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Contact Problem With multivalued Normal compliance and Unilateral constraint

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

﴿ نَرْفَعُ دَرَجَٰتٍ مَّن نَّشَآءُ وَفَوْقَ

كُلِّ ذِى عِلْمٍ عَلِيمٌ ﴾

صدق الله العلي العظيم

سورة يوسف، آية ٧٦

شكر وتقدير

بسم الله الرحمن الرحيم، والصلاة والسلام على أشرف المرسلين.

بدايةً، نحمد الله عز وجل الذي منحنا القوة والصبر لإتمام هذا البحث، فله الحمد والشكر دائماً وأبداً.

إن تخصيص هذه الصفحة للشكر ليس مجرد تقليد أكاديمي أو بروتوكول بالنسبة لنا، بل هو تعبير صادق من أعماق قلوبنا عن الإمتنان العميق لأشخاص استحقوا الثناء والتقدير حقاً.

نتقدم بأسمى عبارات الشكر والتقدير إلى مشرفنا الفاضل

الدكتور السعيد عامر مزيان

الذي لم يبخل علينا يوماً بدعمه غير المشروط وتواجده الدائم. لقد كان لنصائحه القيمة، ورؤيته الثاقبة، ومساهمته الفعالة أثر كبير في توجيه مسار هذا البحث وتجاوز العديد من الصعوبات طوال فترة التحضير، فله منا كل التقدير والإحترام.

كما نود أن نعرب عن خالص شكرنا وتقديرنا لجميع أساتذتنا في قسم الرياضيات والإعلام الآلي، الذين نهلنا من علمهم وكانوا جزءاً من مسيرتنا الأكاديمية. ويمتد شكرنا أيضاً إلى الطاقم الإداري على مساعدتهم وجهودهم التي ساهمت في إنجاح هذا العمل.

وفي الختام، نتوجه بالشكر لكل من يرى نجاحنا نجاحاً له، ولكل من شاركنا هذه اللحظة، وإلى كل من ترك أثراً في هذا العمل.

إهداء

بسم الله الرحمن الرحيم

﴿وَمَا تَوْفِيقِي إِلَّا بِاللَّهِ ۚ عَلَيْهِ تَوَكَّلْتُ وَإِلَيْهِ أُنِيبُ﴾

أهدي تخرجي و سنوات تعبي أولا إلى نفسي الطموحة القوية العظيمة ، إلى تلك التي صبرت و ثابرت و قاومت ، فكنت لها...حتى و إن أبت أتيت بها رغم الصعاب

إلى جنة الله على الأرض إلى من كان دعاؤها سر نجاحي و توفيقي إلى ملاكي و صديقة أيامي إلى من كانت الدافع في كل لحظة ضعف ، المضحية من أجلي معلمتي و سيدتي العظيمة أمي الحبيبة فرحات حفيظة حفظك الله و رعاك

إلى الرجل العظيم ، داعمي الأول في مسيرتي ، سندي و قوتي و ملاذي بعد الله. إلى من انتظر هذه اللحظة ليفتخر بي...إلى فخري واعتزاي أبي العزيز الطاهر أدامك الله ظلا و سندا لنا

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إلى من يرتعش قلبي لذكرها إلى من فارقتني بجسدها و روحها مزالت ترفرف في سماء حياتي ، كل حرف في هذه المذكرة يحمل شوقي إليك وكل نجاح أحق به قلبك الذي كان ينبض بحبنا إلى الروح الطاهرة عمتي فضيلة رحمها الله

قسوم هنية

إهداء

بسم الله الرحمن الرحيم

﴿وَمَا تَوْفِيقِي إِلَّا بِاللَّهِ ۚ عَلَيْهِ تَوَكَّلْتُ وَإِلَيْهِ أُنِيبُ﴾

إلى الذين أضاءوا لي قناديل الأمل حين اشتدّ
المسير، والديّ وإخوتي.. أنتم كنزي الباقي.

إلى روح جدي التي ترفرِف حول نجاحي.. غبتَ
عن العين وما غبتَ عن الوجدان.

إلى نبع الحنان و دعواتها التي كانت تحرس
خطاي، جدتي الغالية.. دمتِ لي نوراً وبركة.

إلى الأصدقاء الذين جعلوا من عثرات الطريق
ضحكات، ومن تعبه ذكريات لا تُنسى.. شكراً
لوفائكم.

وإلى نفسي.. التي آمنت بالحلم حين استصعبه
الآخرون، إليكِ أهدي هذا الوصول.

فار خولة

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General Introduction

In industry as in nature, several frequent phenomena involve contact processes between a deformable body and a foundation, taking into account the piezoelectric effect of the material. Under the hypothesis of small deformations, we study static and quasi-static processes for elastic materials, viscoelastic. The engineering literature on this topic is extensive because of their importance in structural and mechanical systems as well as in metal shaping and extrusion. Due to their inherent completeness, contact phenomena are modeled by nonlinear evolutionary problems that are difficult to analyze.

A general Mathematical Theory of Contact Mechanics (MTCM) is currently emerging. She is concerned with mathematical structures who are at the root of the problems of contact with constituent laws different, that is different materials, various geometries and different contact conditions.

A vast technical literature, mainly in engineering but also in geophysics, covers contact with or without friction. In geophysics, literature focuses on the movement of tectonic plates, particularly earthquakes. There are many publications dealing with friction contact problems, see for example [2, 3, 15]. Other works have considered contact conditions of the normal compliance type with friction, as in [4, 10]. The goal is to provide a clear and rigorous context for the construction of mechanical contact models, the proof of existence and uniqueness results and the establishment of the regularity of the solution.Once the existence, uniqueness, and regularity of solutions are established, important questions arise, such as the mathematical analysis of solutions.

The analysis of models for adhesive contact can be found in [6, 8,14]. We considered an application of the adhesive contact theory in the medical field of prosthetic limbs, there the importance of adhesion between the implanted bone and the tissue has been described since detachment can decrease the capacity of people using the prosthesis or the joint. The piezoelectric effect was predicted by Coulomb and discovered by Becquerel in 1819, but was only correctly explained in 1880 by the brothers Pierre and Jacques Curie (by experimentation on Quartz and Rochelle salt). Although it seems that the first to have observed this phenomenon was Father René Just Haüy. The law of behavior for this type of materials was established by Lippmann in 1881 based on thermodynamic considerations.

New investigations into the study of electromechanical problems have been reported in recent years, for example [4,8], and numerical simulations have been added in [8, 9].

The aim of this memory is to study a frictionless contact problem for rate-type viscoplastic materials within the framework of the Mathematical Theory of Contact Mechanics. We model the materials behavior with a constitutive law of the form

$$\sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds \text{ in } \Omega. \tag{1}$$

where u denotes the displacement field, σ represents the stress tensor and $\xi(t)$ is the linearized strain. Here $\mathcal{A}$ is the elasticity operator, allowed to be nonlinear and $\mathcal{B}$ represents the relaxation operator, assumed to be linear. Quasistatic contact problems for materials following the law (1) can be found in [12] and the references therein. There, the contact was assumed to be frictionless, was modeled with normal compliance and unilateral constraint; the unique weak solvability of the corresponding problems was proved by using arguments of history-dependent variational inequalities.

The memory is composed of three parts that we briefly describe.

The first chapter we introduce the necessary tools for a good understanding of the work. We present the physical frameworks and mathematical models used. Then, we indicate the functional spaces and the main notations used throughout the memory. Finally, we recall some fundamental results of functional analysis concerning strongly monotonic and Lipschitz operators, elliptic quasi-variational inequalities and evolution.

In the second chapter, the model we consider involves a contact condition with multivalued normal compliance and unilateral constraint. This condition takes into account both the deformability and the rigidity of the foundation.

In third we list the assumptions on the data and derive the variational formulation of the problem, we state and prove an existence and uniqueness result, prove a convergence result,

General Notation

$\mathbb{N}$	The set of natural numbers ,
$\mathbb{R}$	The set of real numbers ,
c	A strictly positive real constant,
i.e	That is ,
$\partial_i\psi$	The partial derivative of ψ with respect to the i^{th} variable, $x : \partial_i\psi = \frac{\partial\psi}{\partial x_i}$,
$\nabla\psi$	The gradient of the mapping $\psi : \nabla\psi = (\partial_1\psi, \ldots, \partial_d\psi)$,
$\mathbb{S}^d$	The space of second-order symmetric tensor on $\mathbb{R}^d(d = 2, 3)$,
$\mathrm{Div}\psi$	The divergence of the mapping $,\psi : \mathrm{Div}\psi = \partial_1\psi + \ldots + \partial_d\psi)$,
$(.,.)_{\mathrm{X}}$	The inner product in the space X,
$\| \cdot \|_{\mathrm{X}}$	The norm in the space X,
a.e.	Almost everywhere,
Ω	an open subset of $\mathbb{R}^d$, sometimes referred to as a Hertzian domain.
$\bar{\Omega}$	the closure of Ω,
Γ	the boundary of $\Omega : \Gamma = \partial\Omega$,
Γ_i	The boundary parts of Γ, for $\Gamma, (i = 1, 2, 3)$,
$meas\Gamma_i$	The $(d-1)$-dimensional measure of Γ_i.
$d\Gamma_i$	The $(d-1)$-dimensional Lebesgue measure onΓ_i,
$\boldsymbol{\nu}$	the surface measure on Γ,
$v_\nu, \boldsymbol{v}_\tau$	The normal and tangential components of the vector fieldv_ν defined on $\bar{\Omega}$,
$L^2(\Omega)$	The space of measurable functions$u$ on Ω such that $\int_\Omega \|u\|^2 dx < +\infty$
$\| \cdot \|_{L^2(\Omega)}$	the norm in $L^2(\Omega)$ defined by $\| u \|_{L^2(\Omega)} = (\int_\Omega \|u\|^2 dx)^{\frac{1}{2}}$,
$L^\infty(\Omega)$	The space of measurable functions u on Ωsuch that $\exists c > 0 :\| u \|< c$,e.a. in Ω,
$\mathrm{H}^{\frac{1}{2}}(\Gamma)$	The Sobolev space of order$\frac{1}{2}$ defined on the boundary Γ Time-Dependent and Functional Spaces.

Let $[0, T]$ be a time interval, $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$,

H_Γ	The vector-valued space $(\mathrm{H}^{\frac{1}{2}}(\Gamma))^d$,
$\mathrm{H}_\Gamma^{'}$	The dual space of H_Γ.
$C([0,T];\mathrm{H})$	The space of continuous functions from $[0,T]$ into the Hilbert space H,
$C^1([0,T];\mathrm{H})$	The space of continuously differentiable functions from $[0,T]$ into H,
$L^p([0,T];\mathrm{H})$	The space of measurable functions from $[0,T]$ intoH whose $p-th$ power is integrable.
$\Vert\cdot\Vert_{L^p([0,T];\mathrm{H})}$	The norm associated with the space $L^p([0,T];\mathrm{H})$,
$\mathrm{W}^{k,p}([0,T];\mathrm{H})$	The Sobolev space of functions whose time derivatives up to order $k$and p,
$\Vert\cdot\Vert_{\mathrm{W}^{k,p}([0,T];\mathrm{H})}$	The norm in the Sobolev space $\mathrm{W}^{k,p}([0,T];\mathrm{H})$,
u	The displacement vector field in the domain Ω, The $i-th$ component of the displacen the canonical basis.
σ	The stress tensor associated with the displacement u, σ_i The components of the stress tensorσ_ν in the canonical basis.
φ	The outward unit normal vector to the boundary of Ω,
β	The normal stress on the boundary, defined by Γ_3,
$\dot{u},\ddot{u}$	The first time derivative (velocity) of the displacement u The second time derivative (acceleration) of the displacement u,
$\varepsilon(u)$	The linearized strain tensor associated with $\varepsilon(u)_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i)$,

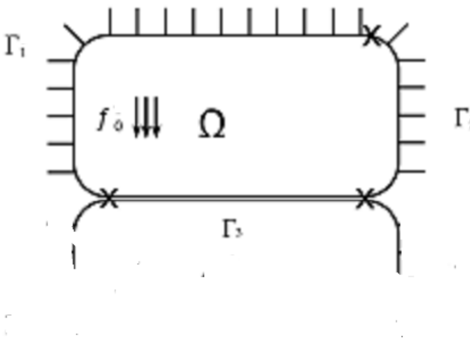
Chapter 1

Modeling

This chapter represents a brief reminder of the mechanics of continuous environments where we are going to introduce the physical frameworks used in this memory ; it is intended to remind the Cauchy equation of motion, to describe elastic laws of behavior, viscoelastic and electro-viscoelastic. Moreover, we specify in this chapter the conditions at the contact limits with friction, with or without accession. The contact phenomena considered in this memory are described by the following two physical frameworks:

1.1 Physical framework.1

(Mechanical problem). Let a material body occupy a bounded domain $\Omega \subseteq \mathbb{R}^{(d=2,3)}$ with a regular boundary surface Γ, partitioned in three measurable parts Γ_1, Γ_2 and Γ_3 , such as $meas(\Gamma_1) > 0$. We note by ν the outgoing unit normal at Γ. The body is embedded at Γ_1 in a fixed structure. On Ω act volume forces of density f_0 (see Physical framework 1).



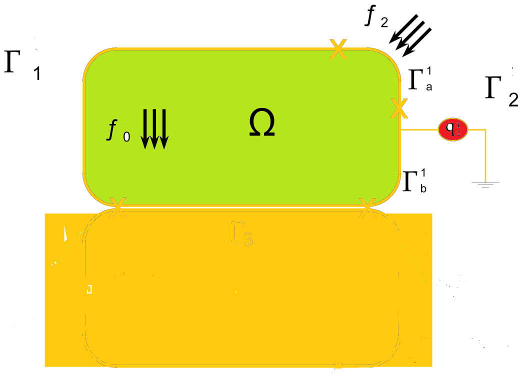
Physical framework 1.

We assume that f_2 and f_0 vary very slowly with respect to time and therefore the process is quasi-static. Let $T > 0$ ($[0, T]$)be the time interval in question. The body is in contact with friction with or without adhesion to an obstacle on the Γ_3 part. We take into consideration the mechanical properties

of the body. Our objective will be to study evolution of these properties over time, under the hypothesis of small transformations.
We will use this physical framework in chapter 2 of this memory.

1.2 Physical framework.2.

(Electro-mechanical problem). Let a material body occupy a bounded domain $\Omega \subseteq \mathbb{R}^{(d=2,3)}$ with a regular boundary surface Γ, partitioned in three measurable parts Γ_1, Γ_2 and Γ_3 , such as $meas(\Gamma_1) > 0$. We note by ν the outgoing unit normal at Γ. The body is embedded on Γ_1 in a fixed structure.On Γ_2 act surface tractions of density f_2 and in Ω act volume forces of density f_0 (see Physical framework 2).



Physical framework 2.

We assume that f_2 and f_0 vary very slowly with respect to time. Let $T > 0$ ($[0, T]$)be the time interval in question. In addition to the action of forces and tractions, the body is subject to the action of electric charges of volume density q and surface electric charges. To describe them, we consider a second partition of the boundary Γ into three measurable parts Γ_a, Γ_b and Γ_3 such that $meas(\Gamma_a) > 0$. We assume that the body is in contact rubbing with or without adhesion with an insulating (or conductive) foundation on Γ_3, the electric potential vanishes on Γ_a and a surface electric charge of density q is represented on Γ_b.
The difference compared to the previous physical framework results from the fact that now we take into consideration the mechanical properties and also the electrical properties of the material body. We study the evolution of these properties in the time interval $[0, T]$ assuming that the process is quasistatic, in the hypothesis of small transformations. We will use this physical framework in chapter 3, of this memory.
Before obtaining the mathematical models that correspond to the physical frameworks presented, here are some ratings and conventions that we will use throughout this memory.

We refer to S^d as the space of symmetric tensors of order two on $\mathbb{R}^d(d = 2, 3)$; $(.)$ and $\|.\|$ represent respectively the inner product and the Euclidean norm on $\mathbb{R}$ and S^d.

$u.v = u_i.v_i, \qquad \parallel v \parallel = (v,v)^{\frac{1}{2}}, \qquad \forall u, v \in \mathbb{R}^d$

$\sigma.\tau = \sigma_{ij}.\tau_{ij}, \qquad \parallel \tau \parallel = (\tau,\tau)^{\frac{1}{2}}, \qquad \forall \sigma, \tau \in \mathbb{S}^d$

with the convention of the silent index.

For a vector v, we denote by v_ν and v_τ the normal and tangential components at the border defined by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu.\nu. \tag{1.1}$$

We designate by $\sigma = \sigma(x,t)$ the field of constraints, by $u = u(x,t)$ the field of displacements and by $\xi(u)$ the field of infinitesimal deformations. For simplify notations, we do not explicitly indicate the dependency of functions in relation to $x \in \bar{\Omega}$ and $t \in [0,T]$.

For a field of constraints σ, we denote by σ_ν and σ_τ the components normal and tangential to the border given by

$$\sigma_\nu = (\sigma.\nu)\nu, \quad \sigma_\tau = \sigma.\nu - \sigma_\nu.\nu. \tag{1.2}$$

Using (1.1) and (1.2), we get the relation

$$(\sigma\nu).v = \sigma_\nu.v_\nu + \sigma_\tau.v_\tau. \tag{1.3}$$

We use standard notation for Lebesgue and Sobolev spaces in Ω and on Γ.

Let

$$V = \{v = (v_i) \in H^1(\Omega)^d, \quad v = 0 \quad on \quad \Gamma_1\}$$

$$Q = \{\tau = (\tau_{ij}) \in L^2(\Omega)^{d\times d}, \quad \tau_{ij} = \tau_{ji}\}.$$

These are real Hilbert spaces endowed with the inner products.

$$(u,v)_V = \int_\Omega \xi(u).\xi(v)dx, \qquad (\sigma,\tau)_Q = \int_\Omega \sigma.\tau dx. \tag{1.4}$$

and the associated norms $\|.\|_V$ and $\|.\|_Q$, respectively.

Here e represents the deformation operator given by

$$\xi(v) = (\xi_{ij}(v)), \qquad \xi_{ij} = \frac{1}{2}(v_{ij} + v_{ji}), \forall v \in H^1(\Omega)^d. \tag{1.5}$$

For an element $v \in V$ we still write v for the trace of v on the boundary and we denote by v_ν and v_τ the normal and tangential components of v on Γ, given by $v_\nu = v.\nu, v_\tau = v - v_v.\nu$. Let Γ_3 be a measurable part of Γ.

Then, by the Sobolev trace theorem,there exists a positive constant c_0 depending on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0\|v\|_V \qquad \forall v \in V. \tag{1.6}$$

For a regular function $\sigma \in Q$ we use the notation σ_ν and σ_τ for the normal and the tangential traces, i.e. $\sigma_\nu = (\sigma\nu).\nu$ and $\sigma_\tau = \sigma\nu - \sigma_v\nu$. Moreover, we recall that with the divergence operator defined by the equality $Div\sigma = (\sigma_{ij,j})$, the following Green$^{'}$s formula holds:

$$\int_\Omega \sigma.\varepsilon(v)dx + \int_\Omega Div\sigma.vdx = \int_\Gamma \sigma.vda \quad \forall v \in V. \tag{1.7}$$

Finally, we denote by Q_∞ the space of fourth order tensor fields given by

$$Q_\infty = \{\varepsilon = (\varepsilon_{ijkl}) : \varepsilon_{ijkl} = \varepsilon_{jikl} = \varepsilon_{klij} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d\}. \tag{1.8}$$

which is a real Banach space with the norm

$$\|\varepsilon\|_{Q_\infty} = \sum_{1\le i,j,k,l \le d} \|\varepsilon_{ijkl}\|_{L_{\infty(\Omega)}}. \tag{1.9}$$

A simple calculation shows that

$$\|\varepsilon\tau\| \le \|\varepsilon\|_{Q_\infty}\|\tau\|_Q \quad \forall \varepsilon \in Q_\infty,\ \tau \in Q. \tag{1.10}$$

For each Banach space X we use the notation $C(\mathbb{R}_+;X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we use the symbol $C(\mathbb{R}_+;K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that $C(\mathbb{R}_+;X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Here we only recall that the convergence of a sequence $(v_k)_k$ to an element v, in the space $C(\mathbb{R}_+;X)$ can be described as follows:

$$\begin{cases} v_k \to v \ in \ C(\mathbb{R}_+;X) \ as \ k \to \infty \text{ if and only if} \\ max\|v_k(t) - v(t)\|_X \to 0 \ as \ k \to \infty, \text{ for all } n \in \mathbb{N}^*. \end{cases} \tag{1.11}$$

Consider now a real Hilbert space X with inner product $(.,.)_X$ and associated norm $\|.\|_X$.

Let K be a subset of X and consider operators $A : K \to X$,

$\mathcal{R} : C(\mathbb{R}_+;X) \to C(\mathbb{R}_+;X)$

as well as functions $j : K \to \mathbb{R},\ f : \mathbb{R}_+ \to X$ with the following properties.

$$K \text{ is a nonempty,closed,convex subset of } X. \tag{1.12}$$

$$A : K \to X. \begin{cases} (a)\text{There exists } m > 0 \text{ such that} \\ \langle Au_1 - Au_2, u_1 - u_2\rangle_X \ge m\|u_1 - u_2\|_X^2 \quad \forall u_1, u_2 \in K. \\ (b)\text{There exists } M > 0 \text{ such that} \\ \|Au_1 - Au_2\|_X \le M\|u_1 - u_2\|_X \quad \forall u_1, u_2 \in K. \end{cases} \tag{1.13}$$

$\mathcal{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$

$$\begin{cases} \text{For every } n \in \mathbb{N}^* \text{ There exists } r_n > 0 \text{ such that} \\ \|(\mathcal{R}u_1(t) - \mathcal{R}u_2(t))\|_X \leq r_n \int_0^t \|u_1(s) - u_2(s)\|_X ds \\ \forall u_1, u_2 \in C(\mathbb{R}_+; X),\ \forall t \in [0, n]\,. \end{cases} \tag{1.14}$$

The function

$$j : K \to \mathbb{R} \qquad \text{is convex and lower semicontinuous.} \tag{1.15}$$

$$f : \mathbb{R}_+ \to X; \qquad f \in C(\mathbb{R}; X). \tag{1.16}$$

Completeness of the space $(V; \|\,.\,\|_V)$ follows from Korns inequality due to the assumption $meas\Gamma_1 > 0$.

Theorem 1.2.1 *Assume (1.12)-(1.16). Then there exists a unique function $u \in C(\mathbb{R}_+, K)$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$(Au(t), v - u(t))_X + ((\mathcal{R}u)(t), v - u(t))_X + j(v) - j(u(t)) \geq (f(t), v - u(t))_X \quad \forall v \in k. \tag{1.17}$$

Theorem 1.2.1 represents a particular case of a more general result proved in [13]. Following the terminology introduced there, we refer to an operator $\mathcal{R}$ satisfying the condition (1.14) as a history-dependent operator. Moreover, (1.17) represents a historydependent quasivariational inequality.

Lets now move on to the description of the mathematical models associated with physical frameworks above.

Mathematical model n 1. The first mathematical model studied in this memory, describes the evolution of the body in the physical framework n 1 . The functions unknowns of the problem are the displacement field $u : \Omega \times [0, T] \to \mathbb{R}$ and the field constraints $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$.

We know that in general, the evolution of a material body is described by the equation of movement of Cauchy

$$Div\sigma + f_0 = \rho.\ddot{u} \in \Omega \times (0, T). \tag{1.18}$$

where $\rho : \Omega \to \mathbb{R}^+$ denotes the mass density; here "Div" represents the divergence operator for tensors, $Div\sigma = (\sigma_(ij, j)$. The evolution process defined by (1.5) is called dynamic process. In some situations, this equation can still be simplified. For example, in the case where the field of velocities u varies very slowly by in relation to time, the term $\rho\ddot{u}$ can be neglected. In this case, equation (1.5) becomes

$$Div\sigma + f_0 = 0 \qquad in \quad \Omega \times (0, T). \tag{1.19}$$

Equation (1.19) is called the equilibrium equation.The evolution process defined by (1.19) is called a quasi-static process.

We recall that in the physical framework $n1$, f_2 and f_0 vary very slowly with respect to time. Therefore, we assume that the accelerations in the system are negligible. We therefore place ourselves in the quasi-static case and we use the equation (1.19).

Chapter 2

Mathematical tools

This chapter is devoted to the description of the functional spaces used throughout this work. We assume that Ω is a bounded Lipschitz domain. This means that its boundary $\mathbb{R}^d, (d = 2, 3)$ can be locally represented as the graph of a Lipschitz function defined on an open subset of Γ. The boundary $\mathbb{R}^{d-1}$ is decomposed into three disjoint measurable parts Γ_1, Γ_2, and Γ_3 on one side, and into two open parts Γ_a and Γ_b on the other side, such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$.

2.1 Normed Spaces

2.1.1 Normed Vector Spaces

Definition 2.1.1 *Let E be a set endowed with two operations:*
An addition $+$ on E such that $(E, +)$ is a commutative group.
An external scalar multiplication of vectors $\vec{v}$ in E by scalar $\lambda \in \mathbb{K}$, satisfying the following properties:

- i $.\lambda\left(\vec{u}+\vec{v}\right) = \lambda.\vec{u} + \lambda.\vec{v}$
- ii $.(\alpha + \lambda).\vec{u} = \alpha.\vec{u} + \lambda.\vec{u}$
- iii $.\lambda.\left(\alpha.\vec{u}\right) = \left(\lambda.\alpha\right).\vec{u}$
- iv $.1.\vec{u} = \vec{u}$

Definition 2.1.2 *Let X be a vector space over field $\mathbb{K}$. A norm $\|.\|$ on the space X is a mapping from X into $\mathbb{R}_+$ that satisfies the following properties:*

1. *for all $x \in X, \|x\| = 0 \Leftrightarrow x = 0$.*
2. *for all $x \in X$ and $\lambda \in \mathbb{K}, \|\lambda x\| = |\lambda|\|x\|$.*
3. *for all $x, y \in X, \|x + y\| \leq \|x\| + \|y\|$.*

This inequality is called the triangle inequality.

Definition 2.1.3 *A normed vector space is a vector space X equipped with a norm $\|.\|$ and the topology induced by this norm.*

2.1.2 Vector Spaces of Functions

Let X be any set and let n be any positive integer.
We denote by $E = \mathbb{F}\left(X, \mathbb{K}^n\right)$,the space of all functions from X with values on $\mathbb{K}^n$.
This space is endowed with two operations: the pointwise addition of functions and the multiplication of a function on X by a scalar (a constant) $\lambda \in \mathbb{K}$.

2.1.3 Banach Spaces

We now introduce some functional spaces and recall several results concerning strongly monotone and Lipschitz operators, as well as variational inequalities and evolution equations.

Definition 2.1.4 *A sequence $(x_k)_k$ of elements of a normed space E is called a Cauchy sequence if:*

$$(\forall \epsilon > 0)(\exists N \geq 1)k, l \geq N \Rightarrow \|x_k - x_l\| \leq \epsilon$$

Every convergent sequence is a Cauchy sequence.

2.2 Contraction

The Banach contraction principle is the most elementary result in fixed point theory. In analysis, these theorems prove to be very useful tools in mathematics, mainly in the field of solving differential equations. The Banach fixed point theorem gives a general criterion in complete metric spaces to ensure that the iteration process of a function converges to a fixed point. We define some definitions which allow us to affirm that a function f satisfies the criteria of the contraction fixed point theorem.

Definition 2.2.1 *Let (X, d) be a metric space, a mapping $f : X \longrightarrow X$ is said to be Lipschitz with constant $k \geq 0$ if:*

$$d(f(x), f(y)) \leq kd(x, y) \qquad \textit{for all } x, y \in X$$

k is called the Lipschitz constant.

Definition 2.2.2 *Let (X, d) be a metric space, a mapping $f : X \longrightarrow X$ is said to be $1-$ Lipschitz if $k = 1$ and:*

$$d(f(x), f(y)) \leq d(x, y) \qquad \textit{for all } x, y \in X$$

And the mapping f is said to be non-expansive.

Definition 2.2.3 *A lipschitz mapping f is called:*

1. *non-expansive if $k \leq 1$*
2. *A contraction if $0 < k < 1$*

Theorem 2.2.1 *Let (X, d)be a complete metric space and let $f : X \longrightarrow X$ be a contraction with Lipschitz constant k.*
Then f admits a unique fixed point $u \in X$.
Moreover ,for every $x \in X$,

$$\lim_{x \longrightarrow \infty} f^n(x) = u$$

$$d(f^n(x), u) \leq \frac{k^n}{1-k} d(x, f(x))$$

2.3 Hilbert spaces

Definition 2.3.1 *Let H be a real vector space and let $(.,.)_H$ be an inner product on H that is $(.,.)_H : H \times H \to \mathbb{R}$ bilinear,symmetric, and positive definite.*

We denote by $\| \cdot \|_H$ the mapping from H to $\mathbb{R}_+$ defined by:

$$\| u \|_H = (u, u)_H^{\frac{1}{2}}, \tag{2.1}$$

Recall that $\| \cdot \|_H$ is a norm on H that satisfies the Cauchy-Schwartz inequality:

$$|(u, v)_H| \leq \| u \|_H \| v \|_H, \ \ \forall u, v \in H. \tag{2.2}$$

We say that H is a Hilbert space if H is complete with respect to the norm defined in (2.1). Let $H^{'}$ be the dual space of H, that is the space of linear and continuous functional on H ,endowed with the norm:

$$\| \eta \|_{H'} = \sup_{v \in H - \{0\}} \frac{< \eta, v >_{H' \times H}}{\| v \|_H},$$

Where $\langle ., . \rangle_{H' \times H}$ denotes the duality pairing between $H^{'}$ and H.

Definition 2.3.2 *Let $(X, \|.\|)$ be a normed vector space and $(x_n)_{n \in \mathbb{N}}$ a sequence in X.*

1. *The sequence $(x_n)_{n \in \mathbb{N}}$ is said to be convergent if there exists $x \in X$ such that* $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$.

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \Rightarrow \|x_n - x\| \leq \epsilon$$

2. the sequence $(x_n)_{n\in\mathbb{N}}$ *is said to be a Cauchy sequence if*

$$\forall\epsilon > 0, \exists N_\epsilon > 0, \forall p, q \geq N_\epsilon \Rightarrow \|x_p - x_q\| \leq \epsilon$$

.

Theorem 2.3.1 ***(Riesz-Fréchet representation theorem**): Let* H *be a Hilbert space and* $H^{'}$ *its dual space. Then, for every*

$$\langle\phi, v\rangle_{H'\times H} = (f, v)_H \quad \forall v \in H.$$

Moreover,

$$\| \phi \|_{H'} = \| f \|_H \ .$$

The importance of this theorem lies in the fact that every continuous linear functional on H can be represented by means of the inner product.The mapping $\phi \mapsto f$ is an isometric isomorphism ,with allows us to identify H with dual space $H^{'}$.

2.4 The spaces $L^p(\Omega)$

Definition 2.4.1 *(Lebesgue space). Let* $p \in \mathbb{R}$ *with* $1 \leq p \leq \infty$*. The Lebesgue space* $L^p(\Omega)$ *is defined as ,*
$L^p(\Omega) = \{v : \Omega \to \mathbb{R}$ *is measurable on* Ω *and* $|v|^p$ *is Lebesgue integrable on* $\Omega\}$*. It is a Banach space when equipped with the norm*

$$\|v\|_{L^p(\Omega)} = \int_\Omega (| \ v(x) \ |^p)^{\frac{1}{p}} dx.$$

if $p = \infty$ *and* $v : \Omega \to \mathbb{R}$*, measurable.*
We define $\|.\|_{L^\infty(\Omega)}$

$$\|v\|_{L^\infty(\Omega)} = supss(v) = inf\{c; |v(x)| \leq c\}.$$

The space $L^\infty(\Omega)$ *is also a Banach space.*

Definition 2.4.2 *Let* $p \in \mathbb{R}$ *with* $1 \leq p \leq \infty$*. A function* $u : \Omega \to \mathbb{R}$ *is said to belong to* $L^p_{loc}(\Omega)$ *if* $u \circ I_K \in L^p$ *for every compact set* $\forall K \subset \Omega$ *Where* I_K *denotes the identity mapping on* K*.*

Theorem 2.4.1 *. For every* $p \in [1, +\infty[$ *,the spaces* $L^p(\Omega)$ *satisfy the following properties:*
1) The spaces $L^p(\Omega)$ *are Banach spaces.*
2) For every function $u \in L^p(\Omega)$ *and every function* $v \in L^q(\Omega)$ *Holder's inequality holds;that is*

$$\int_\Omega | \ u(x)v(x) \ | \ dx \leq \| \ u \ \|_{L^p(\Omega)} \| \ v \ \|_{L^q(\Omega)}, with \ (\frac{1}{p} + \frac{1}{q} = 1)$$

3)The spaces $L^p(\Omega)$ are separable for $p \in [1,+\infty[$

4) The space $L^2(\Omega)$,endowed with the scalar product

$$(u,v) = \int_\Omega u(x)v(x)dx, \forall u,v \in \ L^2(\Omega).$$

It is a Hilbert space.Furthermore , Cauchy-Schwartz inequality associated with Holder's inequality is satisfied ,that is.

$$\int_\Omega \mid u(x)v(x) \mid dx \leq \| u \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} .$$

2.5 Sobolev Spaces

Sobolev spaces were introduced at the beginning of the twentieth century and made it possible to solve a large number of problems related to partial differential equation that previously had no solutions.

We begin with a brief review of some results concerning Sobolev spaces $H^1(\Omega)$ is defined by:

$$H^1(\Omega) = \{u \in \mathrm{L}^2(\Omega) \quad | \quad \partial_i u \in \mathrm{L}^2(\Omega) \, \text{for } i = 1, \cdots, d\}.$$

We denote by ∇u the vector with components $\partial_i u$ $(i = 1, \cdots, d)$.
We have $\nabla u \in \mathrm{L}^2(\Omega)^d$ for all $u \in H^1(\Omega)$.

It is known that $H^1(\Omega)$ is a Hilbert space endowed with inner product:

$$(u,v)_{H^1(\Omega)} = (u,v)_{\mathrm{L}^2(\Omega)} + (\partial_i u, \partial_i v)_{\mathrm{L}^2(\Omega)},$$

and the associated norm:

$$\| u \|_{H^1(\Omega)} = (u,u)^{\frac{1}{2}}_{H^1(\Omega)}, \text{and we write } \| u \|^2_{H^1(\Omega)} = \| u \|^2_{\mathrm{L}^2(\Omega)} + \| \nabla u \|^2_{\mathrm{L}^2(\Omega)^d} .$$

We have the following results:

$$C^1(\bar{\Omega}) \, \text{is dense in } \ H^1(\Omega).$$

Theorem 2.5.1 ***(Rellich's Theorem)***

$$H^1(\Omega) \subset \mathrm{L}^2(\Omega) \, \textit{with compact injection.}$$

Theorem 2.5.2 ***(Sobolev Trace Theorem)***

There exists a linear and continuous operator $\delta : H^1(\Omega) \to \mathrm{L}^2(\Gamma)$ such that $\delta u = u|_\Gamma$ for every $u \in C^1(\bar{\Omega})$.

Remark 2.5.1 *The space $\mathrm{L}^2(\Gamma)$ above denotes the space of real-valued functions on Γ which are $\mathrm{L}^2$with respect to the surface measure dT. The operator δ is called the trace operator, it is defined as the extension by density of the mapping $u \to u|_\Gamma$, for $u \in C^1(\bar{\Omega})$.*

Remark 2.5.2 *We note that the trace operator* $\delta : H^1(\Omega) \to \mathrm{L}^2(\Gamma)$ *this is a compact operator.*

Definition 2.5.1 *For every* $k \in \mathbb{N}$ *and every* $p \in [1, +\infty]$, *we define the Sobolev space* $\mathrm{W}^{k,p}(\Omega)$ *by:*

$$\mathrm{W}^{k,p}(\Omega) = \{u \in \mathrm{L}^p(\Omega) \quad | \ \forall \alpha : \ |\alpha| \leq k; \exists v_\alpha \in \mathrm{L}^p(\Omega) \ \textit{such that} \ v_\alpha = D^\alpha u\},$$

Remark 2.5.3 *We will very often make the abuse of notation consisting in identifying* $D^\alpha u$ *and* v_α.

The norm on the space $\mathrm{W}^{k,p}(\Omega)$ is given by:

$$\| u \|_{\mathrm{W}^{k,p}(\Omega)} = \begin{cases} (\sum_{|\alpha| \leq k} \| D^\alpha u \|^p_{\mathrm{L}^p(\Omega)})^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \| D^\alpha u \|_{\mathrm{L}^\infty(\Omega)} & \text{if} \quad p = \infty. \end{cases}$$

For $p = 2$, the space $\mathrm{W}^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ and the above norm is induced by an inner product.

Theorem 2.5.3 *The Sobolev spaces* $\mathrm{W}^{k,p}(\Omega)$, *for all* $k \in \mathbb{N}$ *and* $p \in [1, +\infty]$, *endowed with the norm* $\| \cdot \|$, *are Banach spaces. Moreover,for every integer* k *the spaces* $H^k(\Omega)$ *are Hilbert spaces.*

For further details on sobolev spaces,we refer the reader to [3].

2.6 Distribution Spaces

Sobolev spaces require some key notions and techniques from the theory of Schwartz distributions, without going into too many details,we introduce the concept of the derivative in the sense of distributions (or weak derivative),as well as distribution spaces.
For more details,see [4]

Definition 2.6.1 *Let* Ω *be an open set* $\mathbb{R}^N$. *A sequence* $(\varphi_n)_{n \in N} \subset D(\Omega)$ *is said to converge to a function* $\varphi \in D(\Omega)$ *if the following conditions are satisfied :*

1. *There exists a compact set* $K \subset\subset \Omega$ *such that* $supp(\varphi_n) \subset K$ *for all* $n \in \mathbb{R}$
2. $\lim_{n \longmapsto \infty} D^\alpha \varphi_n(x) = D^\alpha \varphi(x)$ *uniformly on* K, *for every multi-index* α , *that is to say*

$$\forall \alpha \in \mathbb{N}^N, \lim_{n \longmapsto \infty} \sup_{x \in K} |D^\alpha \varphi_n(x) - D^\alpha \varphi(x)| = 0$$

Definition 2.6.2 *A distribution* T *on* Ω *is a linear function on* $D(\Omega)$ *which satisfies the following continuity property :*

For every compact set K in Ω , there exist an integer $k \in N$ and a constant $C > 0$, such that for every test function $\varphi \in D(\Omega)$ with supp$(\varphi) \subset K$, we have:

$$|\langle T, \varphi\rangle| \leq C \sup_{0\leq l\leq k} \sup_{x\in K} |\varphi^{(l)}(x)|.$$

The space of distributions, denoted by $D'(\Omega)$, is the vector space of continuous linear functional on $D(\Omega)$.

Proposition 2.6.1 *For every $u \in L^1_{loc}(\Omega)$, there exists a distribution $T_u \in D'(\Omega)$ defined by*

$$\langle T_u, \varphi\rangle = \int_\Omega u(x)\varphi(x)dx, \text{ for all } \varphi \in D(\Omega).$$

A function $u \in L^1_{loc}(\Omega)$ is identified with its associated distribution T_u.

Proposition 2.6.2 *If $u \in L^1_{loc}(\Omega)$ and $\int_\Omega u(x)\varphi(x)dx = 0$, for all $\varphi \in D(\Omega)$,then $u = 0$ on Ω.*

Since every function $\varphi \in D(\Omega)$ vanishes identically outside a compact subset of Ω, it is clear, by integration by parts , that for every function $u \in C^1(\Omega)$ the following relation holds :

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\varphi(x)dx = -\int_\Omega \frac{\partial \varphi}{\partial x_i}(x)u(x)dx, \text{for all } 1 \leq i \leq N.$$

2.7 Functional Spaces

We now introduce the following Hilbert spaces , associated with the mechanical unknowns $\boldsymbol{u}$ and $\boldsymbol{\sigma}$:

$$\begin{cases} H = \{\boldsymbol{u} = (u_i) \quad | \quad u_i \in \mathrm{L}^2(\Omega)\} = (\mathrm{L}^2(\Omega))^d, \\ \mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \quad | \quad \sigma_{ij} = \sigma_{ji} \in \mathrm{L}^2(\Omega)\} = (\mathrm{L}^2_s(\Omega))^{d\times d}, \\ H_1 = \{\boldsymbol{u} = (u_i) \quad | \quad u_i \in H^1(\Omega)\} = (H^1(\Omega))^d, \\ \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \quad | \quad \sigma_{ij,j} \in H\}. \end{cases} \tag{2.3}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the following scalar products :

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx, \\ (\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\boldsymbol{\sigma}, \mathrm{Div}\boldsymbol{\tau})_H, \end{cases} \tag{2.4}$$

respectively, where $\varepsilon : H_1 \to \mathcal{H}$ and Div : $\mathcal{H}_1 \to H$ are the deformation and divergence operators , respectively , defined by:

$$\varepsilon(\boldsymbol{u}) = \varepsilon_{ij}(\boldsymbol{u}),\ \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $\| \cdot \|_H, \| \cdot \|_{\mathcal{H}}, \| \cdot \|_{H_1}$ and $\| \cdot \|_{\mathcal{H}_1}$, respectively.

Since boundary Γ is Lipschitz, the outward unit vector $\boldsymbol{\nu}$ is defined almost everywhere on the boundary. For any vector field $\boldsymbol{v} \in H_1$,we use the notation $\boldsymbol{v}$ to denote the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ. Recall that the trace operator $\gamma : H_1 \to \mathrm{L}^2(\Gamma)^d$ it is linear and continuous,but it is not surjective.

Let H_Γ' denote the dual space of H_Γ, and let $(\cdot,\cdot)$ denote the duality pairing between H_Γ' and H_Γ. For any $\boldsymbol{\sigma} \in \mathcal{H}_1$, there exists an element $\boldsymbol{\sigma\nu} \in H_\Gamma'$ such that:

$$(\boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}) = (\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\boldsymbol{\sigma}, \boldsymbol{v})_H \;\; \forall \boldsymbol{v} \in H_1. \tag{2.5}$$

Moreover, if $\boldsymbol{\sigma}$ is sufficiently regular (for example , of class C^1), we have

$$(\boldsymbol{\sigma\nu}, \gamma\boldsymbol{v}) = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} da \;\; \forall \boldsymbol{v} \in H_1. \tag{2.6}$$

Hence , for sufficiently regular $\boldsymbol{\sigma}$, we obtain the following Green's formula :

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} da \;\; \forall \boldsymbol{v} \in H_1, \tag{2.7}$$

Where da denotes the surface measure.

We define the closed subspace of H_1 by

$$\mathrm{V} = \{\boldsymbol{v} \in H_1 \quad | \quad \boldsymbol{v} = 0 \quad \text{on}\,\Gamma_1\}. \tag{2.8}$$

Since $\mathrm{meas}(\Gamma_1) > 0$, Korn's inequality applies to V.Therefore , there exists a constant $c_k > 0$, depending only on Ω and Γ_1 ,such that

$$\| \varepsilon(\boldsymbol{v}) \|_{\mathcal{H}} \geq c_k \| \boldsymbol{v} \|_{H_1} \quad \forall \boldsymbol{v} \in \mathrm{V}. \tag{2.9}$$

On the space V, we consider the inner product defined by;

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathrm{V}} = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathrm{V}, \tag{2.10}$$

Let $\| \cdot \|_{\mathrm{V}}$ denote the associated norm , that is,

$$\| \boldsymbol{v} \|_{\mathrm{V}} = \| \varepsilon(\boldsymbol{v}) \|_{\mathcal{H}} \qquad \forall \boldsymbol{v} \in \mathrm{V}. \tag{2.11}$$

By Korn's inequality ,it follows that the norms $\| \cdot \|_{H_1}$ and $\| \cdot \|_{\mathrm{V}}$ are equivalent on V, and thus $(\mathrm{V}, \| \cdot \|_{\mathrm{V}})$ is a Hilbert space. Moreover, using the Sobolev trace theorem, (2.7) and (2.8), there exists a constant $c_0 > 0$ depending only on Ω, Γ_1 and Γ_3 , such that :

$$\| \boldsymbol{v} \|_{\mathrm{L}^2(\Gamma_3)^d} \leq c_0 \| \boldsymbol{v} \|_{\mathrm{V}} \qquad \forall \boldsymbol{v} \in \mathrm{V}. \tag{2.12}$$

For a scalar function β, with represent the adhesion field on the contact surface Γ_3 , we define the set:

$$\mathcal{Q} = \{\beta : \Gamma_3 \times [0,T] \to \mathbb{R} \mid 0 \leq \beta(t) \leq 1 \text{ on } \Gamma_3\}. \tag{2.13}$$

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by:

$$u.v = u_i v_i \quad \|v\| = \langle u.v\rangle^{1/2} \quad \forall u,v \in \mathbb{R}^d, \tag{2.14}$$

$$\sigma.\tau = \sigma_{ij}\tau_{ij} \quad \|\sigma\| = \langle \sigma.\tau\rangle^{1/2} \quad \forall \sigma,\tau \in \mathbb{S}^d. \tag{2.15}$$

We use standard notation for Lebesgue and Sobolev spaces in Ω and on Γ.

Let

$$V = \left\{v = (v_i) \in H^1 : v = 0 \ on \ \Gamma_1\right\}, \quad Q = \left\{\tau = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ij}\right\}.$$

These are real Hilbert spaces endowed with the inner products

$$\langle u,v\rangle_V = \int_\Omega \varepsilon(u).\varepsilon(v)dx, \quad \langle \sigma,\tau\rangle_Q = \int_\Omega \sigma.\tau dx$$

and the associated norms $\|.\|_V$ and $\|.\|_Q$, respectively. Here ε represents the deformation operator given by

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v)\frac{1}{2}(v_{ij} + v_{ji}) \quad \forall v \in H^1(\Omega)^d. \tag{2.16}$$

Since meas$(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds.

A proof of the Friedrichs-Poincaré inequality can be found in [5].

Let $0 < T < \infty$, and let $(X, \|\cdot\|_X)$ be a real Banach space.

We denote by $C([0,T],\mathrm{X})$ and $C^1([0,T],\mathrm{X})$ the space of continuous and continuously differentiable functions on $[0,T]$ with values in X, respectively, endowed with the norms:

$$\| f \|_{C([0,T],\mathrm{X})} = \max_{t\in[0,T]} \| f(t) \|_{\mathrm{X}},$$

$$\| f \|_{C^1([0,T],\mathrm{X})} = \max_{t\in[0,T]} \| f(t) \|_{\mathrm{X}} + \max_{t\in[0,T]} \| \dot{f}(t) \|_{\mathrm{X}} .$$

We denote by $C_c([0,T],\mathrm{X})$ the set of continuous functions with compact support in $[0,T]$ taking values in X.

Definition 2.7.1 *A function $f : [0,T] \to \mathrm{X}$ is said to be measurable if there exists a subset $\mathrm{E} \subset [0,T]$ of measure zero and a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $C_c([0,T],\mathrm{X})$ such that $\| f_n(t) - f(t) \|_{\mathrm{X}} \longrightarrow 0$ as $n \longrightarrow \infty$, for all $t \in [0,T] \setminus \mathrm{E}$.*

2.7.1 Review of Non linear Analysis in Hilbert Spaces

I- Strongly Monotone Operator

We present some definitions and properties concerning non linear operators and bilinear forms in a Hilbert space endowed with the inner product $(\cdot,\cdot)_{\mathrm{X}}$ and the associated norm $\|\cdot\|_{\mathrm{X}}$

Definition 2.7.2 *Let* $\mathrm{A}:\mathrm{X}\longrightarrow\mathrm{X}$ *be a non-linear operator. The operator* A *is said to be:*

(1) Monotone if

$$(\mathrm{A}u-\mathrm{A}v,u-v)_{\mathrm{X}}\geq 0 \quad \forall u,v\in\mathrm{X}; \tag{2.17}$$

(2) Strongly monotone if there exists a constant $m>0$ *such that*

$$(\mathrm{A}u-\mathrm{A}v,u-v)_{\mathrm{X}}\geq m\parallel u-v\parallel_{\mathrm{X}} \quad \forall u,v\in\mathrm{X}; \tag{2.18}$$

(3) Lipschitz continuous if there exists a constant $M>0$ *such that*

$$\parallel \mathrm{A}u-\mathrm{A}v\parallel_{\mathrm{X}}\leq M\parallel u-v\parallel_{\mathrm{X}} \quad \forall u,v\in\mathrm{X}. \tag{2.19}$$

Let the functional $j:\mathrm{X}\times\mathrm{X}\to\mathbb{R}$ satisfy:

$$\begin{cases}(a)\text{For every}\,\eta\in\mathrm{X},j(\eta,\cdot)\,\text{is convex and lower semicontinuous on}\,\mathrm{X},\\ (b)\text{There exist a constant }\alpha>0\ \text{ such that}\\ j(u_1,v_1)-j(u_1,v_1)+j(u_2,v_1)-j(u_2,v_2)\leq\alpha\parallel u_1-u_2\parallel_{\mathrm{X}}\parallel v_1-v_2\parallel_{\mathrm{X}}.\end{cases} \tag{2.20}$$

The existence and uniqueness of a solution to the problem.

$$(\mathrm{A}u,u-v)_{\mathrm{X}}+j(u,v)+j(u,u)\geq(f,u-v)_{\mathrm{X}} \qquad \forall v\in\mathrm{X} \tag{2.21}$$

is provided the following result.

Theorem 2.7.1 *Assume that hypotheses* (2.18),(2.19) *and* (2.20) *are satisfied. Then* $\alpha<m$, *and for every* $f\in\mathrm{X}$, *there exists a unique solution* $u\in\mathrm{X}$ *to problem* (2.21).

Theorem 2.7.2 ***(Banach fixed point Theorem)*** *Let* K *be a non-empty closed subset of the Banach space* $(\mathrm{X},\|\cdot\|_{\mathrm{X}})$. *Assume that* $\mathbf{\Lambda}:\mathrm{K}\to\mathrm{K}$ *is a contraction, that is , there exists a constant* $c\in]0,1[$ *such that*

$$\parallel\mathbf{\Lambda}(u)-\mathbf{\Lambda}(v)\parallel_{\mathrm{X}}\leq c\parallel u-v\parallel_{\mathrm{X}} \quad \forall u,v\in\mathrm{K}.$$

Then, there exists a unique element $u\in\mathrm{K}$ *such that* $\mathbf{\Lambda}(u)=u$ *; that is, it admits a unique fixed point in* K.

For the operator $\mathbf{\Lambda}^m : \mathrm{K} \to \mathrm{K}$ defined by the relation

$$\mathbf{\Lambda}^m = \mathbf{\Lambda}(\mathbf{\Lambda}^{m-1}) \qquad m \geq 2,$$

we have the following version of the fixed point theorem.

Theorem 2.7.3 *Let* K *be a non-empty closed subset of the Banach space* $(\mathrm{X}, \| \cdot \|_{\mathrm{X}})$*. Assume that* $\mathbf{\Lambda}^m : \mathrm{K} \to \mathrm{K}$ *is a contraction for some positive integer* m*. Then* $\mathbf{\Lambda}$ *admits a unique fixed point in* K*.*

Definition 2.7.3 *A bilinear form* $b : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is said to be continuous there exists a real number* $M > 0$ *such that:*

$$\| b(u,v) \|_{\mathrm{X}} \leq M \| u \|_{\mathrm{X}} \| v \|_{\mathrm{X}}, \quad \forall u, v \in \mathrm{X}.$$

Definition 2.7.4 *A bilinear form* $b : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is said to be coercive if there exists a constant* $m > 0$ *such that:*

$$b(u,u) \geq m \| u \|_{\mathrm{X}}^2, \quad \forall u \in \mathrm{X}.$$

Theorem 2.7.4 ***(Lax-Milgram Theorem)***

Let H *be a Hilbert space ,* $b : \mathrm{H} \times \mathrm{H} \longrightarrow \mathbb{R}$ *a continuous , coercive and bilinear form.*

Let $l : \mathrm{H} \longrightarrow \mathbb{R}$ *be a continuous linear functional. Then, there exists a unique solution* $u \in \mathrm{H}$ *such that:*

$$b(u,v) = l(v), \quad \forall v \in \mathrm{H}. \tag{2.22}$$

Furthermore, if $b(\cdot,\cdot)$ *is symmetric, then* u *is characterized by the variational inequality:*

$$\frac{1}{2} b(u,u) - \langle u, u \rangle_{\mathrm{X}} \leq \frac{1}{2} b(v,v) - \langle v, v \rangle_{\mathrm{X}}, \quad \forall v \in \mathrm{X}. \tag{2.23}$$

II- Ordinary Differential Equation

Theorem 2.7.5 ***(Cauchy-Lipschitz)****: Let* $(\mathrm{X}, \| \cdot \|_{\mathrm{X}})$ *be a real Banach space, and let* $\mathrm{F}(t, \cdot) : \mathrm{X} \to \mathrm{X}$ *be an operator everywhere on* $[0,T]$*, satisfy the following properties:*

$$\begin{cases} (a)\ \textit{There exists}\, \mathrm{L_F} > 0 \,\textit{such that} \\ \| \mathrm{F}(t,x) - \mathrm{F}(t,y) \|_{\mathrm{X}} \leq \mathrm{L_F} \| x - y \|_{\mathrm{X}} \quad \forall x, y \in \mathrm{X}, a.e.\, t \in [0,T]; \\ (b)\, \textit{There exists}\, 1 \leq p \leq \infty \,\textit{such that} \\ \mathrm{F}(\cdot, x) \in \mathrm{L}^p([0,T]; \mathrm{X}) \quad \forall x \in \mathrm{X}. \end{cases}$$

Then, for every $x_0 \in \mathrm{X}$*, there exists a unique function* $x \in \mathrm{W}^{1,p}([0,T]; \mathrm{X})$ *such that;*

$$\dot{x}(t) = \mathrm{F}(t, x(t)),\ a.e.\, t \in [0,T],$$

$$x(0) = x_0.$$

2.7.2 Gronwall's Lemma

We recall here a Gronwall-type lemma , whith appears in many contact problems , in particular to establish the uniqueness the solution.

Lemma 2.7.1 *Let* $m, n \in C([0,T];\mathbb{R})$ *such that* $m(t) \geq 0$ *and* $n(t) \geq 0$ *for all* $t \in [0,T]$, $a \geq 0$ *a constant and* $\psi \in C([0,T];\mathbb{R})$

(1) if

$$\psi(t) \leq a + \int_0^t m(s)ds + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

then

$$\psi(t) \leq (a + \int_0^t m(s)ds)\exp(\int_0^t n(s)ds) \quad \forall t \in [0,T].$$

(2) if

$$\psi(t) \leq m(t) + a\int_0^t \psi(s)ds \quad \forall t \in [0,T],$$

then

$$\int_0^t \psi(s)ds \leq e^{at}\int_0^t m(s)ds \quad \forall t \in [0,T].$$

In the particular case $a = 0$, $n = 1$, part (1) of this lemma.

Corollary 2.7.1 *Let* $m \in C([0,T];\mathbb{R})$ *such that* $m(t) \geq 0$ *for all* $t \in [0,T]$*. If* $\psi \in C([0,T];\mathbb{R})$ *is a function satisfying:*

$$\psi(t) \leq \int_0^t m(s)ds + \int_0^t \psi(s)ds \quad \forall t \in [0,T],$$

then, there exists $c > 0$ *such that*

$$\psi(t) \leq \int_0^t m(s)ds \quad \forall t \in [0,T].$$

Corollary (2.7.1) is often used to prove the uniqueness of the solution.
In the particular case $m = 0$, part (1) of the lemma becomes.

Corollary 2.7.2 *Let* $n \in C([0,T];\mathbb{R})$ *such that* $n(t) \geq 0$ *for all* $t \in [0,T]$*. If* $a \geq 0$ *and* $\psi \in C([0,T];\mathbb{R})$*is a function satisfying;*

$$\psi(t) \leq a + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T].$$

then.

$$\psi(t) \leq (a)exp(\int_0^t n(s)ds) \quad \forall t \in [0,T].$$

The previous corollary is often used to prove the uniqueness of the solution as follows. Assuming that there exist two solutions,and denoting ψ the norm of the difference between these solutions,one then tries to bound ψ in the form

$$\psi(t) \leq \int_{0^t} n(s)\psi(s)ds. \quad \forall t \in [0,T]$$

with some function $n \geqslant 0$.The application of the corollary immediately implies that ψ is identically zero.

Chapter 3

Contact Problem with multivalued Normal compliance and Unilateral constraint

3.1 Problem Formulation P

Find a displacement field $\mathbf{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that for all $t \in \mathbb{R}_+$

$$\sigma(t) = \mathcal{A}\varepsilon(u(t)) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds \text{ in } \Omega. \tag{3.1}$$

$$Div\sigma(t) + f_0(t) = 0 \text{ in } \Omega, \tag{3.2}$$

$$u(t) = 0 \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\sigma(t)\nu = f_2(t) \quad \text{on } \Gamma_2. \tag{3.4}$$

$$\sigma_\tau(t) = 0 \quad \text{on } \Gamma_3 \tag{3.5}$$

and there exists $\xi ; \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ with

$$\left.\begin{aligned}
&u_v(t) \le g,\ \sigma_v(t) + p(u_v(t)) + \xi(t) \le 0\\
&(u_v(t) - g)(\sigma_v(t) + p(u_v(t)) + \xi(t)) = 0,\\
&0 \le \xi(t) \le F,\\
&\xi(t) = 0 \text{ if } u_v(t) < 0,\\
&\xi(t) = F \text{ if } u_v(t) > 0,
\end{aligned}\right\} \text{on} \quad \Gamma_3. \tag{3.6}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable x. Eq.(3.1) represents the viscoelastic constitutive law with long memory introduced in Section 1. Eq(3.2) represents the equation of equilibrium in which Div denotes the divergence operator for tensor valued functions. Conditions (3.3) and (3.4) are the displacement boundary condition and the traction boundary condition, respectively. Condition (3.5) is the frictionless condition and it shows that the tangential stress on the contact surface vanishes.

We now comment on the contact condition (3.6). Here σ_v denotes the normal stress, u_v is the normal displacement and its positive part u_v^+ may be interpreted as the penetration of the bodys surface asperities and those of the foundation. The function p is Lipschitz continuous and increasing, and vanishes for a negative argument,F is a positive function and $g > 0$. This condition can be derived in the following way. Consider the contact at a given time moment $t \in \mathbb{R}$. First, assume that the penetration is bounded by an amount g and, therefore, the normal displacement satises the inequality

$$u_v(t) \leq g \text{ on } \Gamma_3. \tag{3.7}$$

Next, assume that the normal stress has an additive decomposition of the form

$$\sigma_v(t) = \sigma_v^D(t) + \sigma_v^R(t) + \sigma_v^M(t) \text{ on } \Gamma_3, \tag{3.8}$$

in which the function σ_v^D describes the deformability of the foundation and the functions σ_v^R, σ_v^M describe the rigidity of the foundation. We assume that σ_v^D satises a normal compliance contact condition, that is

$$-\sigma_v^D(t) = p(u_v(t)) \text{ on } \Gamma_3. \tag{3.9}$$

The part σ_v^R of the normal stress satises the Signorini condition in the form with a gap function, i.e.

$$\sigma_v^R(t) \leq 0, -\sigma_v^R(u_v(t) - g) = 0 \text{ on } \Gamma_3. \tag{3.10}$$

Finally, the function $\sigma_v^M(t)$ satises the condition

$$|\sigma_v^M(t)| \leq F, \ \sigma_v^M(t) = 0 \text{ if } u_v(t) < 0,$$

$$-\sigma_v^M(t) = F \text{ if } u_v(t) > 0 \text{ on } \Gamma_3. \tag{3.11}$$

We combine (3.8), (3.9) and denote $-\sigma_v^M(t) = \xi(t)$ to see that

$$\sigma_v^R(t) + \sigma_v(t) + p(u_v(t)) + \xi(t) \text{ on } \Gamma_3. \tag{3.12}$$

Then we substitute equality (3.12) in (3.10) and use (3.7), (3.11) to obtain the contact condition (3.6).

We now present additional details of the contact condition (3.6). The inequalities and equalities below in this section are valid at an arbitrary point $x \in \Gamma_3$. First, we recall that (3.6) describes a condition with unilateral constraint, since inequality (3.7) holds at each time moment. Next, assume that at a given moment t there is separation between the body and the foundation, i.e. $u_v(t) < 0$.

Then, since $p(u_v(t)) = 0$, (3.6) shows that $\sigma_v(t) = 0$, i.e. the reaction of the foundation vanishes. Note that the same behavior of the normal stress is described both in the classical normal compliance condition and in the Signorini contact condition, when separation arises. Finally, assume that at a given moment t there is penetration without reaching the bound g, i.e. $0 < u_v(t) < g$. Then (3.6) yields

$$\sigma_v(t) = p(u_v(t)) + F. \tag{3.13}$$

This equality shows that the reaction of the foundation depends on the current value of the penetration and represents a normal compliance condition.

Note that (3.6) also implies that if at the moment t we have penetration with $0 < u_v(t) < g$, then $-\sigma_v(t) \geq F.$. Indeed, if $0 < u_v(t) < g$, then (3.13) holds and this implies that $-\sigma_v(t) \geq F$. We conclude from above that if $-\sigma_v(t) < F$ then there is no penetration and, therefore, F represents the critical value of the normal pressure, under which the penetration is not possible. This describes a rigid-elastic behaviour of the foundation.

In conclusion, condition (3.6) shows the normal stress vanishes where there is separation; the penetration occurs only if the normal stress reaches the critical value F; when there is penetration the contact follows a normal compliance condition of the form (3.13) but up to the limit g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g. For this reason we refer to this condition as a multivalued normal compliance contact condition with unilateral constraint. It can be interpreted physically as follows. The foundation is assumed to be made of a hard material covered by a thin layer composed of a rigid crust and a soft material with thickness g. The thin layer has a rigid-elastic behavior, i.e. is deformable, allows penetration, but only if the normal stress reaches the yield value F; in this case, the contact is modeled with normal compliance, as shown in equality (3.13). The hard material is perfectly rigid and, therefore, it does not allow penetration; the contact with this material is modeled with the Signorini contact condition.

Additional comments on the contact condition (3.6).

3.2 Find the auxiliary Problem

In this section we list the assumptions on the data, derive the variational formulation of the problem $\mathcal{P}$ and then we state and prove its unique weak solvability.To this end we assume that the elasticity operator $\mathcal{A}$ and the relaxation tensor $\mathcal{B}$ satisfy the following conditions.

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that } \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \\ \leqslant L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \text{a.e.} \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \\ \geqslant m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \text{a.e.} \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on} \\ \Omega, \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{3.14}$$

$$\mathcal{B} \in C(\mathbb{R}_+; \mathbf{Q}_\infty). \tag{3.15}$$

The densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{3.16}$$

Finally, the normal compliance function p and the surface yield function F satisfy:

$$\begin{cases} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that } |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \\ \quad \leqslant L_p |r_1 - r_2| \forall r_1, r_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \geqslant 0 \forall r_1, r_2 \in \mathbb{R}, \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}. \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \leqslant 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.17}$$

$$F \in L^2(\Gamma_3), \quad F(\boldsymbol{x}) \geqslant 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{3.18}$$

In what follows we consider the set of admissible displacements defined by

$$U = \{\boldsymbol{v} \in V : v_\nu \leqslant g \text{ on } \Gamma_3\}. \tag{3.19}$$

Moreover, we define the operator $P : V \to V$ and functions $j : V \to \mathbb{R}_+, \boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.20}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} F v_\nu^+ \, da \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.21}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot \boldsymbol{v}\,da \quad \forall \boldsymbol{v}\in V, t\in[0,T]. \tag{3.22}$$

Note that assumptions (3.16)–(3.18) imply that the integrals in (3.20)–(3.22) are well-defined.

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.1)–(3.6) and let $\boldsymbol{v}\in U$ and $t > 0$ be given. First, we use Green's formula (1.7) and the equilibrium Eq. (3.2) to see that

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx = &\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx\\ &+\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da.\end{aligned}$$

We split the surface integral over Γ_1, Γ_2 and Γ_3. Since $\boldsymbol{v}-\boldsymbol{u}(t) = \boldsymbol{0}$ a.e. on Γ_1, $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t)$ on Γ_2, we deduce that:

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))dx = &\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))dx\\ &+\int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))da\\ &+\int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))da.\end{aligned}$$

Moreover, since

$\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))$ on Γ_3, taking into account the frictionless condition (3.5) we obtain

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))dx = &\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))dx\\ &+\int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))da\\ &+\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))da.\end{aligned} \tag{3.23}$$

Write now

$$\begin{aligned}\sigma_\nu(t)(v_\nu - u_\nu(t)) =&(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t))(v_\nu - g) + (\sigma_\nu(t)\\ &+ p(u_\nu(t)) + \xi(t))(g - u_\nu(t))(p(u_\nu(t)) + \xi(t))(v_\nu\\ &- u_\nu(t)) \quad \text{on } \Gamma_3.\end{aligned}$$

Then use the contact conditions (3.6) and the definition (3.19) of the set U to see that

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) \geqslant -(p(u_\nu(t)) + \xi(t))(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3. \tag{3.24}$$

We use (3.6) and the hypothesis (3.18) on function F to deduce that

$$F(v_\nu^+ - u_\nu^+(t)) \geqslant \xi(t)(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3. \tag{3.25}$$

Add the inequalities (3.24) and (3.25) and integrate the result on Γ_3 to find that

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))da \geqslant -\int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))da - \int_{\Gamma_3} F(v_\nu^+ - u_\nu^+(t))da. \tag{3.26}$$

Finally, we combine (3.23) and (3.26) and use the definitions (3.20)–(3.22) to deduce that

$$\begin{aligned}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q+(P\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V+j(\boldsymbol{v})-j(\boldsymbol{u}(t))&\\ \geqslant(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\quad\forall\boldsymbol{v}\in U.&\end{aligned}\tag{3.27}$$

To simplify the notation, we define the operator $\mathcal{S}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;Q)$ by the formula

$$(\mathcal{S}\boldsymbol{v})(t):=\int_0^t\mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{v}(s))ds\quad\forall\boldsymbol{v}\in C(\mathbb{R}_+;V),\ t\in\mathbb{R}_+.\tag{3.28}$$

We now substitute the constitutive law (3.1) in (3.27) to obtain the following variational formulation of Problem $\mathcal{P}$.

3.3 Problem $\mathcal{P}^V$.

Find a displacement field $\boldsymbol{u}:\mathbb{R}_+\to U$ such that for all $t\in\mathbb{R}_+$, the inequality below holds:

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q+((\mathcal{S}\boldsymbol{u})(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q&\\ +(P\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V+j(\boldsymbol{v})-j(\boldsymbol{u}(t))&\\ \geqslant(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\quad\forall\boldsymbol{v}\in U.&\end{aligned}\tag{3.29}$$

In the study of Problem $\mathcal{P}^V$ we have the following existence and uniqueness result.

Theorem 3.3.1 *Under the assumptions (3.14)–(3.18), Problem $\mathcal{P}^V$ has a unique solution which satisfies $\boldsymbol{u}\in C(\mathbb{R}_+;U)$.*

Proof. We apply Theorem (1.2.1) with $X=V,\ K=U$ and the operators $A:V\to V$ and $\mathcal{R}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;V)$ defined by

$$(A\boldsymbol{u},\boldsymbol{v})_V=(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(P\boldsymbol{u},\boldsymbol{v})_V\quad\forall\boldsymbol{u},\boldsymbol{v}\in V,\tag{3.30}$$

$$((\mathcal{R}\boldsymbol{u})(t),\boldsymbol{v})_V=((\mathcal{S}\boldsymbol{u})(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\quad\forall\boldsymbol{u}\in C(\mathbb{R}_+;V),\ \boldsymbol{v}\in V.\tag{3.31}$$

It is easy to see that condition (1.12) holds. Next, we use (3.14), (3.17) and (1.6) to see that the operator A verifies condition (1.13) with $M=L_{\mathcal{A}}+c_0^2L_p$ and $m=m_{\mathcal{A}}$.

Let $n\in\mathbb{N}^*$, $t\in[0,n]$ and let $\boldsymbol{u}_1,\boldsymbol{u}_2\in C(\mathbb{R}_+;V)$. Then, a simple calculation based on assumption (3.15) and inequality (1.10) shows that for any $\boldsymbol{v}\in V$,

$$((\mathcal{S}\boldsymbol{u}_1)(t)-(\mathcal{S}\boldsymbol{u}_2)(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\leqslant r_n\int_0^t\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s))-\boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))\|_Q ds\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q,$$

where

$$r_n=\max_{r\in[0,n]}\|\mathcal{B}(r)\|_{\mathbf{Q}_\infty}.\tag{3.32}$$

Therefore,

$$\|(\mathcal{S}\boldsymbol{u}_1)(t) - (\mathcal{S}\boldsymbol{u}_2)(t)\|_Q \leqslant r_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V ds. \tag{3.33}$$

Inequality (3.33) combined with the definition (3.31) of the operator $\mathcal{R}$ yields

$$\|(\mathcal{R}\boldsymbol{u}_1)(t) - (\mathcal{R}\boldsymbol{u}_2)(t)\|_V \leqslant r_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V ds. \tag{3.34}$$

Thus, the operator $\mathcal{R}$ satisfies the condition (1.14). Next, we use condition (3.18) and inequality (1.6) to see that the functional j defined by (3.21) is a seminorm on V and moreover,

$$j(\boldsymbol{v}) \leqslant c_0\|F\|_{L^2(\Gamma_3)}\|\boldsymbol{v}\|_V \quad \forall \boldsymbol{v} \in V. \tag{3.35}$$

Thus, the seminorm j is continuous on V and, therefore, (1.15) holds. Finally, using assumption (3.16) and definition (3.22) we deduce that $\boldsymbol{f} \in C(\mathbb{R}_+; V)$, i.e. (1.16) holds.

It follows now from Theorem 2.1 that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which satisfies the inequality

$$\begin{aligned} \boldsymbol{u}(t) \in U, \quad (A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + ((\mathcal{R}\boldsymbol{u})(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + j(\boldsymbol{v}) \\ -j(\boldsymbol{u}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U, \end{aligned} \tag{3.36}$$

for all $t \in \mathbb{R}_+$. In other words, there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ such that (3.29) holds for all $t \in \mathbb{R}_+$, which concludes the proof.

With the solution $\boldsymbol{u}$ of Problem $\mathcal{P}^V$, define $\boldsymbol{\sigma}$ by (3.1). Then, it follows from (3.14) and (3.15) that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Moreover, it is easy to see that (3.27) holds for all $t \in \mathbb{R}_+$ and, using standard arguments, it results from here that

$$\mathrm{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \forall t \in \mathbb{R}_+. \tag{3.37}$$

Therefore, using the regularity $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)$ in (3.16) we deduce that $\mathrm{Div}\boldsymbol{\sigma} \in C(\mathbb{R}_+; L^2(\Omega)^d)$ which implies that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$. Here,

$$Q_1 = \left\{\boldsymbol{\tau} \in Q : \mathrm{Div}\boldsymbol{\tau} \in L^2(\Omega)^d\right\},$$

which is a real Hilbert space with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{Div}\boldsymbol{\sigma}, \mathrm{Div}\boldsymbol{\tau})_{L^2(\Omega)^d},$$

and the associated norm $\|\cdot\|_{Q_1}$.

A pair of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfying (3.1), (3.29) for all $t \in \mathbb{R}_+$ is called a *weak solution* of the contact problem $\mathcal{P}$. 3.3.1 provides the unique weak solvability of Problem $\mathcal{P}$. Moreover, we have the regularity $\boldsymbol{u} \in C(\mathbb{R}_+; U), \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$.

3.4 A convergence result

We now study the dependence of the solution of Problem $\mathcal{P}^V$ with respect to perturbations of the data. To this end, we assume in what follows that (3.14) (3.18) hold and we denote by $\boldsymbol{u}$ the solution of Problem $\mathcal{P}^V$ obtained in 3.3.1. For a parameter $\rho > 0$, let $\boldsymbol{f}_{0\rho}, \boldsymbol{f}_{2\rho}, p_\rho$ and F_ρ represent perturbations of $\boldsymbol{f}_0, \boldsymbol{f}_2, p$ and F, respectively, which satisfy conditions (3.16) (3.18). With these data, we define the operator $P_\rho : V \to V$ and functions $j_\rho : V \to \mathbb{R}_+$, $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equalities

$$(P_\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\rho(u_\nu) v_\nu \, da \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.38}$$

$$j_\rho(\boldsymbol{v}) = \int_{\Gamma_3} F_\rho v_\nu^+ \, da \quad \forall \boldsymbol{v} \in V, \tag{3.39}$$

$$(\boldsymbol{f}_\rho(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0\rho}(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_{2\rho}(t) \cdot \boldsymbol{v} \, da \quad \forall \boldsymbol{v} \in V, \; t \in [0, T]. \tag{3.40}$$

Then, we consider the following perturbation of Problem $\mathcal{P}^V$.

Problem.$\mathcal{P}_\rho^V$.

Find a displacement field $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U$ such that for all $t \in \mathbb{R}_+$, the inequality below holds:

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + ((\mathcal{S}\boldsymbol{u}_\rho)(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\
&\quad + (P_\rho \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V + j_\rho(\boldsymbol{v}) - j_\rho(\boldsymbol{u}_\rho(t)) \\
&\quad \geqslant (\boldsymbol{f}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \quad \forall \boldsymbol{v} \in U.
\end{aligned} \tag{3.41}$$

It follows from 3.3.1that, for each $\rho > 0$.

Problem $\mathcal{P}_\rho^V$ has a unique solution $\boldsymbol{u}_\rho$ with the regularity $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; U)$.

Introduce the following assumptions:

$$\begin{cases}
\text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \alpha \in \mathbb{R}_+ \text{ such that} \\
\text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \leqslant G(\rho)(|r| + \alpha) \\
\quad \forall r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for each } \rho > 0. \\
\text{(b) } G(\rho) \to 0 \text{ as } \rho \to 0.
\end{cases} \tag{3.42}$$

$$F_\rho \to F \quad \text{in } L^2(\Gamma_3) \text{ as } \rho \to 0. \tag{3.43}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in } C(\mathbb{R}_+; L^2(\Omega)^d) \text{ as } \rho \to 0. \tag{3.44}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in } C(\mathbb{R}_+; L^2(\Gamma_2)^d) \text{ as } \rho \to 0. \tag{3.45}$$

We have the following convergence result.

Theorem 3.4.1 *Under the assumptions* (3.42) (3.45)*, the solution* $\boldsymbol{u}_\rho$ *of Problem* $\mathcal{P}_\rho^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{P}^V$*, i.e.*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in } C(\mathbb{R}_+; V) \text{ as } \rho \to 0. \tag{3.46}$$

Proof. Let $\rho > 0, n \in \mathbb{N}^*$ and let $t \in [0, n]$. We take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (3.41), $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in (3.29), and add the resulting inequalities to obtain

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\
&\quad \leqslant ((\mathcal{S}\boldsymbol{u}_\rho)(t) - (\mathcal{S}\boldsymbol{u})(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (P_\rho \boldsymbol{u}_\rho(t) \\
&\qquad - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V + j_\rho(\boldsymbol{u}(t)) - j_\rho(\boldsymbol{u}_\rho(t)) + j(\boldsymbol{u}_\rho(t)) \\
&\qquad - j(\boldsymbol{u}(t)) + (\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V.
\end{aligned} \tag{3.47}$$

Let us bound each term on the right side of the inequality (3.47). First, from assumption (3.14) and inequality (3.33) it follows that

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\
&\quad \geqslant m_{\mathcal{A}} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
&((\mathcal{S}\boldsymbol{u}_\rho)(t) - (\mathcal{S}\boldsymbol{u})(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\
&\quad \leqslant r_n \left(\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u_2}(s)\|_V \, ds \right) \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.
\end{aligned} \tag{3.49}$$

Next, we use the definitions (3.38) and (3.44), the monotonicity of the function p_ρ and assumption (3.42) to see that

$$\begin{aligned}
&(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \\
&= \int_{\Gamma_3} (p_\rho(u_{\rho\nu}(t)) - p(u_\nu(t)))(u_\nu(t) - u_{\rho\nu}(t)) \, da \\
&\leqslant \int_{\Gamma_3} (p_\rho(u_\nu(t)) - p(u_\nu(t)))(u_\nu(t) - u_{\rho\nu}(t)) \, da \\
&\leqslant \int_{\Gamma_3} |p_\rho(u_\nu(t)) - p(u_\nu(t))||u_\nu(t) - u_{\rho\nu}(t)| \, da \\
&\leqslant \int_{\Gamma_3} G(\rho)(|u_\nu(t)| + \alpha)|u_\nu(t) - u_{\rho\nu}(t)| \, da.
\end{aligned}$$

Then using the trace inequality (3.15) , after some elementary calculations we find that

$$\begin{aligned}
&(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \\
&\leqslant G(\rho) \left(c_0^2 \|\boldsymbol{u}(t)\|_V + c_0 \alpha \mathrm{meas}(\Gamma_3)^{\frac{1}{2}} \right) \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.
\end{aligned} \tag{3.50}$$

On the other hand, definitions (3.39) and (3.21) combined with (1.6) imply

$$\begin{aligned}&j_\rho(\boldsymbol{u}(t)) - j_\rho(\boldsymbol{u}_\rho(t)) + j(\boldsymbol{u}_\rho(t)) - j(\boldsymbol{u}(t))\\ &\leqslant c_0\|F_\rho - F\|_{L^2(\Gamma_3)}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.\end{aligned} \tag{3.51}$$

Finally, we note that

$$(\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \leqslant \delta_{\rho n}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V, \tag{3.52}$$

where

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V. \tag{3.53}$$

We combine now inequalities (3.47) (3.52) to deduce that

$$\begin{aligned}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leqslant &\frac{r_n}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds\\ &+ \frac{G(\rho)}{m_{\mathcal{A}}}\left(c_0^2\|\boldsymbol{u}(t)\|_V + c_0\alpha \operatorname{meas}(\Gamma_3)^{\frac{1}{2}}\right)\\ &+ \frac{c_0}{m_{\mathcal{A}}}\|F_\rho - F\|_{L^2(\Gamma_3)} + \frac{\delta_{\rho n}}{m_{\mathcal{A}}}.\end{aligned} \tag{3.54}$$

Let

$$\xi_{n,u} = \max\left\{\frac{1}{m_{\mathcal{A}}}\left(c_0^2\max_{r\in[0,n]}\|\boldsymbol{u}(r)\|_V + c_0\alpha\operatorname{meas}(\Gamma_3)^{\frac{1}{2}}\right), \frac{c_0}{m_{\mathcal{A}}}, \frac{1}{m_{\mathcal{A}}}\right\}.$$

Then from (3.54),

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leqslant \left(G(\rho) + \|F_\rho - F\|_{L^2(\Gamma_3)} + \delta_{\rho n}\right)\xi_{n,u} + \frac{r_n}{m_{\mathcal{A}}}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds.$$

Applying the Gronwall inequality we obtain

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leqslant \left(G(\rho) + \|F_\rho - F\|_{L^2(\Gamma_3)} + \delta_{\rho n}\right)\xi_{n,u}e^{\frac{r_n}{m_{\mathcal{A}}}t}$$

and thus,

$$\max_{t\in[0,n]}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leqslant (G(\rho) + \|F_\rho - F\|_{L^2(\Gamma_3)} + \delta_{\rho n})\xi_{n,u}e^{\frac{nr_n}{m_{\mathcal{A}}}}. \tag{3.55}$$

By assumptions (3.42)(b), (3.43) (3.45) and definition (3.53),

$$G(\rho) \to 0, \quad \|F_\rho - F\|_{L^2(\Gamma_3)} \to 0 \quad \text{and} \quad \delta_{\rho n} \to 0 \quad \text{as } \rho \to 0. \tag{3.56}$$

Combining the convergence (3.56) with inequality (3.55) we obtain

$$\max_{t\in[0,n]}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \quad \text{as } \rho \to 0. \tag{3.57}$$

Since the convergence (3.57) holds for each $n \in \mathbb{N}^*$, we deduce from (1.11) that (3.46) holds, which concludes the proof.

The convergence result on the displacement in 3.4.1 leads to a corresponding result on the stress function. Indeed, let $\boldsymbol{\sigma}$ be the function defined by (3.1) and define

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))\,ds, \quad t \in \mathbb{R}_+. \tag{3.58}$$

Then, arguments similar to those used at the end of Section 4 imply that $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q_1)$ and moreover,

$$\text{Div}\ \boldsymbol{\sigma}_\rho(t) + \boldsymbol{f}_{\rho 0}(t) = \boldsymbol{0} \quad \forall t \in \mathbb{R}_+. \tag{3.59}$$

We combine equalities (3.1), (3.37), (3.58)and (3.59), and use assumptions (3.14), (3.15) and notation (3.32) to see that

$$\begin{aligned}\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_{Q_1} &= \|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \\ &\leqslant L_\mathcal{A}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + r_n \int_0^n \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds,\end{aligned}$$

for all $n \in \mathbb{N}^*$ and $t \in [0, n]$. Thus,

$$\max_{t\in[0,n]} \|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_{Q_1} \leqslant (L_\mathcal{A} + n r_n) \max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \quad \forall n \in \mathbb{N}^*. \tag{3.60}$$

We now use (3.60) and the convergence (3.57)to see that

$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in } C(\mathbb{R}_+; Q_1) \text{ as } \rho \to 0. \tag{3.61}$$

In addition to the mathematical interest in the convergence result (3.46), (3.61), it is of importance from mechanical point of view, since it states that the weak solution of the problem (3.1) (3.6) depends continuously on the normal compliance function, the surface yield function, and the densities of body forces and surface tractions.

ملخص

- في مذكرتنا تطرقنا إلى حالة من حالات التلامس اللزج لجسم صلب مع سطح أملس .
- **في البداية تطرقنا إلى بعض خواص الدوال التحليلية من ضمنها الدوال اللبشتيزية و بعض نظريات المعادلات التفاضلية المتراجحات في فضاءات مختلفة**
- **و من ثم طرح المسألة التغيرية المراد دراستها و تكون عادة صعبة فبحثنا عن مسألة مساعدة لتسهيل البحث عن الحل و تقاربه.**
- **في الأخير أثبتنا وجود الحل البسيط و تقاربه.**

كلمات مفتاحية:

تلامس _ إحتكاك _ قاعدة(أساس) _ عملية شبه ساكنة _ مواد لزجة مرنة _ معادلة تغايرية تكافئية .

General summary

We presented a model for the quasi-static process of contact between an electro-viscoelastic body and a foundation. The contact was modeled with the normal compliance condition and the associated dry friction Coulomb law. The new characteristic of the model was the normal conformity as presented in the difference equation (3.7). The difficulty in solving this type of problem lies not only in the coupling of viscoelastic and electrical aspects, but also in the non-linearity of the boundary conditions modeling this type of physical phenomena (contact conditions), which gives us a quasi-variational inequality and type of non-linear parabolic variational equalities. The existence of the weak solution of the problem was established using arguments from the theory of evolutionary variational inequalities, parabolic inequalities, and convergences of solutions.

Keywords: contact, foundation, friction, quasi-static process, viscoelastic, parabolic variational.

Résumé

Nous avons présenté un modèle pour le processus quasi-statique de contact entre un corp électro-viscoélastiques et un fondation. Le contact a été modélisé avec la condition de conformité normale et la loi de Coulomb de friction sèche associée. La nouvelle caractéristique du modèle était la conformité normale que présentée dans léquation de la différence (3.7). La difficulté de résoudre ce type de problème réside non seulement dans le couplage des aspects viscoélastiques, et électriques, mais aussi dans la non-linéarité des conditions aux limites modélisant ce type de phénomènes physiques (conditions de contact), ce qui nous donne une inégalités quasi variationnelles et type dégalités variationnelles paraboliques non linéaires. Lexistence de la solution unique faible du problème a été établie en utilisant des arguments de la théorie des inégalités variationnelles évolutives, des inégalités paraboliques et de la convergences de solution .

Mots-clés: contact , fondation, Frottement, processus quasistatique ,viscoélastiques, variationnelles paraboliques.

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