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Error estimation for a piezoelectric contact problem

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أمال

Dedications

I dedicate my graduation to the one blessed by God with dignity and majesty, to the one who taught me the art of giving without waiting, to the one whose name I proudly carry "***my dear father***".

To my angel in life, to the embodiment of love, tenderness, and devotion, to the smile of life and the essence of existence, and to the one whose prayer has been the secret to my success "***my dearest mother***".

To those who have greatly encouraged me, from whom I learned perseverance and diligence, and from whom I learned the meaning of life "***my brothers and sisters***".

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To the little hands that knock on my door as visitors, bringing joy and life into my days " **to the buds of the family**."

MAAMIR IKHLAS

Contents

Preface

Contact problems with or without friction involving deformable bodies are very common in industry as well as in everyday life. The contact of the piston with the liner, the wheel on the landing strip of an airplane and a shoe with the ground represent only three examples among many others. Considering the fact that these phenomena play an important role in mechanical structures and systems. Accompanied by complex physical and surface phenomena, contact processes are modeled by very difficult nonlinear boundary problems. The consideration of different contact and friction conditions associated with increasingly complex behavioral laws leads to new and non-standard models, expressed using variational evolution inequalities. Significant progress has been made recently in the variational and numerical analysis of these models.

Located at the crossroads of several scientific disciplines, the main characteristic of mathematical modeling in contact mechanics is the cross-fertilization between mechanical models and applications in engineering sciences, on the one hand, and mathematical and numerical analysis , on the other hand. Research in this area is motivated by significant application possibilities in different industrial sectors (notably in the metallurgical industry and the automobile industry) but also in civil engineering and medicine.

Piezoelectric materials were discovered at the beginning of the century by the Curie couple. These are particular dielectrics which make it possible to transform elastic deformation energy into electrical energy, and vice versa. More precisely, piezoelectricity is the ability of certain materials to polarize when they are mechanically stressed, the charge appearing on their surface being proportional to the deformation generated. The inverse piezoelectric effect is the obtaining of a deformation by application of an electric field.

Piezoelectric materials are very numerous. The best known is undoubtedly quartz, still used in watches to generate clock pulses. But it is synthetic ceramics, PZT (lead, zirconate, titanate) which are most widely used today in industry.

More generally, the direct effect can be used to create sensors (pressure sensors, etc.) while the opposite effect can be used to create actuators (piezoelectrically controlled injectors in automobiles, nanomanipulators).

The use of piezoelectricity has exploded in recent years and is expanding rapidly. The ability of these materials to convert mechanical energy into electrical energy and vice versa is invaluable for acoustic transducers, medical ultrasound, and for the high precision of pumps and motors. High piezoelectric performance has also opened up new possibilities for "energy harvesting", using ambient motion and vibration to generate electricity where batteries or other energy sources are impractical or unavailable [1].

The object of this memoir is the study of a frictionless contact problem, between a deformable piezoelectric body with a foundation and with normal compliance. The foundation is deformable. We place ourselves in the setting of small deformations and we study quasi-static processes for electro-elastic materials.

This memoir is organized into two parts which we will briefly describe in order to facilitate its reading.

The **first part** of this work is devoted to the preliminary results that we need for the study of the contact problem considered in the second part of the manuscript. It constitutes the basis of this dissertation and includes three chapters.

The *first chapter* presents the functional spaces used in the study of the variational problem considered. We briefly discuss some reminders about the functional spaces used in contact mechanics. Then, we present some results on strongly monotonic and Lipschitz operators. Next, we recall a result on elliptic variational equalities which we use later in this manuscript. To close this chapter, we recall the Gronwall type lemma used to obtain error estimates.

The *second chapter* of this part is essentially based on modeling. We begin by specifying the physical setting, then we present the electro-elastic behavioral law considered in contact problem. Finally, we describe the contact conditions that we employ in the considered contact problem.

In **the second part**, we proceed to the study of a frictionless contact problem between an electro-elastic body and an obstacle assimilated to a foundation. This part has two chapters which are structured as follows.

The *first chapter* is devoted to the analytical study of a frictionless contact problem for electro-elastic materials with normal compliance in a quasi-static process. Contact with a deformable base. The weak formulation of the problem is formulated as a system formed by a variational equation with respect to the displacement field and a variational equation with respect to the electric field. We establish a result of existence and uniqueness of the solution. The proof is based on the theory of elliptic variational equations.

In *the second chapter*, we present a numerical approximation of this contact problem based on an approximation in both time and space, using respectively a uniform temporal discretization and a finite element method. At the end of these discretizations, we give a result on the estimation of the error.

Notations

If Ω is a domain of $\mathbb{R}^d\,(d=2,3)$, we note by:

Γ	the boundary of Ω supposed to be regular.
$\Gamma_i\left(i=\overline{1,3},a,b\right)$	the part of the boundary Γ.
$mes\ \Gamma_i$	the Lebesgue measure $(d-1)$ dimensional of Γ_i.
$\boldsymbol{\nu}$	unit outward normal on the boundary of Ω.
$\upsilon_\nu,\ \boldsymbol{\upsilon}_\tau$	the normal and tangential components of the vector field $\boldsymbol{\upsilon}$ defined on $\overline{\Omega}$.
H	$\{\mathbf{u}=(u_i)\mid u_i\in L^2(\Omega)\}$.
$\mathcal{H}$	$\{\boldsymbol{\sigma}=(\sigma_{ij})\mid \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\}$.
H_1	$\{\mathbf{u}\in H\mid \boldsymbol{\varepsilon}(\mathbf{u})\in\mathcal{H}\}$.
$\mathcal{H}_1$	$\{\boldsymbol{\sigma}\in\mathcal{H}\mid Div\boldsymbol{\sigma}\in H\}$.
V	$\{\boldsymbol{\upsilon}\in H_1\mid \boldsymbol{\upsilon}=\mathbf{0}\text{ on }\Gamma_1\}$.
W	$\{\phi\in H^1(\Omega)\mid \phi=0\text{ on }\Gamma_a\}$.
$\mathcal{W}$	$\{\mathbf{D}\in H\mid div\mathbf{D}\in L^2(\Omega)\}$.
$H^1(\Omega)$	the Sobolev space of order 1 on Ω.
$H^{\frac{1}{2}}(\Gamma)$	the Sobolev space of order $\frac{1}{2}$ on Γ.
H_Γ	the space $H^{\frac{1}{2}}(\Gamma)^d$.
$H^{-\frac{1}{2}}(\Gamma)$	the dual space of $H^{\frac{1}{2}}(\Gamma)$.
H'_Γ	the dual space of $H_\Gamma=H^{-\frac{1}{2}}(\Gamma)^d$.
$\gamma: H_1\to H_\Gamma$	the trace application for vector functions.

If H is a real Hilbert space and $d\in\mathbb{N}^*$, we use the following notations:

H^d	the space $\{\mathbf{x}=(x_i)\ \mid x_i\in H,\ i=\overline{1,d}\}$.
$H_s^{d\times d}$	the space$\{\mathbf{x}=(x_{ij})\ \mid x_{ij}=x_{ji}\in H,\ i,j=\overline{1,d}\}$.
$(.,.)_H$	the scalar product of H.
$\lvert . \rvert_H$	the norm of H.

If more $[0,T]$ a time interval, $k\in\mathbb{N}$ and $1\le p\le+\infty$, we note by

$\mathcal{C}(0,T;H)$	the space of continuous functions on $[0,T]$ on H.
$\lvert . \rvert_{0,H}$	the norm of $\mathcal{C}(0,T;H)$.

For a function f, we note by

$f_{,i}$	the partial derivative of f compared to the i th component x_i.
∇f	gradient operator of f.
$\boldsymbol{\varepsilon}(f)$	linearized deformations operator of $f = \frac{1}{2}(\nabla f + \nabla^T f)$.
$Div f$	divergence operator of f.

Other notations

$\mathbb{S}^d$	the space of second-order symmetric tensors on $\mathbb{R}^d = \mathbb{R}_s^{d\times d}$.
c	a strictly positive generic constant.
a.e	almost everywhere.partout.
i.e	identically equal.
r_+	positive part of r.
Π_h	finite element interpolation operator.
$O(h)$	for any constant $c > 0$ independent of h such that $\lvert O(h)\rvert \leq ch$.

Part I

Preliminaries

This part, in which we present the functional analysis tools and contact mechanics notations, contains two chapters. In the first chapter we introduce certain functional spaces for electro-mechanics that we need in the study of the contact problem that we address in the second part and we present a brief reminder of monotone operators. Next, we review results concerning variational equations which will be of great use for demonstrations. In the second chapter we are interested in contact problem modeling. We present the law of behavior and the contact conditions.

Chapter 1

Elements of nonlinear analysis

In this chapter, we recall some results concerning functional spaces, a theorem on elliptic variational equations which will be of great use for the demonstrations and Gronwall's lemma.

1.1 Function spaces

The variational analysis of electro-mecanics problems requires the introduction of specific function spaces. In this section we introduce the functional spaces in mechanics associated with the divergence and deformation operators.

We define the following spaces:

$$\begin{cases} H = \left\{ \mathbf{u} = (u_i) \mid u_i \in L^2(\Omega) \right\}, \\ \mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \\ H_1 = \left\{ \mathbf{u} = (u_i) \mid \boldsymbol{\varepsilon}(\mathbf{u}) \in \mathcal{H} \right\}, \\ \mathcal{H}_1 = \left\{ \boldsymbol{\sigma} \in \mathcal{H} \mid Div\boldsymbol{\sigma} \in H \right\}. \end{cases} \tag{1.1}$$

The spaces H, $\mathcal{H}$, H_1 et $\mathcal{H}_1$ are real Hilbert spaces provided with scalar products given, respectively, by

$$\begin{cases} (\mathbf{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} dx, \\ (\mathbf{u}, \boldsymbol{v})_{H_1} = (\mathbf{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H, \end{cases} \tag{1.2}$$

where $\boldsymbol{\varepsilon} : H_1 \to \mathcal{H}$, $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(u)), \ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ Div\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The norms on the spaces H, $\mathcal{H}$, H_1 et $\mathcal{H}_1$ are denoted by $|.|_H$, $|.|_{\mathcal{H}}$, $|.|_{H_1}$ and $|.|_{\mathcal{H}_1}$, respectively.

Since the boundary Γ is Lipschitzian, the exterior normal vector $\boldsymbol{\nu}$ at the boundary is defined almost everywhere. For any vector field $\boldsymbol{v} \in H_1$ we use the notation $\boldsymbol{v}$ to designate $\gamma\boldsymbol{v}$ the trace of $\boldsymbol{v}$ on Γ.

We recall that the trace application $\gamma : H_1 \to L^2(\Gamma)^d$ is linear and continuous, but is not surjective. The image of H_1 by this application is denoted by $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$, this subspace is continuously injected into $L^2(\Gamma)^d$.

Furthermore, if $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ is sufficiently regular, we recall Green's formula:

$$\int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega Div\boldsymbol{\sigma}.\boldsymbol{v}dx = \int_\Gamma \boldsymbol{\sigma\nu}.\boldsymbol{v}da \ \forall \boldsymbol{v} \in H_1. \tag{1.3}$$

We now introduce a closed subspace of H_1, the definition of which is given below

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\}. \tag{1.4}$$

Since $mes\ \Gamma_1 > 0$, Korn's inequality applies to V, there exists a constant $c_K > 0$ depending only on Ω and Γ_1 such that

$$|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \geq c_K |\boldsymbol{v}|_{H_1} \forall \boldsymbol{v} \in V. \tag{1.5}$$

On V, we consider the scalar product given by

$$(\mathbf{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \ \forall \mathbf{u}, \boldsymbol{v} \in V, \tag{1.6}$$

and let $|.|_V$ be the associated norm; that's to say

$$|\boldsymbol{v}|_V = |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \ \forall \boldsymbol{v} \in V. \tag{1.7}$$

By Korn's inequality (1.5) and (1.7), it follows that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and thus $(V, (.,.)_V)$ is a Hilbert space.

Furthermore, according to (1.5), (1.7) and Sobolev's trace theorem, let us find that there exists a constant $c_0 > 0$ depending only on Ω, Γ_1 and Γ_3 such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)^d} \leq c_0 \, |\boldsymbol{v}|_V \;\; \forall \boldsymbol{v} \in V. \tag{1.8}$$

Then, we introduce the following spaces to the electric field:

$$W = \left\{ \phi \in H^1(\Omega) \mid \phi = 0 \text{ sur } \Gamma_a \right\}, \tag{1.9}$$

$$\mathcal{W} = \left\{ \mathbf{D} = (D_i) \mid D_i \in L^2(\Omega), \; div\mathbf{D} \in L^2(\Omega) \right\}, \tag{1.10}$$

where $div\mathbf{D} = (D_{i,i})$. Since $mes \; \Gamma_a > 0$, the Friedrichs-PoincarÃ© inequality is verified thus there exists a constant $c_F > 0$ depending only on Ω and Γ_a such that

$$|\nabla \phi|_H \geq c_F \, |\phi|_{H^1(\Omega)} \;\; \forall \phi \in W. \tag{1.11}$$

The spaces W and $\mathcal{W}$ are real Hilbert spaces provided with scalar products given by

$$(\varphi, \phi)_W = (\nabla \varphi, \nabla \phi)_H \,, \;\; (\mathbf{D}, \mathbf{E})_{\mathcal{W}} = (\mathbf{D}, \mathbf{E})_H + (div\mathbf{D}, div\mathbf{E})_{L^2(\Omega)} \,, \tag{1.12}$$

let $|.|_W$ and $|.|_{\mathcal{W}}$ be the associated norms; that's to say

$$|\phi|_W = |\nabla \phi|_H \,, |\mathbf{D}|^2_{\mathcal{W}} = |\mathbf{D}|^2_H + |div\mathbf{D}|^2_{L^2(\Omega)} \,. \tag{1.13}$$

Furthermore, according to (1.11), (1.13) and Sobolev's trace theorem, find that there exists a constant $\tilde{c}_0 > 0$ depending only on Ω, Γ_a and Γ_3 such that

$$|\phi|_{L^2(\Gamma_3)} \leq \tilde{c}_0 \, |\phi|_W \;\; \forall \phi \in W. \tag{1.14}$$

In addition, if $\mathbf{D} : \Omega \to \mathbb{R}^d$ is sufficiently regular, we also recall Green's formula below:

$$\int_\Omega \mathbf{D} . \nabla \phi dx + \int_\Omega div\mathbf{D} \phi dx = \int_\Gamma \mathbf{D} \boldsymbol{\nu} \phi da \; \forall \phi \in H^1(\Omega) \,. \tag{1.15}$$

1.2 Elliptic equation and Gronwall's Lemma

In this section, we present a result concerning elliptic variational equations which is involved in the study of the problem considered subsequently. Then, we recall Gronwall's lemma which will be used later.

1.2.1 Elliptic equation

In this paragraph, we consider a Hilbert space X with the scalar product $(.,.)_X$ and the associated norm $|.|_X$.

DÃ©finition 1.1. *An operator $A : X \to X$ is said:*

(1) monotone if

$$(Au - Av, u - v)_X \geq 0 \quad \forall u, v \in X;$$

(2) strictly monotone if

$$(Au - Av, u - v)_X > 0 \ \forall u, v \in X, u \neq v;$$

(3) strongly monotone if there exists $m > 0$ such that

$$(Au - Av, u - v)_X \geq m\,|u - v|_X^2 \ \forall u, v \in X;$$

(4) Lipschitz continuous if there exists $M > 0$ such that

$$|Au - Av|_X \leq M\,|u - v|_X \ \forall u, v \in X;$$

Proposition 1.2. *Let $A : X \to X$ be a strongly monotone and Lipschitz operator. Then for all $f \in X$ there exists a unique element $u \in X$ such that $Au = f$.*

The previous result is a special case of the Minty-Browder theorem (see for example [4] p.88).

1.2.2 Gronwall's Lemma

We recall here Gronwall's lemma which intervenes subsequently in this memoir.

Lemma 1.3. *Let $T > 0$ be given and $N > 0$ be an integer. We define $k = \frac{T}{N}$. Suppose that $\{e_n\}_{n=0}^N$ and $\{g_n\}_{n=0}^N$ are two sequences of non-negative numbers satisfying*

$$e_n \leq \bar{c} g_n + \bar{c} \sum_{j=0}^{n} k e_j, \ n = 0, ..., N,$$

with, $\bar{c}$ is a positive constant independent of N and k. Then, if k is sufficiently small, there exists a positive constant c, independent of N and k, such that

$$\max_{0 \leq n \leq N} e_n \leq c \max_{0 \leq n \leq N} g_n$$

The proof of Lemma 1.3 can be found in [17].

Chapter 2

Modeling

In this chapter, we first introduce the physical setting of the contact problem as well as a brief reminder of the mechanics of continuous media, we recall in particular here the Cauchy equation of motion and the equation of conservation of the charge. Then, we describe the electro-elastic behavior law. Finally, we present the different contact boundary conditions which will be used in the following.

2.1 Physical setting and mathematical models

In this section, we will introduce the physical setting and the mathematical models which will serve as a basis for the study of the contact problem between a deformable piezoelectric body and a deformable foundation.

Physical setting. We consider a deformable material body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$,, with a Lipschitzian boundary Γ, partitioned into three measurable parts Γ_1, Γ_2 and Γ_3, corresponding to the mechanical boundary conditions, such that $mes\ \Gamma_1 > 0$. The body can come into contact on Γ_3 with a deformable foundation. If the foundation is electrically isolated, we propose to divide Γ into two measurable parts Γ_a and Γ_b with $\Gamma_3 \subset \Gamma_b$, otherwise, we partition it into three measurable parts Γ_a, Γ_b and Γ_3, corresponding to the electrical boundary conditions, such that my $mes\ \Gamma_a > 0$. We denote by $\boldsymbol{\nu}$ the outgoing unit normal at Γ. Let $T > 0$ and let $[0, T]$ be the time interval in question. The body is embedded on Γ_1 in a fixed structure. On Γ_2 surface tractions of density $\mathbf{f}_2$ act and in Ω volume forces of density $\mathbf{f}_0$ and electric charges of volume density q_0 act. We assume that $\mathbf{f}_2$ and $\mathbf{f}_0$ vary very slowly with respect to time. The body is subjected to the action of zero potential on the part Γ_a of the boundary as well as to the action of electric charges of surface density q_2, acting on the part Γ_b.

Notations. We denote by $\mathbb{S}^d$ the space of symmetric tensors of order two on $\mathbb{R}^d$ ($d = 2, 3$), "$.$", $|.|$ represent respectively the dot product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$. So,

$$\mathbf{u}.\boldsymbol{v} = u_i v_i, \;\; |\boldsymbol{v}| = (\boldsymbol{v}.\boldsymbol{v})^{\frac{1}{2}}, \; \forall \mathbf{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \;\; |\boldsymbol{\tau}| = (\boldsymbol{\tau}.\boldsymbol{\tau})^{\frac{1}{2}}, \; \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d,$$

with the dumb index convention.

We denote by $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ the displacements field, $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ the stress field, $\varphi : \Omega \times [0,T] \to \mathbb{R}$ the electric potential field and $\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$ the electric displacement field. $\boldsymbol{\varepsilon}(\mathbf{u})$ and $\mathbf{E}(\varphi)$ are the linearized strain field and the electric field, respectively.

For the displacement field $\mathbf{u}$, we designate by u_ν and $\mathbf{u}_\tau$ the normal and tangential components of $\mathbf{u}$ on the boundary given by

$$u_\nu = \mathbf{u}.\boldsymbol{\nu}, \quad \mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}. \tag{2.1}$$

We define in a similar way, σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and tangential components of the Cauchy stress tensor $\boldsymbol{\sigma\nu}$

$$\sigma_\nu = (\sigma\nu)_\nu, \; \boldsymbol{\sigma}_\tau = (\boldsymbol{\sigma\nu})_\tau,$$

thus

$$\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu}, \; \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Using (2.1) and the previous equalities, we have the relation

$$(\boldsymbol{\sigma\nu}).\mathbf{u} = \sigma_\nu u_\nu + \boldsymbol{\sigma}_\tau.\mathbf{u}_\tau,$$

which will intervene throughout this memoir to establish the variational formulation of mechanical problem.

In this manuscript, to designate the partial derivatives $\frac{\partial u}{\partial x_i}$ of a function u, we adopt the notation $u_{,i}$. The time derivative of u will be denoted $\dot{u}$.

In addition, $\dot{\mathbf{u}}$ designates the velocity field and $\ddot{\mathbf{u}}$ designates the acceleration field.

In this memoir, the electric field is given by

$$\mathbf{E}\left(\varphi\right) = -\nabla\varphi,$$

either again

$$\mathbf{E}\left(\varphi\right) = \left(E_i\left(\varphi\right)\right), \; E_i\left(\varphi\right) = -\varphi_{,i}, \; 1 \leq i \leq d.$$

We now move on to describing the mathematical models associated with the Physical setting above.

Mathematical models. We start with the mathematical models that describes the evolution of the body in the above Physical setting. The fundamental law of the mechanics of continuous media expressing the equivalence between external forces and the tensor of accelerations for any system leads to the Cauchy equation of motion

$$Div\boldsymbol{\sigma} + \mathbf{f}_0 = \rho\ddot{\mathbf{u}} \text{ in } \Omega \times \left(0, T\right), \tag{2.2}$$

where $\rho : \Omega \to \mathbb{R}_+$ denotes the mass density. The evolutionary processes defined by (2.2) are called dynamic processes. In certain situations, this equation can be further simplified. For example, in the case where $u = 0$, it is a static process. In the case where the velocity field varies slowly with respect to time, that is to say the term $\rho\ddot{\mathbf{u}}$ can be neglected, we are in the presence of a quasistatic process. In these two cases equation (2.2) becomes

$$Div\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \text{ in } \Omega \times \left(0, T\right).$$

In the case of a piezoelectric material, we have a new unknown, the electric field $\mathbf{E}$, hence the need to introduce another balance equation to manage it. This is the Maxwell-Gauss equation or charge conservation equation.

$$div\mathbf{D} = q_0 \text{ in } \Omega \times \left(0, T\right). \tag{2.3}$$

Equations (2.2) and (2.3) are equivalent to $d + 1$ scalar relations. Consequently, mathematically these equations are not sufficient to model the body balance problem because there are more unknowns than equations. Equations (2.2) and (2.3) express a universal process valid for all piezoelectric materials and therefore they are not sufficient to determine all the different electro-mechanical behaviors of continuous media. Consequently, the equations are insufficient, on their own, to describe the balance of piezoelectric material bodies. They must then be supplemented by other relationships which characterize the behavior of each type of material and which we designate under the general term of law of behavior.

2.2 Constitutive law

In this section, we consider the constitutive law for electroelastic materials. This law is used in numerous works relating in particular to the mathematical study of contact problems.

Electroelastic materials. We consider here a category of materials where the stress tensor $\boldsymbol{\sigma}$ and the electric displacement vector $\mathbf{D}$ are linked by the constitutive law

$$\begin{cases} \boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{E}^*\mathbf{E}(\varphi), \\ \mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{B}\mathbf{E}(\varphi), \end{cases} \tag{2.4}$$

where $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator. $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ is the piezoelectric tensor which reflects the proportionality between the charge and the deformation at constant or zero field and $\mathbf{B} = (B_{ij})$ is the zero-strain dielectric tensor which constitutes a positive definite symmetric tensor. Moreover $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$, denotes the transpose of the tensor $\mathcal{E}$ such that

$$\mathcal{E}\boldsymbol{\sigma}.\boldsymbol{v} = \boldsymbol{\sigma}.\mathcal{E}^*\boldsymbol{v} \ \forall\boldsymbol{\sigma} \in \mathbb{S}^d, \ \forall\boldsymbol{v} \in \mathbb{R}^d.$$

Furthermore, we note that the operator $\mathcal{A}$ also depends on the spatial variable $\mathbf{x}$, but for simplicity, we do not explicitly report this dependence, therefore, everything that follows is valid for non-homogeneous materials. We use the notation $\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u})$ for $\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}(\mathbf{u}))$. In particular, we assume that the stress tensor $\boldsymbol{\sigma}$ is a linear function of the small strain tensor $\boldsymbol{\varepsilon}$ and the gradient of the electric potential or the electric field $\mathbf{E}$, i.e.

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(u) + e^*_{ijk}\varphi_{,k},$$

where $\mathcal{A} = (a_{ijkl})$ is a four order tensor, its components a_{ijkl} are called elasticity coefficients and $\mathcal{E} = (e_{ijk})$ is the constants piezoelectric tensor. In the non-homogeneous case a_{ijkl} and e_{ijk} depend on the point $\mathbf{x} \in \Omega$ and in the homogeneous case a_{ijkl} and e_{ijk} are constants.

We now move on to the contact conditions used in Chapters 3 and 4.

2.3 Boundary Conditions

In this section, we introduce the contact conditions used in the boundary problems that we will consider in the memoir. We place ourselves in the physical setting above. We now define the mechanical and electrical boundary conditions on each part of Γ.

The displacement boundary condition. The body is embedded in a fixed position on the part Γ_1, the field of displacements there is therefore zero

$$\mathbf{u} = 0 \text{ on } \Gamma_1 \times (0, T)\,. \tag{2.5}$$

The traction boundary condition. A surface traction of density $\mathbf{f}_2$ acts on Γ_2 and consequently the Cauchy stress vector $\boldsymbol{\sigma\nu}$ satisfies

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \text{ on } \Gamma_2 \times (0, T)\,. \tag{2.6}$$

Contact with normal compliance. We assume that the normal stress satisfies the normal compliance condition

$$-\sigma_\nu = (u_\nu)_+ \ \text{ on } \ \Gamma_3 \times (0, T)\,, \tag{2.7}$$

Frictionless contacts. We assume that we have perfect slip. This results in equality

$$\boldsymbol{\sigma}_\tau = 0 \ \text{ on } \ \Gamma_3 \times (0, T)\,, \tag{2.8}$$

Electrical boundary conditions. We assume that the electric potential is zero on the part Γ_a

$$\varphi = 0 \text{ on } \Gamma_a \times (0, T)\,. \tag{2.9}$$

The electric charges of surface density q_2 act on the part Γ_b and suppose that the foundation is electrically insulating with ($\Gamma_3 \subset \Gamma_b$). Then, the normal electrical displacement satisfies

$$\mathbf{D}.\boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0, T)\,, \tag{2.10}$$

and

$$q_2 = 0 \text{ on } \Gamma_3 \times (0, T)\,. \tag{2.11}$$

We will use these conditions in chapter 3 of this memoir.

Part II

Piezoelectric problem with normal compliance

In this part, we are interested in the study of a piezoelectric contact problem in small deformations. This part is divided into two chapters. In the first chapter, we study an electro-elastic contact problem in a quasi-static process with normal compliance and without friction. For this problem, we give the strong formulation of the problem as well as the assumptions on the data in order to obtain the variational formulation of the problem. Then, we provide a result of existence and uniqueness of the weak solution under a smallness assumption. Then, we prove this result of existence and uniqueness of the weak solution of the problem. In the next chapter, we estimate the spatial and temporal discretization error.

Chapter 3

Analytical study of the problem

In this chapter, we are interested in the theoretical study of a quasi-static problem of contact without friction between a body of electro-elastic constitutive law and an obstacle assimilated to an electrically insulating deformable foundation. The problem is formulated by a system of partial differential equations containing the law of behavior of the material, the equilibrium equations of the body and the boundary conditions to which it is subjected.

This chapter has three sections. In the first section, we start by formulating the mechanical problem and we specify the appropriate assumptions about the data in order to obtain the variational formulation. Then, in the second section, we state the variational formulation of the mechanical problem. Then, in the third section, we state and prove our main result of existence and uniqueness of the weak solution. The techniques employed are based on the results of variational equations and the theory of monotone operators.

3.1 Problem statement

We place ourselves in the physical setting above. We consider that the body is electro-elastic, more precisely, we use a constitutive law of the form (2.4). Regarding contact, we model by normal compliance without friction.

Under these considerations, the mechanical problem we study is as follows.

Problem P. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$ *and an electric displacement field*

$\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathcal{E}^*\nabla\varphi \ in \ \Omega \times (0,T)\,, \tag{3.1}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{B}\nabla\varphi \ in \ \Omega \times (0,T)\,, \tag{3.2}$$

$$Div\boldsymbol{\sigma} + \mathbf{f}_0 \ = \mathbf{0} \ in \ \Omega \times (0,T)\,, \tag{3.3}$$

$$div\mathbf{D} = q_0 \ in \ \Omega \times (0,T)\,, \tag{3.4}$$

$$\mathbf{u} \ = \mathbf{0} \ \ on \ \ \Gamma_1 \times (0,T)\,, \tag{3.5}$$

$$\boldsymbol{\sigma\nu} \ = \mathbf{f}_2 \ \ on \ \Gamma_2 \times (0,T)\,, \tag{3.6}$$

$$-\sigma_\nu = (u_\nu)_+, \ \boldsymbol{\sigma}_\tau = \mathbf{0} \ on \ \ \Gamma_3 \times (0,T)\,, \tag{3.7}$$

$$\varphi = 0 \ on \ \Gamma_a \times (0,T)\,, \ q_2 = 0 \ on \ \Gamma_3 \times (0,T)\,, \tag{3.8}$$

$$\mathbf{D}.\boldsymbol{\nu} = q_2 \ \ on \ \Gamma_b \times (0,T)\,, \tag{3.9}$$

Here and below, in order to simplify the notation, we do not explicitly report the dependence of different functions on the spatial variable $\mathbf{x}$. We now describe the equations and boundary conditions of problem P.

First, (3.1) and (3.2) represent the electro-elastic constitutive law. Equations (3.3) and (3.4) represent the equilibrium equations for the stress and electric-displacement fields while (3.5) and (3.6) are the displacement and traction boundary condition, respectively. Condition (3.7) represents the normal compliance conditions where r_+ is a positive part of r. (3.8) and (3.9) represent the electric boundary conditions.

To obtain the variational formulation of the problem (3.1) – (3.9), we introduce the closed subspace V of $H^1(\Omega)^d$ defined by (1.4). Since $meas(\Gamma_1) > 0$, Korn's inequality (1.5) holds. On the space V we consider the inner product and the associated norm given by (1.6) – (1.7) . It follows from Korn's inequality that $|.|_{H_1}$and $|.|_V$ are equivalent norms on V. Therefore $(V, |.|_V)$ is a real Hilbert space.

We also introduce the spaces W and $\mathcal{W}$ defined by (1.9) – (1.10) . The associated inner products given by (1.12). Moreover, when $\mathbf{D} \in \mathcal{W}$ is a regular function, the following Green's type formula (1.15) holds. Notice also that, since $meas(\Gamma_a) > 0$, the Friedrichs-PoincarÃ© inequality (1.11) holds.It fallows from (1.11) – (1.12) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therfore $(W, |.|_W)$ is a real Hilbert space.

In the study of the mechanical problem (3.1) – (3.9), we now list assumptions on the data. Assume that the operators $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, $\mathbf{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfy the following conditions with $L_{\mathcal{A}}, m_{\mathcal{A}}$, m_B and $c_{\mathcal{E}}$ being positive constants:

$$\begin{cases} \text{(a)} \ |\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} \ (\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(c)} \ \mathbf{x} \to \mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \ \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d)} \ \mathbf{x} \to \mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.10}$$

$$\begin{cases} \text{(a)} \ \mathbf{B}(\mathbf{x})\mathbf{E} = (b_{ij}(x)E_j) \\ \qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} \ b_{ij} = b_{ji}\,, \ b_{ij} \in L^\infty(\Omega). \\ \text{(c)} \ \mathbf{BE}.\mathbf{E} \geq m_B \, |\mathbf{E}|^2 \\ \qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Omega. \end{cases} \tag{3.11}$$

$$\begin{cases} \text{(a)} \ \mathcal{E}(\mathbf{x})\boldsymbol{\tau} = (e_{ijk}(\mathbf{x})\tau_{jk}) \\ \qquad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b)} \ e_{ijk} = e_{ikj} \in L^\infty(\Omega), \\ \qquad c_{\mathcal{E}} = \max_{1 \leq i,j,k \leq d} \{e_{ijk}\}\,. \end{cases} \tag{3.12}$$

The body forces, surface tractions, volume electric charges and surface electric charges have the regularity

$$\mathbf{f}_0 \in C(0,T;H), \quad \mathbf{f}_2 \in C(0,T;L^2(\Gamma_2)^d), \tag{3.13}$$

$$q_0 \in C(0,T;L^2(\Omega)), \quad q_2 \in C(0,T;L^2(\Gamma_b)). \tag{3.14}$$

Next we define the three mappings $\mathbf{f} : [0,T] \to V$, $q : [0,T] \to W$ and $j : V \times V \to \mathbb{R}$, respectively, by

$$(\mathbf{f}(t), \boldsymbol{v})_V = \int_\Omega \mathbf{f}_0(t).\boldsymbol{v} dx + \int_{\Gamma_2} \mathbf{f}_2(t).\boldsymbol{v} \, da, \tag{3.15}$$

$$(q(t), \phi)_W = \int_\Omega q_0(t) \ \phi \ dx - \int_{\Gamma_b} q_2(t) \ \phi \ da, \tag{3.16}$$

$$j(\mathbf{u},\boldsymbol{\upsilon}) = \int_{\Gamma_3} (u_\nu)_+ \upsilon_\nu \, da \tag{3.17}$$

We note that condition (3.13) and (3.14) imply that

$$\mathbf{f} \in C(0,T;V),\ q \in C(0,T;W). \tag{3.18}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.1) − (3.9).

3.2 Variational formulation

In this section, we are interested in the variational formulation of Problem P1. We assume in what follows that $\mathbf{u}$ and $\boldsymbol{\sigma}$ are sufficiently regular functions which satisfy (3.1), (3.3) and (3.6) − (3.7). Let $\boldsymbol{\upsilon} \in V$. We use Green's formula (1.3) and the equation of motion (3.3), in order to write

$$\int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}\left(\boldsymbol{\upsilon}\right) dx + \int_\Omega Div\boldsymbol{\sigma}.\boldsymbol{\upsilon} dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\boldsymbol{\upsilon} da.$$

We divide the surface integral Γ_1, Γ_2 et Γ_3 and using the definition of V with (3.6) and (3.23), we obtain

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}\,.\boldsymbol{\varepsilon}\left(\boldsymbol{\upsilon}\right) dx &= \int_\Omega \mathbf{f}_0.\boldsymbol{\upsilon} dx + \int_{\Gamma_2} \mathbf{f}_2.\boldsymbol{\upsilon} da + \int_{\Gamma_3} \boldsymbol{\sigma}\,\boldsymbol{\nu}.\boldsymbol{\upsilon} da \\ &= \left(\mathbf{f},\boldsymbol{\upsilon}\right)_V + \int_{\Gamma_3} \boldsymbol{\sigma}\,\boldsymbol{\nu}.\boldsymbol{\upsilon} da.\end{aligned}$$

On Γ_3, we have

$$\boldsymbol{\sigma}\,\boldsymbol{\nu}.\boldsymbol{\upsilon} = \sigma_\nu \upsilon_\nu + \boldsymbol{\sigma}_\tau.\boldsymbol{\upsilon}_\tau.$$

From (3.7), we have

$$\sigma_\nu \upsilon_\nu = -\left(u_\nu\right)_+ \upsilon_\nu,$$

Furthermore, we have

$$\boldsymbol{\sigma}_\tau.\boldsymbol{\upsilon}_\tau = \mathbf{0}$$

From all of the above, using (3.17), we obtain

$$\left(\,\boldsymbol{\sigma}\,,\boldsymbol{\varepsilon}\left(\boldsymbol{\upsilon}\right)\right)_{\mathcal{H}} + j\left(\mathbf{u},\boldsymbol{\upsilon}\right) = \left(\mathbf{f},\boldsymbol{\upsilon}\right)_V. \tag{3.19}$$

Furthermore, let φ and $\mathbf{D}$ be sufficiently regular functions which satisfy (3.2), (3.4) and (3.9). Let $\phi \in W$. We use Green's formula (1.15), we have

$$\int_\Omega \mathbf{D}.\nabla\phi dx + \int_\Omega div\mathbf{D}\phi dx = \int_\Gamma \mathbf{D}.\boldsymbol{\nu}\phi da,$$

We divide the surface integral Γ_a and Γ_b, using (3.4) and the definition of W, we find

$$\int_\Omega \mathbf{D}.\nabla\phi dx + \int_\Omega q_0\phi dx = \int_{\Gamma_b} \mathbf{D}.\boldsymbol{\nu}\phi da,$$

according to (3.9), we have

$$\int_{\Gamma_b} \mathbf{D}.\boldsymbol{\nu}\phi da = \int_{\Gamma_b} q_2\phi da \quad \forall\phi \in W.$$

We combine the two previous equalities with (3.16) in order to see that

$$(\mathbf{D}, \nabla\phi)_H = -(q, \phi)_W. \tag{3.20}$$

We now bring together the constitutive law (3.1) – (3.2), the equalities (3.19) and (3.20) to establish the following variational formulation of the problem P.

Problem PV. *Find a displacement field* $\mathbf{u} : [0, T] \to V$ *and an electric potential field* $\varphi : [0, T] \to W$ *such that for all* $t \in [0, T]$,

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

$$+j(\mathbf{u}(t), \boldsymbol{v}) = (\mathbf{f}(t), \boldsymbol{v})_V \ \forall\boldsymbol{v} \in V, \tag{3.21}$$

$$(\mathbf{B}\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \nabla\phi)_H = (q(t), \phi)_W \ \forall\phi \in W, \tag{3.22}$$

The element $(\mathbf{u}, \varphi)$ which satisfies (3.21) – (3.22) is called the weak solution of Problem P. The solvability of this system will be the subject of the following section.

3.3 Existence and uniqueness of the solution

In this section, we state and prove the following existence and uniqueness result.

Theorem 3.1. *Suppose that assumptions* (3.10) – (3.14) *are satisfied. Then, the PV problem admits a unique solution* $(\mathbf{u}, \varphi)$, *and if* $c_{\mathcal{E}}^2 < m_{\mathcal{A}} m_B$ *it has the regularity*

$$\mathbf{u} \in C(0, T; V), \tag{3.23}$$

$$\varphi \in C(0,T;W), \tag{3.24}$$

We conclude that under assumptions (3.10) – (3.14), the mechanical problem (3.1) – (3.9) has a unique weak solution. The regularity of the weak solution is given by (3.23) – (3.24) .

The regularity of the weak solution is given by (3.23) – (3.24) and in terms of constraints

$$\boldsymbol{\sigma} \in C\left(0,T;\mathcal{H}_1\right), \tag{3.25}$$

$$\mathbf{D} \in C(0,T;\mathcal{W}). \tag{3.26}$$

The proof of Theorem 3.1 will be given in several steps. To present it, we assume in what follows that the hypotheses (3.10) – (3.14) are verified, c designates a positive constant which depends on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{M}$, $\mathcal{E}$, $\mathbf{B}$ and T whose value can change from one place to another.

Proof of the theorem 3.1. Let $t \in [0,T]$, we consider the product space $X = V \times W$ with the inner product:

$$(x,y)_X = (\mathbf{u},\boldsymbol{v})_V + (\varphi,\phi)_W \ \ \forall x = (\mathbf{u},\varphi),\ y = (\boldsymbol{v},\phi) \in X$$

and the associated norm $|.|_X$. Let $A_t : X \to X$ be the operator given by

$$(A_t x, y)_X = (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}\,(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathbf{B}\nabla\varphi\,(t)\,,\nabla\phi)_H + (\mathcal{E}^*\nabla\varphi\,(t)\,,\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

$$-(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}\,(t)),\nabla\phi)_H + j(\mathbf{u}(t),\boldsymbol{v}) \ \forall x = (\mathbf{u},\varphi),\ y = (\boldsymbol{v},\phi) \in X \tag{3.27}$$

We consider the element $F \in X$ given by

$$F = (\mathbf{f}, q) \in X$$

We have the following equivalence result: the couple $x = (\mathbf{u},\varphi)$ is a solution to the problem PV if and only if it is a solution to the following equation

$$(A_t x\,(t)\,,y)_X = (F\,(t)\,,y)_X \ \ \forall y \in X,\ t \in [0,T]\,.$$

we will see that the operator A_t is strongly monotone and Lipschitz continuous.

Indeed, for $x_1 = (\mathbf{u}_1,\varphi_1)$ and $x_2 = (\mathbf{u}_2,\varphi_2)$, for all $y = (\boldsymbol{v},\phi) \in X$. We use (3.27), we have

$$(A_t x_1 - A_t x_2, y)_X = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$

$$+(\mathbf{B}\nabla\varphi_1 - \mathbf{B}\nabla\varphi_2, \nabla\phi)_H + (\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$$
$$-(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}\,(\mathbf{u}_2)\,, \nabla\phi)_H + j(\mathbf{u}_1, \boldsymbol{v}) - j(\mathbf{u}_2, \boldsymbol{v}), \tag{3.28}$$

using Cauchy Schwarz inequality, we see that

$$\begin{aligned}(A_t x_1 - A_t x_2, y)_X \;\; \leq \;\; & |\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2)|_{\mathcal{H}}\, |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \\ & + |\mathbf{B}\nabla\varphi_1 - \mathbf{B}\nabla\varphi_2|_H\, |\nabla\phi|_H \\ & + |\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2|_{\mathcal{H}}\, |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \\ & + |\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_2)|_{\mathcal{H}}\, |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \\ & + |j(\mathbf{u}_1, \boldsymbol{v}) - j(\mathbf{u}_2, \boldsymbol{v})|\,.\end{aligned}$$

We use (1.2) , (3.10) – (3.14), (1.7) and (1.13) , we obtain

$$\begin{aligned}(A_t x_1 - A_t x_2, y)_X \;\; \leq \;\; & c\,(|\mathbf{u}_1 - \mathbf{u}_2|_V\, |\boldsymbol{v}|_V + |\varphi_1 - \varphi_2|_W\, |\phi|_W \\ & + |\varphi_1 - \varphi_2|_W\, |\boldsymbol{v}|_V + |\mathbf{u}_1 - \mathbf{u}_2|_V\, |\boldsymbol{v}|_V) \\ & + |j(\mathbf{u}_1, \boldsymbol{v}) - j(\mathbf{u}_2, \boldsymbol{v})|\,.\end{aligned}$$

From (3.17), Cauchy Schwarz inequality, $|v_\nu| \leq |\boldsymbol{v}|$ and (1.8), we have

$$\begin{aligned}|j(\mathbf{u}_1, \boldsymbol{v}) - j(\mathbf{u}_2, \boldsymbol{v})| \;\; & \leq \;\; \int_{\Gamma_3} |((u_{1\nu})_+ - (u_{2\nu})_+)\, v_\nu|\, da \\ & \leq \;\; c_0^2\, |\mathbf{u}_1 - \mathbf{u}_2|_V\, |\boldsymbol{v}|_V\,,\end{aligned}$$

therefore

$$(A_t x_1 - A_t x_2, y)_X \leq c\,(|\mathbf{u}_1 - \mathbf{u}_2|_V\, |\boldsymbol{v}|_V + |\varphi_1 - \varphi_2|_W\, |\phi|_W)\,,$$

from the definition of $|.|_X$, we conclude that

$$(A_t x_1 - A_t x_2, y)_X \leq c\, |x_1 - x_2|_X\, |y|_X$$

by taking $y = A_t x_1 - A_t x_2$, we see that

$$|A_t x_1 - A_t x_2| \leq c\, |x_1 - x_2|_X\,.$$

which proves that A_t is Lipschitz.

Furthermore, we have

$$\begin{aligned}(A_t x_1 - A_t x_2, x_1 - x_2)_X &= (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2), \boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2))_{\mathcal{H}} \\ &\quad + (\mathbf{B}\nabla\varphi_1 - \mathbf{B}\nabla\varphi_2, \nabla(\varphi_1 - \varphi_2))_H \\ &\quad + (\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2))_{\mathcal{H}} \\ &\quad - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_2), \nabla(\varphi_1 - \varphi_2))_H \\ &\quad + j(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - j(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)\end{aligned}$$

we know that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_2), \nabla(\varphi_1 - \varphi_2))_H = (\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2))_{\mathcal{H}}$$

and, we have

$$\begin{aligned}j(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - j(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) &= j(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \\ &\geq \int_{\Gamma_3} ((u_{1\nu})_+ - (u_{2\nu})_+)(u_{1\nu} - u_{2\nu})\, da \\ &\geq 0,\end{aligned}$$

thus, by using (1.2), (1.7) and (1.13) we obtain

$$\begin{aligned}(A_t x_1 - A_t x_2, x_1 - x_2)_X &\geq c\left(|\boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2)|_{\mathcal{H}}^2 + |\nabla(\varphi_1 - \varphi_2)|_H^2\right) \\ &\geq c\left(\left|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\right|_V^2 + |\varphi_1 - \varphi_2|_W^2\right) \\ &\geq c\,|x_1 - x_2|_X^2 .\end{aligned}$$

therefore, A_t is strongly monotone. Therefore, using a standard result on the elliptic variational equalities (Proposition 1.2) that there exists a unique element $x = (\mathbf{u}, \varphi) \in X$ which solves (3.21) – (3.22).

Now let us show that $\mathbf{u} \in C(0, T; V)$, $\varphi \in C(0, T; W)$. Let $t_1, t_2 \in [0, T]$ and denote $\mathbf{u}(t_i) = \mathbf{u}_i$, $\varphi(t_i) = \varphi_i$, $\boldsymbol{\sigma}(t_i) = \boldsymbol{\sigma}_i$, $\mathbf{D}(t_i) = \mathbf{D}_i$, $\mathbf{f}(t_i) = \mathbf{f}_i$, $q(t_i) = q_i$ for $i = 1, 2$. Using the relations (3.21) and (3.22), we find

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}_2), \boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} + (\mathcal{E}^*\nabla(\varphi_1 - \varphi_2), \boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}} \\ &\qquad + j(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = (\mathbf{f}_1 - \mathbf{f}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \qquad (3.29)\end{aligned}$$

$$(\mathbf{B}\nabla(\varphi_1-\varphi_2),\nabla(\varphi_1-\varphi_2))_H-(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1-\mathbf{u}_2),\nabla(\varphi_1-\varphi_2))_H$$

$$=(q_1-q_2,\varphi_1-\varphi_2)_W,$$

and, by using the assumption (3.10) – (3.12) on $\mathcal{A}$, B and $\mathcal{E}$, the definition (3.16) of the functional j, we obtain

$$m_{\mathcal{A}}\left|\mathbf{u}_1-\mathbf{u}_2\right|_V\leq c_{\mathcal{E}}\left|\varphi_1-\varphi_2\right|_W+\left|\mathbf{f}_1-\mathbf{f}_2\right|_V \tag{3.30}$$

and

$$m_B\left|\varphi_1-\varphi_2\right|_W\leq c_{\mathcal{E}}\left|\mathbf{u}_1-\mathbf{u}_2\right|_V+\left|q_1-q_2\right|_W. \tag{3.31}$$

It follows from inequalities (3.30) – (3.31) and the fact that $c_{\mathcal{E}}^2<m_{\mathcal{A}}m_B$ that

$$\left|\mathbf{u}_1-\mathbf{u}_2\right|_V\leq c\left(\left|q_1-q_2\right|_W+\left|\mathbf{f}_1-\mathbf{f}_2\right|_V\right) \tag{3.32}$$

The regularity of the function q, $\mathbf{f}$ and the relations (3.31) – (3.32) show that (3.23) – (3.24) are satisfed.

Which completes the proof of Theorem 3.1. □

In the next, we pose, for all $t\in[0,T]$

$$\boldsymbol{\sigma}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t))+\mathcal{E}^*\nabla\varphi(t), \tag{3.33}$$

$$\mathbf{D}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t))-\mathbf{B}\nabla\varphi(t), \tag{3.34}$$

Let $t_1,t_2\in[0,T]$, from (3.33), (3.10), (3.12) and (1.7), (1.13) we see that

$$\left|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\right|_{\mathcal{H}}\leq c\left(\left|\mathbf{u}_1-\mathbf{u}_2\right|_V+\left|\varphi_1-\varphi_2\right|_W\right)$$

from the regularity of $(\mathbf{u},\varphi)$ given by (3.23) – (3.24), we imply that

$$\boldsymbol{\sigma}\in C(0,T;\mathcal{H}). \tag{3.35}$$

We choose $\boldsymbol{\upsilon}\in\mathcal{D}(\Omega)^d$ in (3.21) and use (3.33), (3.15) to find

$$Div\boldsymbol{\sigma}(t)+\mathbf{f}_0(t)=\mathbf{0}\ \forall t\in[0,T].$$

From the previous equality with (3.13), we conclude that

$$Div\boldsymbol{\sigma}\in C(0,T;H), \tag{3.36}$$

the relations (3.35) – (3.36) give us

$$\boldsymbol{\sigma} \in C\left(0,T\,;\mathcal{H}_1\right).$$

From (3.34) with (3.11) – (3.12) and (1.7) – (1.13), we deduce that

$$\left|\mathbf{D}_1 - \mathbf{D}_2\right|_H \leq c\left(\left|\varphi_1 - \varphi_2\right|_W + \left|\mathbf{u}_1 - \mathbf{u}_2\right|_V\right),$$

from the regularity of $(\mathbf{u}, \varphi)$ given by (3.23) – (3.24), we can write

$$\mathbf{D} \in C\left(0,T\,;H\right). \tag{3.37}$$

We choose $\phi \in \mathcal{D}\left(\Omega\right)$ in (3.22) and use (3.34), (3.16) to find

$$div\mathbf{D}\left(t\right) = q_0\left(t\right) \;\; \forall t \in \left[0,T\right].$$

The regularity (3.14) gives us

$$div\mathbf{D} \in C\left(0,T\,;L^2\left(\Omega\right)\right). \tag{3.38}$$

the relations (3.37) – (3.38) give us

$$\mathbf{D} \in C\left(0,T\,;\mathcal{W}\right).$$

This means that the regularity (3.25) – (3.26) is satisfied.

Chapter 4

Numerical study of the problem

In this chapter, we will consider a numerical approximation of the PV variational problem, based on a temporal and spatial discretization. For these discretized diagrams we will obtain a result of the error estimation.

4.1 Approximate variational formulation

In this section, we introduce a discrete numerical scheme of Problem PV. Everywhere below we assume that conditions (3.10) – (3.14) hold. Thus, it follows from Theorem 3.1 that Problem PV admits a unique solution. More precisely, we are interested in solving Problem PV on a finite time interval $[0, T]$, with $T > 0$ arbitrary but fixed. So, let N be a positive integer; we define the size of the time step $k = \frac{T}{N}$ and we consider uniform temporal discretization

$$t_n = nk, \ 0 \leq n \leq N,$$

where N is a sufficiently large integer. For a continuous function $v(t)$ with values in a functional space, we write $v_j = v(t_j)$, $0 \leq j \leq N$. For spatial discretization, we consider a polygonal domain Ω.

Let $\mathcal{H}^h$ and B^h be the finite element spaces of piecewise constants. The spaces $\mathcal{H}$ and $L^2(\Gamma_3)$ are approximated by $\mathcal{H}^h$ and B^h, respectively.

The spaces V and W are approximated respectively by the spaces of the following finite elements:

$$V^h = \left\{ \boldsymbol{v}^h \in \left[C\left(\overline{\Omega}\right)\right]^d \ \mid \ \boldsymbol{v}^h|_K \in \left[P_1(K)\right]^d \ \forall K \in \mathcal{T}_h, \ \boldsymbol{v}^h = 0 \text{ on } \Gamma_1 \right\},$$

$$W^h = \left\{ \phi^h \in C\left(\overline{\Omega}\right) \ \mid \ \phi^h|_K \in P_1(K) \ \forall K \in \mathcal{T}_h, \ \phi^h = 0 \text{ on } \Gamma_a \right\},$$

where $\mathcal{T}_h$ is an element resulting from the triangularization of $\overline{\Omega}$, $P_1(K)$ is the space of polynomials of degree less than or equal to one over K and h designates the spatial discretization parameter which is defined as

$$h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K),$$

where $\operatorname{diam}(K) = \max\{|x-y|\, ; \ x, y \in K\}$.

For all $\boldsymbol{\tau} \in \mathcal{H}$, we denote $\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}$ its orthogonal projection of finite elements on $\mathcal{H}^h$,

$$\left(\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h\right)_{\mathcal{H}} = \left(\boldsymbol{\tau}, \boldsymbol{\tau}^h\right)_{\mathcal{H}} \ \forall \boldsymbol{\tau}^h \in \mathcal{H}^h.$$

Let $\mathbf{u}_0^h \in V^h$ be a finite element approximation of $\mathbf{u}_0$ which is the unique solution of (3.21) – (3.22) for $t = 0$ and let $\boldsymbol{v}(t) = \dot{\mathbf{u}}(t)$ be the velocity variable.

We use the following discrete displacement field

,

$$\mathbf{u}_n^{hk} = \mathbf{u}_0^h + k \sum_{j=1}^{n} \boldsymbol{v}_j^{hk} \ n \geq 1,$$

We now consider the following approximate variational problem.

Problem PV^{hk}. *Find a discrete velocity field* $\mathbf{u}^{hk} = \left\{\mathbf{u}_n^{hk}\right\}_{n=0}^{N} \subset V^h$ *and a discrete electric potential field* $\varphi^{hk} = \left\{\varphi_n^{hk}\right\}_{n=0}^{N} \subset W^h$ *such that: for* $n \geq 0$,

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}}$$

$$+j(\mathbf{u}_n^{hk}, \boldsymbol{v}^h) = (\mathbf{f}_n, \boldsymbol{v}^h)_V \ \forall \boldsymbol{v}^h \in V^h, \tag{4.1}$$

$$(\mathbf{B}\nabla\varphi_n^{hk}, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\phi^h)_H$$

$$= (q_n, \phi^h)_W \ \ \forall \phi^h \in W^h, \tag{4.2}$$

Now we announce the following existence and uniqueness result.

Theorem 4.1. *Suppose that the conditions stated in Theorem 3.1 are satisfied. Then the problem* PV^{hk} *has a unique solution.*

Proof. From a discrete analogue of Theorem 3.1, it follows that (4.1) – (4.2) admits a unique solution $\left(\mathbf{u}_n^{hk}, \varphi_n^{hk}\right)_{n=0}^{N} \in V^h \times W^h$. □

We now move on to the analysis of the error between the solutions of Problems PV and PV^{hk}.

4.2 Error estimation

This section is devoted to deriving error estimates for the discrete solution. We assume the following assumptions of regularity of the solution:

$$(\mathbf{u}, \varphi) \in C(0, T; V \times W), \tag{4.3}$$

$$(\mathbf{u}, \varphi) \in C(0, T; H^2(\Omega)^d \times H^2(\Omega)), \tag{4.4}$$

In what follows, c denotes a generic positive constant whose value can change from one occurrence to another, but is independent of the discretization parameters h and k.

First, we make an error estimate on the electric potential.

By applying (3.22) to time $t = t_n$ and $\phi = \phi^h$, we obtain for all $n \geq 0$,

$$\begin{aligned} &(\mathbf{B}\nabla\varphi_n, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n), \nabla\phi^h)_H \\ &\quad = (q_n, \phi^h)_W \ \forall \phi^h \in W^h, \end{aligned} \tag{4.5}$$

we subtract (4.2) from (4.5), for all $\phi^h \in W^h$, we see that

$$(\mathbf{B}\nabla\varphi_n - \mathbf{B}\nabla\varphi_n^{hk}, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\phi^h)_H = 0,$$

therefore, we can write

$$(\mathbf{B}\nabla\varphi_n - \mathbf{B}\nabla\varphi_n^{hk}, \nabla\left(\phi^h - \varphi_n^{hk}\right))_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\left(\phi^h - \varphi_n^{hk}\right))_H = 0,$$

while, using the writing $\phi^h = \phi^h + \varphi_n - \varphi_n$, we see that

$$\begin{aligned} &(\mathbf{B}\nabla\varphi_n - \mathbf{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ = \ &(\mathbf{B}\nabla\varphi_n - \mathbf{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \phi^h\right))_H - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\left(\varphi_n - \phi^h\right))_H. \end{aligned}$$

Using (3.11) to see that

$$\begin{aligned} m_B \left|\varphi_n - \varphi_n^{hk}\right|_W^2 &\leq (\mathbf{B}\nabla\varphi_n - \mathbf{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ &= (\mathbf{B}\nabla\varphi_n - \mathbf{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \phi^h\right))_H \\ &\quad +(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ &\quad -(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_n^{hk}), \nabla\left(\varphi_n - \phi^h\right))_H, \end{aligned}$$

using the Cauchy-Schwarz inequality, (3.11) – (3.12) and the following inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \ \forall\epsilon > 0, \tag{4.6}$$

we obtain

$$\left|\varphi_n - \varphi_n^{hk}\right|_W^2 \leq c\left(\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V^2 + \left|\varphi_n - \phi^h\right|_W^2\right). \tag{4.7}$$

In what follows, we proceed to estimate the errors on the displacement field.

We now state a relation that we will use in error estimation (see [16])

$$\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V^2 \leq ck^2 + \left|\mathbf{u}_0 - \mathbf{u}_0^h\right|_V^2 + ck\sum_{j=1}^{n}\left|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right|_V^2,$$

since $\boldsymbol{v}$ is a constant, then, we have

$$\boldsymbol{v}_j^{hk} = \boldsymbol{v}_j \implies \boldsymbol{v}_j - \boldsymbol{v}_j^{hk} = \mathbf{0},$$

therefore

$$\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V^2 \leq ck^2 + \left|\mathbf{u}_0 - \mathbf{u}_0^h\right|_V^2. \tag{4.8}$$

We apply (3.21) and taking $t = t_n$, for all $\boldsymbol{v} = \boldsymbol{v}^h \in V^h$ and $n \geq 0$, we obtain

$$\begin{gathered} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_n, \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} \\ +j(\mathbf{u}_n, \boldsymbol{v}^h) = (\mathbf{f}_n, \boldsymbol{v}^h)_V, \end{gathered} \tag{4.9}$$

we substract (4.9) to (4.1) to obtain

$$\begin{gathered} (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} \\ = j(\mathbf{u}_n^{hk}, \boldsymbol{v}^h) - j(\mathbf{u}_n, \boldsymbol{v}^h) \end{gathered}$$

therefore, we can write

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \mathbf{u}_n^{hk}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \mathbf{u}_n^{hk}))_{\mathcal{H}}$$

$$= j(\mathbf{u}_n^{hk}, \boldsymbol{v}^h - \mathbf{u}_n^{hk}) - j(\mathbf{u}_n, \boldsymbol{v}^h - \mathbf{u}_n^{hk})$$

while, using the writing $\boldsymbol{v}^h = \boldsymbol{v}^h + \mathbf{u}_n - \mathbf{u}_n$, we see that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{u}_n - \mathbf{u}_n^{hk}))_{\mathcal{H}} &= (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_n^{hk}), \boldsymbol{\varepsilon}(\mathbf{u}_n - \boldsymbol{v}^h))_{\mathcal{H}} \\ &\quad + (\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{u}_n^{hk} - \mathbf{u}_n))_{\mathcal{H}} \\ &\quad + (\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\mathbf{u}_n - \boldsymbol{v}^h))_{\mathcal{H}} \\ &\quad + j(\mathbf{u}_n^{hk}, \boldsymbol{v}^h - \mathbf{u}_n^{hk}) - j(\mathbf{u}_n, \boldsymbol{v}^h - \mathbf{u}_n^{hk}).\end{aligned} \tag{4.2.1}$$

we have

$$\begin{aligned}&\left|j(\mathbf{u}_n^{hk}, \boldsymbol{v}^h - \mathbf{u}_n^{hk}) - j(\mathbf{u}_n, \boldsymbol{v}^h - \mathbf{u}_n^{hk})\right| \\ &= \left|\int_{\Gamma_3} (u_{n\nu}^{hk})_+ \left(v^h - u_n^{hk}\right)_\nu \, da - \int_{\Gamma_3} (u_{n\nu})_+ \left(v^h - u_n^{hk}\right)_\nu \, da\right| \\ &\leq \int_{\Gamma_3} \left|\left((u_{n\nu}^{hk})_+ - (u_{n\nu})_+\right) \left(v^h - u_n^{hk}\right)_\nu \right| da \\ &\leq \int_{\Gamma_3} \left|\mathbf{u}_n^{hk} - \mathbf{u}_n\right| \left|\boldsymbol{v}^h - \mathbf{u}_n^{hk}\right| \, da \\ &\leq \int_{\Gamma_3} \left|\mathbf{u}_n^{hk} - \mathbf{u}_n\right| \left|\boldsymbol{v}^h - \mathbf{u}_n\right| \, da \\ &+ \int_{\Gamma_3} \left|\mathbf{u}_n^{hk} - \mathbf{u}_n\right| \left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right| \, da\end{aligned}$$

according to the Cauchy-Schwarz and Sobolev trace inequalities (1.8), we obtain

$$\begin{aligned}&\left|j(\mathbf{u}_n^{hk}, \boldsymbol{v}^h - \mathbf{u}_n^{hk}) - j(\mathbf{u}_n, \boldsymbol{v}^h - \mathbf{u}_n^{hk})\right| \\ &\leq c_0^2 \left|\mathbf{u}_n^{hk} - \mathbf{u}_n\right|_V \left|\boldsymbol{v}^h - \mathbf{u}_n\right|_V + c_0^2 \left|\mathbf{u}_n^{hk} - \mathbf{u}_n\right|_V^2\end{aligned} \tag{4.11}$$

we use $(4.10)-(4.11)$, $(3.10)\,(b)$, (3.12), the inequality (4.6) with the hypothesis $c_0^2 < m_{\mathcal{A}}$ we have

$$\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V^2 \leq c\left(\left|\varphi_n - \varphi_n^{hk}\right|_W^2 + \left|\boldsymbol{v}^h - \mathbf{u}_n\right|_V^2\right) \tag{4.12}$$

Adding (4.7) and (4.12) with the use of (4.8) to obtain for all $n \geq 0$

$$\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V^2 + \left|\varphi_n - \varphi_n^{hk}\right|_W^2$$

$$\leq ck^2 + c\left|\mathbf{u}_0 - \mathbf{u}_0^h\right|_V^2 + c\left[\left|\varphi_n - \phi^h\right|_W^2 + \left|\mathbf{u}_n - \boldsymbol{v}^h\right|_V^2\right]$$

$$+ck\sum_{j=0}^{n}\left[\left|\mathbf{u}_j - \mathbf{u}_j^{hk}\right|_V^2 + \left|\varphi_j - \varphi_j^{hk}\right|_W^2\right].$$

thus

$$\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V + \left|\varphi_n - \varphi_n^{hk}\right|_W$$

$$\leq ck + c\left|\mathbf{u}_0 - \mathbf{u}_0^h\right|_V + c\left[\left|\varphi_n - \phi^h\right|_W + \left|\mathbf{u}_n - \boldsymbol{v}^h\right|_V\right]$$

$$+ck\sum_{j=0}^{n}\left[\left|\mathbf{u}_j - \mathbf{u}_j^{hk}\right|_V + \left|\varphi_j - \varphi_j^{hk}\right|_W\right].$$

From this inequality, by applying Lemma 2.4 of Gronwall in order to see that

$$\max_{0\leq n\leq N}\left\{\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V + \left|\varphi_n - \varphi_n^{hk}\right|_W\right\}$$

$$\leq ck + c\left|\mathbf{u}_0 - \mathbf{u}_0^h\right|_V + c\max_{0\leq n\leq N}\left[\inf_{\phi^h\in W^h}\left|\varphi_n - \phi^h\right|_W + \inf_{\boldsymbol{v}^h\in V^h}\left|\mathbf{u}_n - \boldsymbol{v}^h\right|_V\right]$$

Under assumptions (4.3) and (4.4), we can apply the standard theory of finite element interpolation (see for example [16]) in order to see that

$$\left|\mathbf{u}_0 - \mathbf{u}_0^h\right|_V \leq ch\left|\mathbf{u}_0\right|_{H^2(\Omega)^d},$$

$$\max_{0\leq n\leq N}\inf_{\boldsymbol{v}^h\in V^h}\left|\mathbf{u}_n - \boldsymbol{v}^h\right|_V \leq ch\left|\mathbf{u}\right|_{C\left(0,T;H^2(\Omega)^d\right)},$$

$$\max_{0\leq n\leq N}\inf_{\phi^h\in W^h}\left|\varphi_n - \phi^h\right|_W \leq ch\left|\varphi\right|_{C(0,T;H^2(\Omega))}.$$

In conclusion, we showed the following result.

Theorem 4.2. *Suppose k is sufficiently small. Then, under the regularity assumptions (4.3) – (4.4) and $c_0^2 < m_{\mathcal{A}}$, we have the following error estimate*

$$\max_{0\leq n\leq N}\left[\left|\mathbf{u}_n - \mathbf{u}_n^{hk}\right|_V + \left|\varphi_n - \varphi_n^{hk}\right|_W\right] \leq c\left(h + k\right),$$

where c is a constant dependent on a certain norm of the solution.

This optimal error estimate completes this section and chapter.

Conclusion

In this memoir, we treated a frictionless contact problem between a deformable body and a foundation with normal compliance. We considered the case of electro-elastic materials. To begin, we proposed a variational formulation of the contact model. Then, we obtained a result of existence and uniqueness of the weak solution. To achieve this, we used arguments from elliptic variational equations. We considered a numerical approximation of the contact problem, using a uniform temporal discretization and a spatial discretization by the finite element method. At the end of this discretization, we showed the existence and uniqueness of the approximate variational problem. Finally, for this scheme, we obtained an error estimation result under assumptions of regularity of the solution.

Bibliography

[1] R.C. Batra and J.S. Yang, Saint-Venant's principle in linear piezoelectricity. Journal of Elasticity, **38**(2): 209-218, 1995.

[2] D. Braess, Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics, third ed, Cambridge University Press, Cambridge, 2007.

[3] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, third ed., Springer-Verlag, New York, 2008.

[4] H. BrÃ©zis, Analyse Fonctionnelle, ThÃ©orie et Application, Masson (1987).

[5] P.G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, vol. **II**, North-Holland, Amsterdam, pp. 17–351,1991.

[6] J.R. Fernandez, M. Shillor and M. Sofonea, Analysis and Numerical Simulations of A Dynamic Contact Problem with Adhesion, Math. Comput. Modelling **37**: 1317-1333, (2003).

[7] W. Han, M. Shillor and M. Sofonea, Variational and Numerical Analysis of A Quasistatic Viscoelastic Problem with Normal Compliance, Friction and Damage, J. Comput. Appl. Math. **137**: 377-398, (2001).

[8] T. Ikeda, Fundamentals of piezoelectricity. Oxford University Press, Oxford, 1990.

[9] L. Jianu, M. Shillor and M. Sofonea, A Viscoelastic Bilateral Frictionless Contact Problem with Adhesion, Applic. Anal. **80**: 233-255, (2001).

[10] F. Maceri and P. Bisegna, The unilateral frictionless contact of a piezoelectric body with a rigid support. Mathematical and Computer Modelling, **28(4-8)**: 19-28, 1998.

[11] M.S. Mesai Aoun, Analyse mathÃ©matique de quelques problÃ©mes aux limites en piÃ©zoÃ©lectricitÃ©. MÃ©moire de Magister UniversitÃ© Ouargla (2013).

[12] M.S. Mesai Aoun, M. Selmani and A. Azeb Ahmed, Variational Analysis of a Frictional Contact Problem with Wear and Damage. Mathematical Modelling and Analysis **26 (2)**: 170-187, 2021.

[13] M. Rochdi, M. Shillor and M. Sofonea, Quasistatic viscoelastic contact with normal compliance and friction. Journal of Elasticity, **51(2)**: 105-126, 1998.

[14] M. Selmani, Etude mathÃ©matique de quelques problÃ©mes aux limites en mÃ©canique du contact. ThÃ¨se de doctorat UniversitÃ© sÃ©tif (2006).

[15] M. Selmani, A piezoelectric frictionless contact problem with adhesion, Studia Univ. "Babes-Bolai". Mathematica. Volume **LIV** . Number **2**, June 2009..

[16] M. Sofonea, W. Han and M. Shillor, Analysis and approximation of contact problems with adhesion or damage. CRC Press, 2006.

[17] M. Sofonea & A. Matei, Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Note Series. **398** (2012), Cambridge University Press, Cambridge.

[18] M. Sofonea and Y. Souleiman, A viscoelastic sliding contact problem with normal compliance, unilateral constraint and memory term. Mediterranean Journal of Mathematics, **13**: 2863-2886, 2016.

[19] Y. Souleiman, Analyse de quelques problÃ¨mes de contact glissant. ThÃ¨se de Doctorat UniversitÃ© de Perpignan via Domitiat (2017).

[20] B. Souraya and A.A. Abdelaziz, Analysis of a dynamic contact problem for electro-viscoelastic materials. Milan Journal of Mathematics, **86(1)**: 105-124, 2018.

[21] N. Stromberg, Continuum thermodynamics of contact, friction and wear. Ph.D. Thesis. Linkoping University, Sweeden, 1995.

[22] N. Stromberg, L. Johansson and A. Klarbring, Derivation and analysis of a generalized standard model for contact, friction and wear. International Journal of Solids and Structures, **33(13)**: 1817-1836, 1996.

Abstract: In this memoir, we are interested in the modeling, the variational analysis and numerical analysis of a contact problem, for electro-elastic materials. The first part of this memoir concern some preliminary results, in particular the mathematical and mechanical tools necessary to carry out the continuation of this work. The second part is devoted to the study of a contact problem under various conditions of contact without friction. For this problem, we introduce strong formulation and variational formulation. Then, we obtain existence and uniqueness results for weak solutions. Finally, we propose a numerical approximation of a problem using a discretized scheme. For this scheme, we obtain an error estimation result.

Key words: electro-elastic, contact without friction, normal compliance, variational equality, weak solution, error estimate.

Résumé: Dans ce mémoire, nous nous intéressons à la modélisation, à l'analyse variationnelle et à l'analyse numérique d'un problème de contact, pour les matériaux électro-élastiques. La première partie de ce mémoire concerne quelques résultats préliminaires, notamment les outils mathématiques et mécaniques nécessaires pour mener à bien la suite de ces travaux. La deuxième partie est consacrée à l'étude d'un problème de contact dans diverses conditions de contact sans frottement. Pour ce problème, nous introduisons la formulation forte et la formulation variationnelle. Ensuite, nous obtenons un résultat d'existence et d'unicité pour la solution faible. Enfin, nous proposons une approximation numérique du problème à l'aide d'un schéma discrétisé. Pour ce schéma, nous obtenons un résultat d'estimation d'erreur.

Mots clés: électro-élastique, contact sans frottement, conformité normale, égalité variationnelle, solution faible, estimation d'erreur.

ملخص: في هذه المذكرة، نحن مهتمون بالنمذجة والتحليل التبايني والتحليل العددي لمسألة اتصال للمواد الكهربائية المرنة. يتعلق الجزء الأول من هذه المذكرة ببعض النتائج الأولية، ولا سيما الأدوات الرياضية والميكانيكية اللازمة لمواصلة هذا العمل. أما الجزء الثاني فقد خصص لدراسة مسألة تلامس في ظل ظروف التلامس المختلفة دون احتكاك. لهذه المسألة، نقدم صياغة قوية وصياغة متباينة. ومن ثم نحصل على نتائج الوجود والتفرد للحل الضعيف. وأخيرا، نقترح تقريب عددي للمسألة باستخدام مخطط منفصل. بالنسبة لهذا المخطط، نحصل على نتيجة تقدير الخطأ.

كلمات مفتاحية: المرونة الكهربائية، الاتصال بدون احتكاك، الامتثال الطبيعي، المساواة التباينية، الحل الضعيف، تقدير الخطأ.