :

$$|G(jw)| \approx \frac{Kw_n^2}{w^2} \text{ ,so } |G(jw)|_{dB} = 20log(K) + 40log(w_n) - 40log(w) \text{ , } \varphi(w) \approx -\pi.$$

The gain curve therefore follows an asymptotic direction which is a straight line with a slope −40 dB/decade at high frequencies.

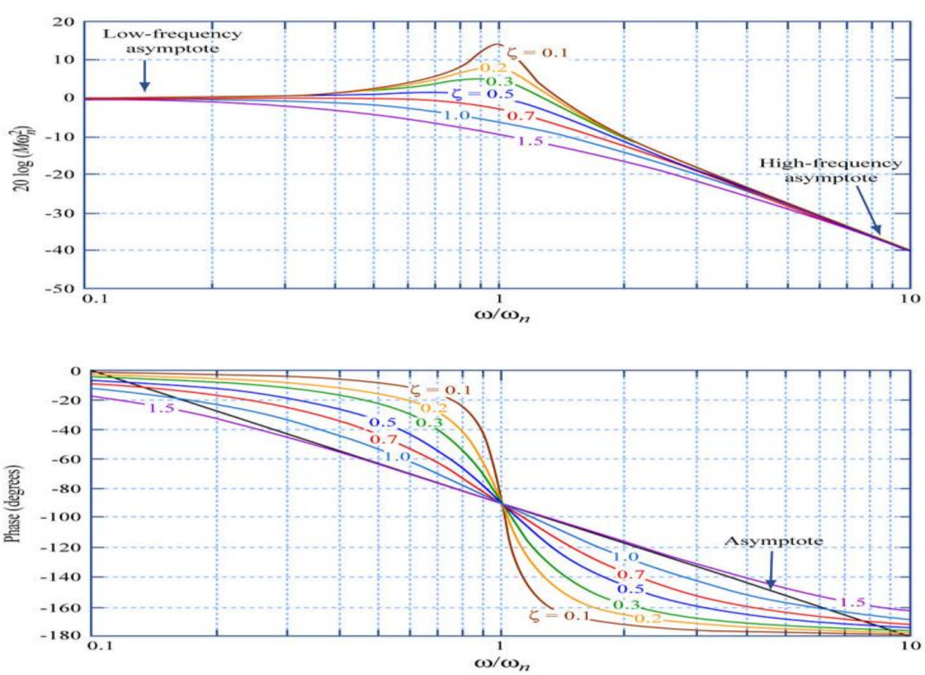
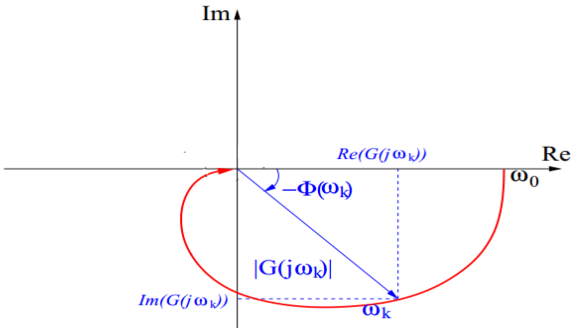


Figure 4.4: Bode diagram of a second-order system

4.3 Diagram of Nyquist

The Nyquist diagram corresponds to the representation in the complex plane of the complex transfer function *G(jω) = Re(G(jω)) + Im(G(jω))* when ω varies from 0 to +∞ (Figure 4.5). This diagram is always oriented in the direction of increasing ω



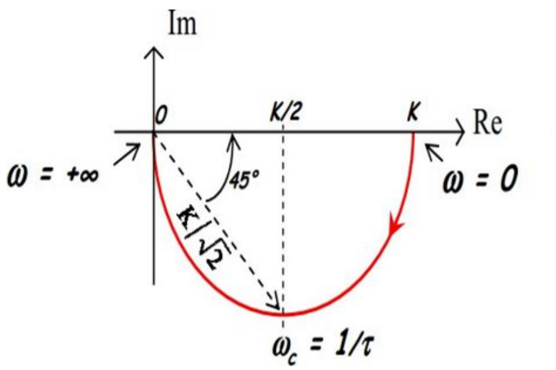


Fig. 4.5 Diagram of Nyquist **Figure 4.6:** Nyquist diagram of a first-order system

4.3.1 Nyquist diagram of First-order systems

To draw the Nyquist diagram, it is necessary to decompose G(jω) into real part and imaginary part, we have:

$$G(jw) = \frac{K}{1+j\tau w} = \frac{K(1-j\tau w)}{(1+j\tau w)(1-j\tau w)} = \frac{K(1-j\tau w)}{(1+j\tau^2 w^2)}$$

Therefore : $$G(jw) = \frac{K}{(1+j\tau^2 w^2)} + j\frac{K\tau w}{(1+j\tau^2 w^2)} = X + jY \quad (4.8)$$

The Nyquist locus of a first-order system is a semicircle with center (K/2.0) and radius K/2 as shown in Figure 4.6.

4.3.2 Nyquist diagram of Second-order systems

The Nyquist diagram of a second-order system is defined from the following expression:

$$G(jw) = \frac{K\left(1-\left(\frac{w}{w_n}\right)^2\right)}{\left(1-\left(\frac{w}{w_n}\right)^2\right)^2+4\xi\left(\frac{w}{w_n}\right)^2} - 2j\frac{K\xi\left(\frac{w}{w_n}\right)}{\left(1-\left(\frac{w}{w_n}\right)^2\right)^2+4\xi\left(\frac{w}{w_n}\right)^2} \quad (4.9)$$

The figure 4.9 illustrates the plots obtained according to the different cases. Note that all the tracks have the same starting and ending points. At high frequencies (ω tends to infinity), the asymptotic analysis shows that the phase tends to –π i.e. the curve is tangent to the real axis, from the left, at point 0. A possible resonance ($\xi < 0.7$) is observed by a curve which then presents a maximum gain .

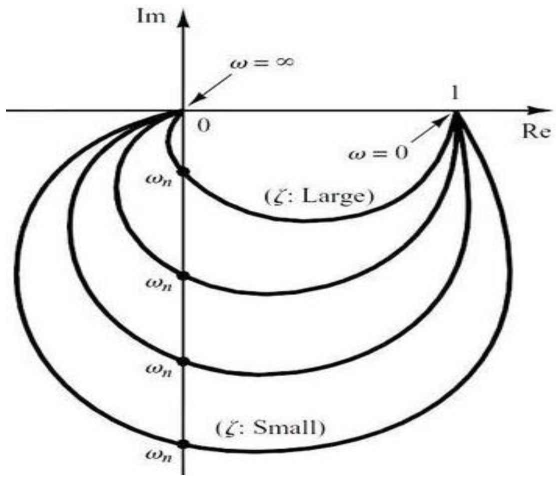


Figure 4.7: Nyquist diagram of a second-order system

4.4 Phase margin and gain margin

4.4.1 Phase margin

The phase margin is defined by [4, 7, 9, 11]:

$$\varphi_t = \varphi(w_c) + 180^o \tag{4.10}$$

where w_c is the cut-off frequency defined by :

$$20log|G(jw)|_{w=w_c} = 0dB \quad So: \ |G(jw)|_{w=w_c} = 1 \tag{4.11}$$

4.3.2 Gain margin

The gain margin G_t is the factor by which the loop gain is defined by [4, 7, 9, 11]:

$$M = \frac{1}{|L(w_\pi)|} \ So \ MdB = -20log\left|G(jw)\right|_{w=w_\pi}, \tag{4.12}$$

Where $w_\pi : \ \varphi(w)_{w=w_\pi} = -180^o$ (4.13)

The gain and phase margins defined previously can also be represented on the Bode diagram (Figures 4.8 and 4.9)

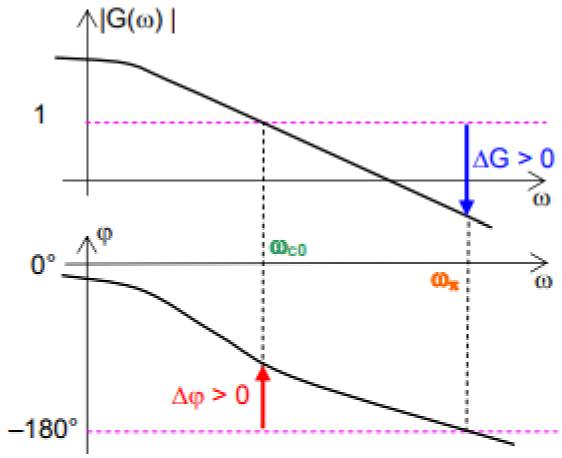


Fig. 4.8: illustration of profit margins and phase on the Bode diagram

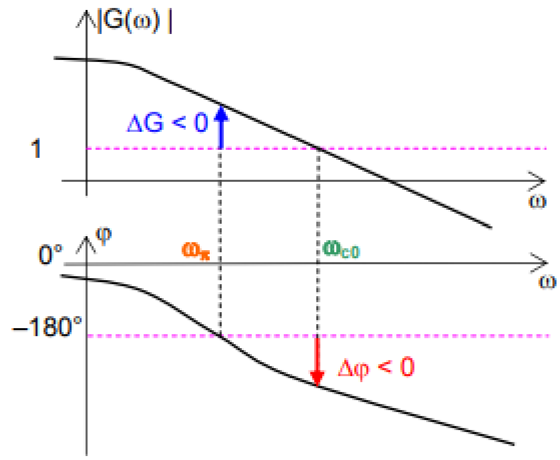


Fig. 4.9: illustration of profit margins and phase on the Bode diagram

Exercises of chapter 4

Frequency responses of linear systems

Exercice 02 : Soit un système avec une fonction de transfert: $H(p) = \frac{5}{p+5}$

1- Write *H(p)* in the form: $H(jw) = \frac{k}{(1+j\tau w)}$

2- Deduce the cut-off pulse $\boldsymbol{w_c}$

3- Give the modulus and the phase shift of the transfer function *H (jw)*: $|H(jw)|_{dB}, \varphi(w)$

4- Fill module variation $|H(jw)|_{dB}$ and phase shift $\varphi(w)$ according to the following table:

w	$w \ll w_c$	$w = w_c$	$w \gg w_c$
$\|H(jw)\|_{dB}$			
$\varphi(w)$			

5- Plot the asymptotic and real *Bode* diagram ($|H(jw)|_{dB}$ et $\varphi(w)$) :

6- Same steps for: $H(p) = \frac{10000}{(p+10)^3}$

Exercise 02: Consider a system with a transfer function: $T(p) = \frac{1+0.01P}{P^2+2P+1}$

1- Write *T(p)* in the form of two transfer functions: $T(p) = T_1(p)\, T_2(p)$

2- Define the modulus and the phase shift in Bode representation.

3- Calculate the cut-off pulsations and fill in the table

4- Plot the asymptotic and real Bode diagrams.

Exercice 3 : Consider a system with a transfer function: $F(p) = \frac{10000}{p(p+10)(p+100)}$

- Draw the asymptotic Bode diagram of this system and deduce its Nyquist locus.

Exercice 4 : Consider a system with a transfer function: $F(p) = \frac{5}{\left(1+\frac{p}{100}\right)^3}$

- Calculate the phase margin and gain margin.

Chapter 5

Stability of Linear Control Systems

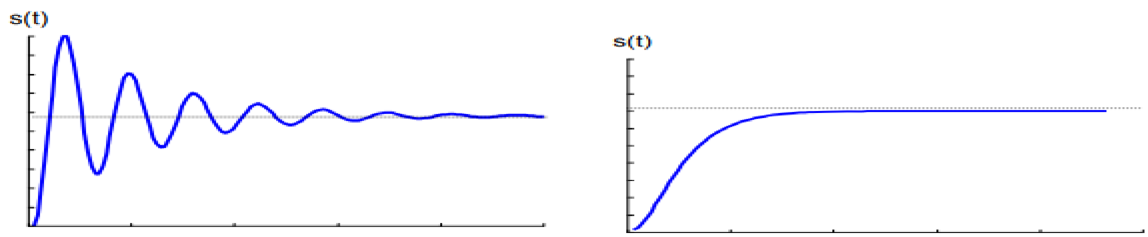
Chapter 5

Stability of Linear Control Systems

5.1 Definition of stability

The definition of stability implies that only the forced response remains as the natural response approaches zero. A system is stable if every bounded input yields a bounded output. We call this statement a bounded-input, bounded-output (BIBO) definition of stability. A linear system is stable if, after having subjected its input to a sudden variation (unit step, for example):

- the movement initiated by its output remains limited in amplitude (i.e. the output retains a finite value)
- this movement is damped more or less quickly and the output tends towards a state of equilibrium



Figuer 5.1: Stable systems, a: with oscillatory b: non-oscillatory

5.2 Method of Determine Stability

Stability of a system is solely based on the characteristic equation of the system. Consider a simple feedback system shown in Fig. 2

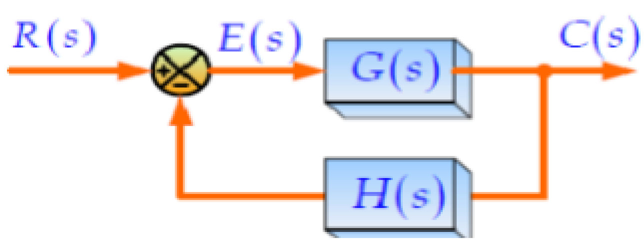


Figure 5.2 closed-loop control system

The transfer function is given as :

$$\frac{C(s)}{R(S)} = \frac{G(s)}{1+G(S)H(S)} \tag{5.1}$$

The characteristic equation is of the above system :

$$1 + G(S)H(S) = 0 \tag{5.2}$$

The roots of the characteristic equation are called closed loop poles.

Example 5.1: Study the stability of the systems :

$G(s) = \frac{20}{(s+1)(s+2)(s+3)}$, Poles at s = -1, -2, -3; therefore stable system

$G(s) = \frac{20(s+1)}{(s-1)(s^2+2s+2)}$, Poles at $s = +1$, therefore unstable system

$G(s) = \frac{20(s-1)}{(s+2)(s^2+4)}$ Poles at $s = -2, \pm j2$, therefore Marginally unstable system

The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the s-plane. The system is stable if all poles are located in left hand side *(LHS)*of the imaginary axis of *s*-plane. If one of the poles is located or placed on the imaginary axis *(RHS)*, then the system is said to marginally stable or marginally unstable.

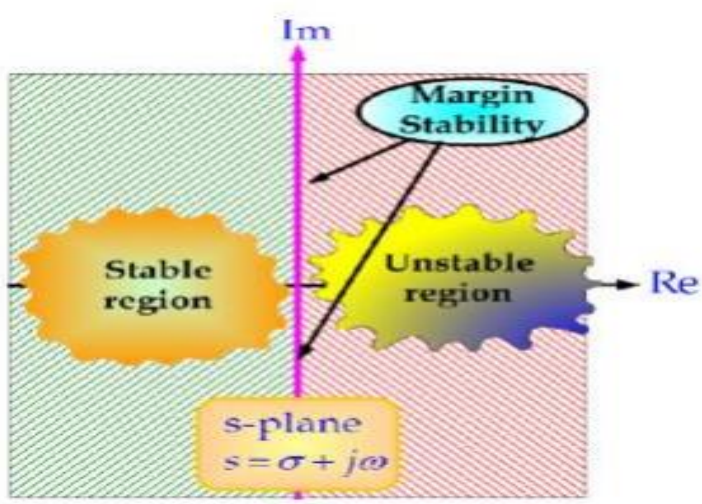


Figure5.3 Stability condition based on the location of the closed loop poles

5. 3 Stability criterion using Routh-Hurwitz and Nyquist

However, some time in design the roots are unknown or there are some variable parameters embedded in the characteristic equation of the system. These two criteria are normally been used to determine the system stability without solving the roots of characteristic equation:

- Routh-Hurwitz criterion
- Nyquist criterion

5.3.1 Routh-Hurwitz Criterion

For a transfer function,

$$G(s)=\frac{K\left(b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s^1+b_0\right)}{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s^1+a_0} \qquad \text{where } n\geq m \tag{5.3}$$

Routh table is:

$$\begin{array}{lcllll}
s^n & : & a_n & a_{n-2} & a_{n-4} & \dots & 0 \\
s^{n-1} & : & a_{n-1} & a_{n-3} & a_{n-5} & \dots & 0 \\
s^{n-2} & : & c_1 & c_2 & c_3 & \dots & 0 \\
s^{n-3} & : & d_1 & d_2 & d_3 & \dots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \\
s^1 & : & g_1 & 0 & & & \\
s^0 & : & h_1 & 0 & & &
\end{array}$$

Where : $c_1=\dfrac{a_{n-1}a_{n-2}-a_n a_{n-3}}{a_{n-1}}$; $\quad c_2=\dfrac{a_{n-1}a_{n-4}-a_n a_{n-5}}{a_{n-1}}$

$$c_3=\frac{a_{n-1}a_{n-6}-a_n a_{n-7}}{a_{n-1}}, \quad d_1=\frac{c_1 a_{n-3}-c_2 a_{n-1}}{c_1}; \quad d_2=\frac{c_1 a_{n-5}-c_3 a_{n-1}}{b_1} \tag{5.4}$$

1. If all elements in the first column have the same sign, the system is stable and all poles are on ***LHS*** of *s*-plane.
2. The number of changes of signs is equal to the number of unstable poles or poles on ***RHS*** of *s*-plane. System is unstable.
3. If the first element of a row is zero, change it with a small positive value $'\varepsilon'$.
4. If all elements of a row are zero, poles are on the imaginary real axis or on ***LHS*** of the imaginary axis of *s*-plane.

Example 5.2: Determine the stability of the system using Routh-Hurwitz criterion. If the characteristic equation of a system is,

$$s^5+s^4+6s^3+5s^2+12s+20=0$$

Routh's Table

$$\begin{array}{lcccc} s^5: & 1 & 6 & 12 & 0 \\ s^4: & 1 & 5 & 20 & 0 \\ s^3: & 1 & -8 & 0 & \\ s^2: & 13 & 20 & 0 & \\ s^1: & \frac{-124}{13} & 0 & & \\ s^0: & 20 & 0 & & \end{array}$$

/ 2 sign changes in the first column, Therefore, the system is unstable.

Example 5.3 : Study the stability of the systems using Routh-Hurwitz criterion :

$$s^5 + s^4 + 5s^3 + 5s^2 + 12s + 10 = 0$$

Routh's Table

$$\begin{array}{lcccc} s^5: & 1 & 5 & 12 & 0 \\ s^4: & 1 & 5 & 10 & 0 \\ s^3: & \varepsilon & 2 & 0 & \\ s^2: & \frac{5\varepsilon-2}{\varepsilon} = -\infty & 10 & 0 & \\ s^1: & \frac{-2\infty-10\varepsilon}{-\infty} \approx 2 & 0 & & \\ s^0: & 10 & 0 & & \end{array}$$

/

There are 2 sign changes in the first column, thus the system has 2 poles on RHS of *s*-plane. Therefore, the system is unstable.

Example 5.4 : Study the stability of the systems using Routh-Hurwitz criterion :

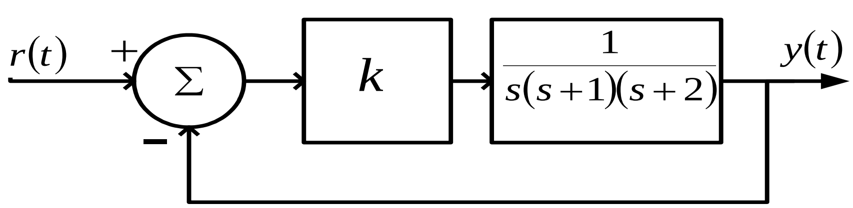
$$s^3 + 3s^2 + 2s + 5 = 0$$

Routh's Table:

$$\begin{array}{lccc} s^3: & 1 & 2 & 0 \\ s^2: & 3 & 5 & 0 \\ s^1: & 1/3 & 0 & \\ s^0: & 5 & 0 & \end{array}$$

/ There is no sign change in the first column, hence the system stable.

Example 5.5 : Determine the range of *k* so that the system will remain stable.



Characteristic equation,

$$1+GH=0$$

$$1+\frac{k}{s(s+1)(s+2)}=s(s+1)(s+2)+k$$

$$=s^3+3s^2+2s+k=0$$

Routh's Table

$$\begin{array}{llll} s^3: & 1 & 2 & 0 \\ s^2: & 3 & K & 0 \\ s^1: & \frac{6-K}{3} & 0 & \\ s^0: & K & 0 & \end{array}$$

The system will remain stable if there is no sign change in the first column, or

$$\frac{6-K}{3}>0 \quad \text{and} \quad K>0$$

$$\frac{6}{3}>\frac{K}{3} \qquad \therefore \qquad K<6$$

Hence, overall $0<K<6$ is the range of *K* for the system to remain stable.

5.4 Graphic criterion of stability

The graphic criteria make it possible to study the stability of a servo-control from the harmonic representation of the open loop transfer function (FTBO). This section presents the Graphic criterion which can be deduced from the *Bode* and *Nyquist* diagrams to judge the stability of the system control (feedback loop).

Consider the system control described by the following functional diagram:

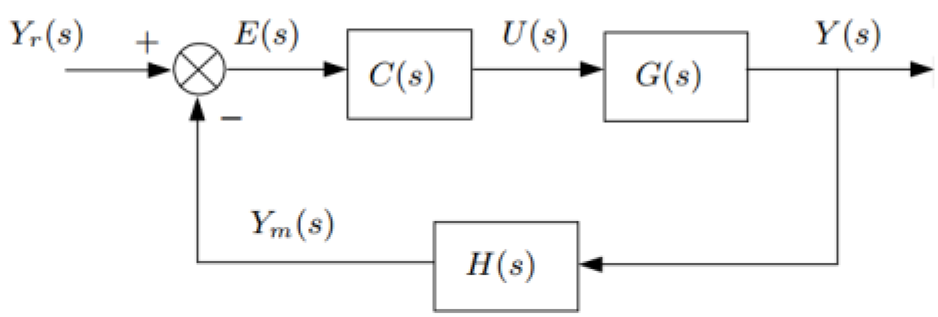


Figure 5.4: Ideal continuous-time of the system control

The open loop transfer function (FTBO) is *C(s)G(s)H(s)* . This is the input-output relationship when the return chain is not connected to the comparator. The closed loop transfer function is :

C(s)G(s)/(1+H(s) C(s)G(s)) (5.5)

Theorem 5.3: (Reverse criterion)

If the open loop system is stable and with minimum phase (poles and zeros with strictly negative real part) then the closed loop system is stable if, by traversing in the direction of increasing pulsations the place of transfer in the Nyquist plane of the FTBO we leave the critical point (−1, 0) on the left (Figure 5.5). It is unstable otherwise (Figure 5.6).

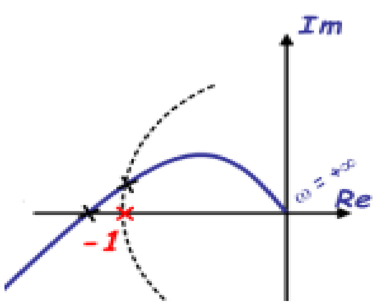


Fig. 5.5 Nyquist plane
(stable system)

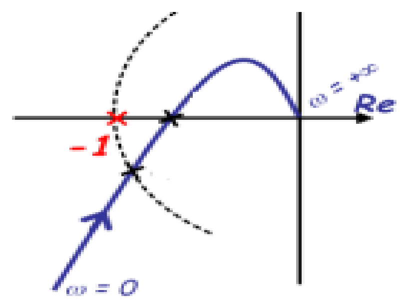


Fig. 5.6: Nyquist plane
(unstable system)

In the Bode plane, a controlled system is stable if, for the pulsation w_c defined by $|FTBO(jw)|_{w=w_c} = 1$ (i.e. 0dB), the phase shift is greater than −180° (Figure 5.4).

5.4. Gain and Phase margins

Beyond the fact that a system is stable, we are also interested in seeing how far we are from the borderline of stability. Several measures exist to characterize how far a stable system is from being unstable. The phase margin defines the value of the phase angle needed to decrease the phase at the cut-off frequency to achieve borderline stability. The phase margin can be expressed analytically:

$$\varphi_t = \varphi(w_c) + 180^o \quad (5.6)$$

where w_c is the cut-off frequency defined by : w_c *is such that* $|L(jw)|_{w=w_c} = 1$

- ✓ A closed-loop system is *stable* if the phase margin is positive.

Exemple 5.4 : if $\varphi = -120^o$ at the cut-off frequency, the phase margin is

$$\varphi_t = \varphi(w_c) + 180^o = -120^o + 180^o = 60^o$$

This means that the closed-loop system is stable. For design purposes, 60^o for the phase margin is a typical prescription.

The gain margin G_t is the factor by which the loop gain is to be multiplied to push a closed-loop system to borderline stability, i.e

$$G_t = \frac{1}{|L(w_\pi)|} \text{ , Where } w_\pi : \varphi(w)_{w=w_\pi} = -180^o \tag{5.7}$$

Here w_π is that frequency where the phase angle is -180^o .

- ✓ The closed-loop system is *stable* if the gain margin exceeds 1.

The gain and phase margins defined previously can also be represented on the Bode diagram (Fig. 5.7 and Fig.5.7) .

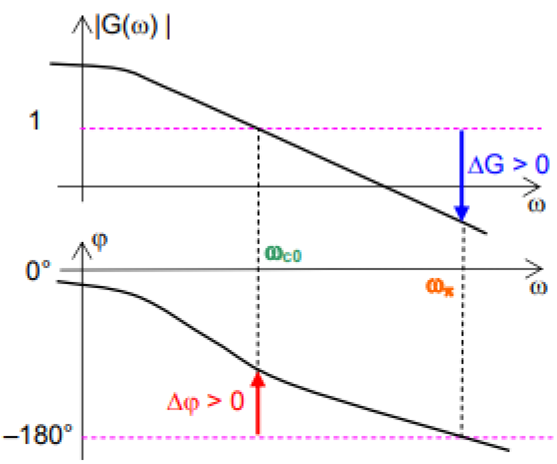


Fig. 5.7: illustration of profit margins and phase on the Bode diagram (stable system: ΔG > 0 and Δϕ > 0)

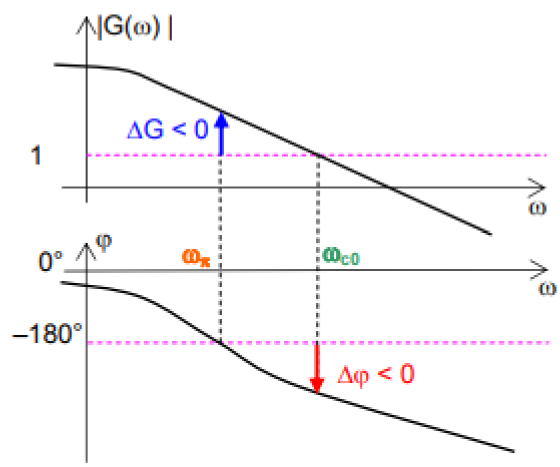


Fig. 5.8: illustration of profit margins and phase on the Bode diagram (unstable system: ΔG < 0 and Δϕ < 0)

5.6 Sstability of the system using MATLA commands

The stability criterion can be used to check the stability of linear systems. In practice a natural choice is to apply a unit step excitation as a bounded input.

Example 5.6.1 Assume we have the following closed-loop transfer function:

$$T(s) = \frac{s+5}{s^5 - 3s^4 + 4s^3 + 10s^2 + 5s - 10}.$$

Applying a unit step input, check the stability of the closed-loop system. Use MATLAB commands, such as :

```
num=[1, 5]
den=[1, -3, 4, 10, 5, -10]
T=tf(num,den)
 step(T)
```

It can be seen that the step response will not be bounded, meaning that the closed-loop system is unstable.

One way to obtain the system output in analytical form is to derive the partial fraction expansion form of the LAPLACE transform of the transfer function. Then the analytical solution of the output signal in the time domain is simply obtained by performing an inverse LAPLACE transformation.

- Check the location of the poles in this analysis:

```
poles=roots(den)
      poles = 2.1150 + 2.1652i
       2.1150 - 2.1652i
      -0.9824 + 0.7214i
       -0.9824 - 0.7214i
      0.7348
```

or in another form :

[zeros,poles,KonstGain]=zpkdata(T,'v')

Either way, it can be seen that the system has poles in the right half-plane. Alternatively, the pzmap command shows the location of both the zeros and poles in graphical mode:

pzmap(T)

Note that complex poles produce oscillations in the output response. However, this can only barely be seen because of the dominant exponential growth.

- **NYQUIST Stability Criterion using MATLAB**

Example 5.6.2 Assume the open-loop transfer function is given by:

$$L(s) = \frac{10}{(1 + 10s)(1+s)}.$$

Use negative unity feedback. Check the stability of the closed-loop system using the simplified NYQUIST criterion:

s=zpk('s')

*L=10/((1+10*s)*(1+s))*

[z,p,k]=zpkdata(L,'v')

It can be seen that the open-loop system has no unstable pole, thus the simplified NYQUIST criterion is applicable.

nyquist(L),grid

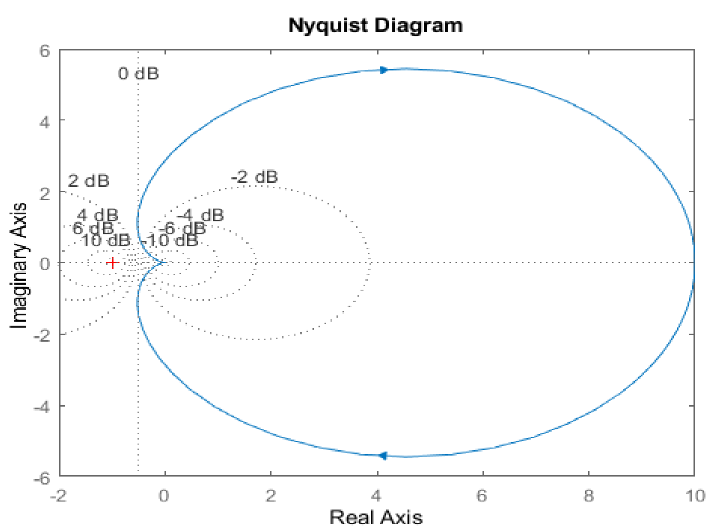


Figure : 5.9: NYQUIST diagram

The NYQUIST diagram does not encircle the point (1 þ 0j), so the closed-loop system is stable. Furthermore, we can conclude that the closed-loop system is structurally stable.

- **Phase Margin, Gain Margin using MATLAB**

Example 5.6.3 Given the open-loop transfer function:

$$L(s) = \frac{1}{(0.5 + s)(s^2 + 2s + 1)},$$

Apply negative unity feedback. What can we say about the stability?

Find the phase margin (pm) and the gain margin (gm), as well as the cut-off frequency wc.

s=zpk('s')

L=1/((0.5+s)*(s^2+2*s+1))

- ***Method_1:*** Use the margin command

[gm,pm,wg,wc]=margin(L)

gm=4.5001

pm=72.227

wg=1.4142

wc=0.5675

To get a graphical evaluation (see Fig. 5.10) we have:

margin(L)

Note that if a graphical evaluation is selected, the gain margin Gm is given in decibels.

*Gm=20*log10(gm)*

- ***Method_2***: use the BODE and NYQUIST diagrams

nyquist(L)

bode(L)

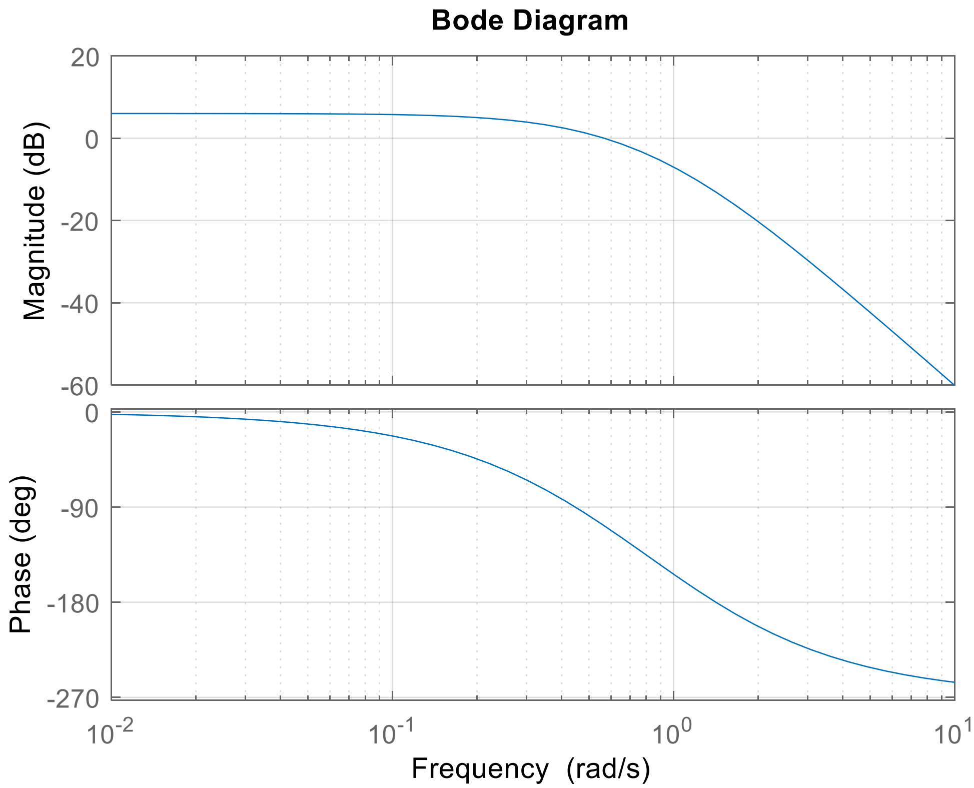


Figure : 5.10 Characteristic points of the BODE diagram

- ***Method_3:*** Read from a frequency-amplitude-phase table Store the calculated points in a table then read the margins:

  w=logspace(-1,1,100);

  [num,den]=tfdata(L,'v')

  [mag,phase]=bode(num,den,w);

  Tabl=[mag, phase,w']

  [mag,phase]=bode(L,w);

  Tabl=[mag(:), phase(:), w']

Exercises Chapter 5

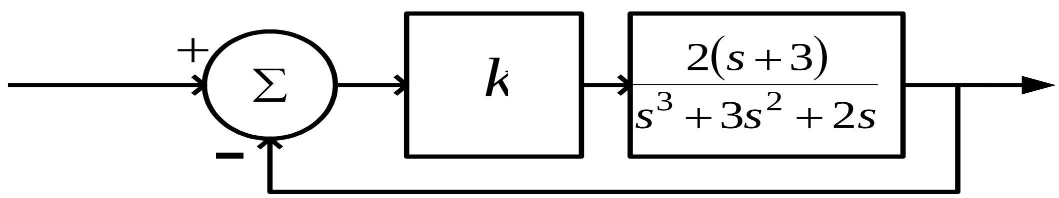
Stability of Linear Control Systems

Exercise 1 : Study the stability of the following systems:

$H_1(p) = \frac{(p+1)(p-2)}{(3p+2)}$, $H_2(p) = \frac{12p}{(5p-1)}$, $H_3(p) = \frac{(p+1)(p+12)}{(p-2)(p+1)p}$

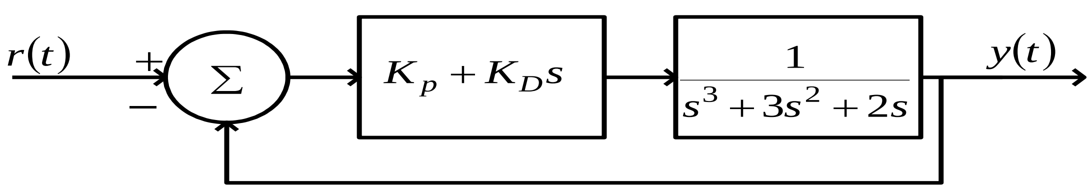
$H_4(p) = \frac{(p+1)}{(p^2+2p+2)}$, $H_5(p) = \frac{1}{(p^2-p-6)}$

Exercise 2 : Determine the range of *k* for the following system to remain stable:



$$Char.\ Eqn. = 1 + GH = 1 + \frac{2k(s+3)}{s^3+3s^2+2s} = 0 \quad , \quad s^3+3s^2+(2k+2)s+6k=0$$

Exercise 3 : Determine the ranges of K_p and K_D so that the system will be stable.



$$Char.\ Eqn. = 1 + GH = 1 + \frac{K_p + K_D s}{s^3+3s^2+2s} = 0 \quad , \quad s^3+3s^2+(2+K_D)s+K_p=0$$

Exercise 4 : Determine the range of *k* so that the system is boundary stable and determine the oscillation frequency of the system if the char. Eqn. of the system is: $s^4+25s^3+15s^2+20s+k=0$

Exercise 5 : Given the open-loop transfer function: $H(p) = \frac{1}{(p+0.5)(p^2+2p+1)}$;

Apply negative unity feedback. What can we say about the stability? Find the phase margin and the gain margin , as well as the cut-off frequency w_c.

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