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Dedication

To everyone whose forehead has been drenched in sweat, and to those who taught me that success comes only through patience and determination ...

To the light that illuminated my path and the lamp whose glow never fades in my heart...

...My father...

To the one who made Paradise lie beneath her feet and eased my hardships with her prayers...

To the great woman who always longed to see this day with her own eyes...

...My mother...

To those who strengthened were the springs from which I drank, the best of my days and their purity, the joy of my eyes... My dear siblings...

*To the loyal **friends** and companions of the years .*

I dedicate this this humble work, which you have always wished for. Today, i have completed and fulfilled, exalted and Almighty.

So praise be to Allah for what He has ordained, for making me blessed, and for aiding me wherever I am .

Praise be to Allah, in gratitude and appreciation, for the beginning and the end .

Our final supplication is praise be to Allah, Lord of the worlds.

ATALLAH AYA

Dedication

All praise and thanks are due to Allah, the Most Merciful, the All-Knowing, who guided me, strengthened me, and made this achievement possible.

To my beloved father, whose wisdom and support have been a light in my life thank you for being my strength and example.

To my dearest mother, whose prayers never ceased and whose love never wavered may Allah reward you for every tear, every prayer, and every sacrifice.

To my brothers and sisters, who stood by me with love, laughter, and endless encouragement your presence made the journey easier and brighter.

And to **my husband**, my partner and steadfast supporter, who believed in me when I doubted myself, and held my hand through every challenge... This milestone is ours. May Allah bless you for your patience, love, and unwavering support.

O Allah, let this knowledge be light upon my path, a means to please You, and a tool to serve my faith and community.

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Notations

$\mathbb{K}$	: Field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.
$C([0, 1], \mathbb{R})$	: The space continuous functions from $[0, 1]$ to $\mathbb{R}$.
$D(T)$	: Domaine de T
$C(X)$	: The set of continuous functions from x to $\mathbb{R}$.
$B(H)$	: Space of bounded linear operators on H
$ a $	: The absolute value of a
$\ a \ $	: The norm of a .
$r(a)$	: Rayon spectral de T .
a^*	: The adjoint operator of a
a^{-1}	: The inverse operator of a .
$1_{\mathcal{A}}$	: The identity element in set $\mathcal{A}$.
$\sigma(a)$	: The spectrum of a .
$\sigma(a)$	: The spectrum of a .
$\rho(a)$	: The resolvent of a
$\mathcal{A}_{sa}$	: Hermitian elements
$\mathcal{A}^+$	: Positive elements

Introduction

The study of operators in C^* -algebras has been a central topic in functional analysis and operator theory for decades, with profound implications across mathematics, quantum mechanics, and theoretical physics. C^* -algebras, named for their closure under the involution operation $(*)$ and the norm topology, provide a natural framework for investigating bounded linear operators on Hilbert spaces. These algebras generalize the concept of algebras of continuous functions on compact spaces and play a pivotal role in the mathematical formulation of quantum mechanics, where observables are represented by self-adjoint operators within such algebras.

The theory of self-adjoint, normal and positive operators has its roots in the early 20th century, with groundbreaking contributions by David Hilbert, John von Neumann, and Marshall Stone. **Self-adjoint** operators, characterized by the property $a^* = a$, are indispensable in quantum mechanics, where they model physical observables such as position, momentum, and energy. **Normal operators**, satisfying $a^*a = aa^*$, encompass a broader class that includes unitary and self-adjoint operators, and their spectral decomposition theorems are cornerstones of operator theory. In addition **positive operators** constitute a crucial class in the study of C^* -algebras. An operator a in a C^* -algebra is said to be positive (often denoted $a \geq 0$) if it is self-adjoint and has non-negative eigenvalues. This definition is equivalent to the existence of an operator b in the C^* -algebra such that $a = b^*b$. Positive operators play a central role in C^* -algebra theory, where they are used to define traces and non-commutative geometry, and they are fundamental in the mathematical formulation of quantum mechanics, particularly in representing physical states and measurements.

In recent decades, the study of these operators has expanded significantly, driven by their applications in quantum field theory, non-commutative geometry, and signal processing. For instance, the spectral theorem for normal operators provides a powerful tool for diagonalizing operators and understanding their action on Hilbert spaces. Meanwhile, self-adjoint operators are central to the formulation of quantum statistical mechanics and the study of partial differential equations. Memory included three chapters:

- . **Chapter 1:** We touched on some concepts of Banach space, Hilbert space and talked about some of the basic properties and concepts of operators in Hilbert space with some examples.
- . **Chapter 2:** We provided a comprehensive introduction to C^* -algebra theory, focusing on the fundamental algebraic and analytical properties of these mathematical structures.
- . **Chapter 3:** This chapter focuses on studying special classes of operators in C^* -algebras, which are distinguished by their important algebraic and spectral properties. Among these classes, we explore self-adjoint, positive and normal operators .

Preliminaries

In this chapter, we will discuss some information related to our topic, which is represented in some definitions, basic concepts, and theories used in the proposed study. For more details, we invite the reader to view [4, 5, 6, 7, 8, 9, 15].

1.1 Banach spaces

Definition 1.1.1. A normed space $(E, \| \cdot \|)$ is called a **Banach space** if it is complete under the associated metric i.e. if each Cauchy sequence converges.

A useful property of Banach spaces is the following: if (x_n) is a sequence so that $\sum \|x_n\| < \infty$, then $\sum x_n$ converges i.e. the partial sums $\sum_{k=1}^n x_k$ converge to a point x in the space. For it follows from the triangle inequality, that the sequence of partial sums is Cauchy. Series with the above property are called **absolutely convergent**.

The following normed spaces are Banach spaces:

$C(K)$ (K a compact space);

$C^n(I)$ (I a compact interval, $n \in \mathbb{N}$);

$L(E, F)$ (E a normed space, F a Banach space).

In particular, the dual E' of a normed space is always a Banach space.

Proof. The completeness of $C(K)$: this is essentially a restatement of the fact that the uniform limit of continuous functions is continuous. Similarly, the completeness of $C^n(I)$

follows from that of $C(I)$ and the fact that if a sequence of functions in $C^n(I)$ converges together with their derivatives up to order n , then the limit function is in $C^n(I)$. $\square$

Example 1.1.1. *Prove that if F is a Banach space, so is $L(E, F)$. Show that if E, F are normed spaces with E non-trivial (i.e. $E \neq 0$) then F is isomorphic to a closed subspace of $L(E, F)$.*

In fact: *Deduce that F is complete if $L(E, F)$ is (choose a non-zero continuous linear form f on E and consider the mapping $y \rightarrow (x \rightarrow f(x)y)$ from F into $L(E, F)$)*

Example 1.1.2. *Show that the space $C([0, 1])$ with norm $\|x\| = \int_0^1 |x(t)| dt$ is not a Banach space.*

It is clear that a closed subspace of a Banach space is a Banach space with the induced norm. Conversely if a normed subspace of a normed space is a Banach space, then it is closed. Also a product of a finite family of Banach spaces is a Banach space.

Just as we can embed non complete metric spaces in complete spaces, so every non complete normed space can be embedded in a Banach space as we show below .

Proposition 1.1.1. *Let L be a dense subspace of a normed space $(E, \|\cdot\|)$, T a continuous linear mapping from L (with the induced norm) into a Banach space $(F, \|\cdot\|_2)$. Then there is a unique continuous linear mapping $\hat{T}$ from E into F which extends T . The norms of $\hat{T}$ and T coincide .*

Proof. If $x \in E$, there is a sequence (x_n) in L so that $x_n \rightarrow x$. (x_n) is Cauchy in L and so (Tx_n) is Cauchy in F . Let $y := \lim Tx_n$. y is independent of the choice of the approximating sequence (x_n) (the reader should check this) and so the correspondence $x \rightarrow y$ can be used to define a mapping $\hat{T}$ from E into F . We omit the details required to show that T has the required properties. $\square$

Proposition 1.1.2. *Let E be a normed space. Then E can be embedded as a dense subspace of a Banach space E_1 so that the following property holds: every continuous linear operator T from E into a Banach space F has a unique extension to a continuous linear operator $\hat{T}$ from E_1 into F . The norms of $\hat{T}$ and T coincide .*

Proof. We identify E with $J_E(E) \subseteq E''$ and let E_1 be the closure of $J_E(E)$ in E'' .

The space E_1 is unique in the following sense. Suppose that we can embed E in a second space E_2 with the same properties. Then we can extend both of these embeddings to get linear contractions between E_1 and E_2 which are the identity of E and so are mutually inverses. Hence E_1 and E_2 are in this natural way isometrically isomorphic. This justifies the name completion for the above space which we denote by $\widehat{E}$. $\square$

Example 1.1.3.

- A. Show that $B_{\widehat{E}}$ is the closure of B_E in $\widehat{E}$.
- B. Let F be a dense subspace of E . Show that if $x \in E$ there is a sequence (x_n) in F which is such that $\sum \|x_n\| < \infty$ and $\sum_{k=1}^n x_k = x$. Deduce that a normed space E is a Banach space if and only if for every sequence (x_n) in E which is such that $\sum \|x_n\| < \infty$, the series $\sum x_n$ converges in E .

Proposition 1.1.3. Let F be a closed subspace of a Banach space E . Then the quotient E/F is also a Banach space.

Proof. Let (y_n) be sequence in E/F so that $\sum \|y_n\| < \infty$. Then there is a sequence (x_n) in E so that for each n $\pi_F(x_n) = y_n$ and $\|x_n\| \leq \|y_n\| + 2^{-n}$. Then $\sum \|x_n\| < \infty$ and so $\sum x_n$ converges in E , say to x . Clearly $\sum y_n$ converges in E/F to $\pi_F(x)$

We now give a useful criterium for demonstration the completeness of concrete Banach sequence or function spaces. $\square$

Proposition 1.1.4. Let S be a set, A is closed, absolutely convex subset of $\mathbb{C}^S$ (with the product topology) which is bounded i.e. such that for each $t \in S$ there is a $K > 0$ so that $|x(t)| \leq K$ whenever $x \in A$. Then if $E_A = \bigcup_n nA$, $(E_A, \|\cdot\|_A)$ is a Banach space.

Proof. Of course E_A is a normed space we show that it is complete. If (x_n) is an $\|\cdot\|$ -Cauchy sequence, each $(x_n(t))$ is Cauchy (this follows from the fact that A is bounded the reader should check this). Hence by the completeness of $\mathbb{C}$ there is a function x so that $x_n(t) \rightarrow x(t)$ for each $t \in S$. We show that $x \in E_A$ and $x_n \rightarrow x$ in E_A . For each $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $x_m - x_n \in \epsilon A$ if $m, n \geq N$. Letting $n \rightarrow \infty$ and using the fact that A is pointwise closed we see that $x_m - x \in \epsilon A$ for $m \geq N$. From this it follows easily that

$x \in E_A$ and $\|x_m - x\|_a \rightarrow 0$

□

As an application of this result consider the sets

$$A_p = \left\{ x \in \mathbf{C}^S : \sum_{s \in S} |x(s)|^p \leq 1 \right\}.$$

for $1 \leq p < \infty$. We know that A_p is closed and absolutely convex in $S^{\mathbf{C}}$. Hence the corresponding normed spaces (i.e. $(\ell^p(S), \|\cdot\|_p)$) are complete. In the special case where $S = \mathbb{N}$ we obtain the sequence space ℓ^p and so we see that these are Banach spaces. Similarly,

$$A_\infty = \{x \in S^{\mathbf{C}} : \sup_{s \in S} |x(s)| \leq 1\}$$

generate

$$\ell^\infty(S) = \{x \in S^{\mathbf{C}} : \text{for each } \epsilon > 0 \{t \in S : |x(t)| \geq \epsilon\} \text{ is finite}\}$$

of $c_0(S)$ is closed in $\ell^\infty(S)$ and so is a Banach space. (for if $x_n \rightarrow x$ in $\ell^\infty(S)$ and each $x_n \in c_0(S)$, then for $\epsilon > 0$ we choose an x_N with $\|x_N - x\| \leq \epsilon/3$. There is a finite $A_1 \subseteq S$ so that the values of x_N outside of A_1 have absolute value at most $\epsilon/3$. Then the values of x outside of A_1 have absolute values at most ϵ . This method of proof is known as an $\epsilon/3$ argument for obvious reasons.)

One of the most important motivations for the study of Banach spaces and their linear operators was the classical theory of integral equations and we conclude this section with some remarks on that subject. We begin with an elementary criterion for the invertibility of an operator which is the basis for many perturbation results:

1.2 Hilbert spaces

Recall that any inner product space X has an associated norm defined by:

$$\|x\| = \sqrt{\langle x, x \rangle} \tag{1.1}$$

Thus an inner product space can be viewed as a special kind of normed vector space. In particular, every inner product space X has a metric defined by:

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}.$$

Definition 1.2.1. A **Hilbert space** is an inner product space whose associated metric is complete.

That is, a Hilbert space is an inner product space that is also a Banach space. For example, $\mathbb{R}^n$ is a Hilbert space under the usual dot product:

$$\|x, y\| = x_1y_1 + \dots + x_ny_n.$$

More generally, a finite-dimensional inner product space is a Hilbert space.

Example 1.2.1. The standard inner product on $\mathbb{C}^n$ is given by

$$\langle x, y \rangle = \sum_{j=1}^n \overline{x_j} y_j,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, with $x_j, y_j \in \mathbb{C}$. This space is complete, and therefore it is a finite-dimensional Hilbert space.

Example 1.2.2. Let $C([a, b])$ denote the space of all complex-valued continuous functions defined on the interval $[a, b]$. We define an inner product on $C([a, b])$ by

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) dx.$$

where $f, g : [a, b] \rightarrow \mathbb{C}$ are continuous functions. This space is not complete, so it is not a Hilbert space. The completion of $C([a, b])$ with respect to the associated norm,

$$\|f\| = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}.$$

is denoted by $L^2([a, b])$. The spaces $L^p([a, b])$, are not Hilbert spaces when $p \neq 2$.

Similarly, if $\mathbb{T}$ is the circle, then $L^2(\mathbb{T})$ is the Hilbert space of square-integrable functions $f : \mathbb{T} \rightarrow \mathbb{C}$ with the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}} \overline{f(x)} g(x) dx.$$

Theorem 1.2.1. (Cauchy-schwarz) If $x, y \in X$, where X is an inner prod space, then

$$|(x, y)| \leq \|x\| \|y\| \tag{1.2}$$

where the norm $\|\cdot\|$ is defined in (1.1)

Proof. By the nonnegativity of the inner product, we have for all $x, y \in X$ and $\lambda, \mu \in \mathbb{C}$. Expansion of the inner product, and use of (1.1) implies that

$$0 \leq (\lambda x - \mu y, \lambda x - \mu y).$$

$$\bar{\lambda}\mu(x, y) + \lambda\bar{\mu}(y, x) \leq |\lambda|^2\|x\|^2 + |\mu|^2\|y\|^2.$$

If $(x, y) = re^{i\varphi}$, where $r = |(x, y)|$ and $\varphi = \arg(x, y)$, then we choose

$$\lambda = \|y\|e^{i\varphi}, \quad \mu = \|x\|$$

It follows that

$$2\|x\|\|y\||x, y| \leq 2\|x\|^2\|y\|^2,$$

which proves the result. An inner product space is a normed space with respect to the norm defined in (1.1). The converse question of when a norm is derived from an inner product in this way is answered by the following theorem. $\square$

Theorem 1.2.2. *A normed linear space X is an inner product space with a norm derived from the inner product by (1.1) if and only if*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X. \quad (1.3)$$

Proof. Use of (1.1) to write norms in terms of inner products, and expansion of the result, implies that (1.3) holds for any norm that is derived from an inner product. Conversely, if a norm satisfies (1.3), then the equation

$$(x, y) = \frac{1}{4} \{ \|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2 \}. \quad (1.4)$$

defines an inner product on X . We leave a detailed verification of this fact to the reader.

The relation (1.3) is called the parallelogram law. Its geometrical interpretation is that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals. As the polarization formula (1.4) shows, an inner product is uniquely determined by its values on the diagonal, that is, by its values when the first and second arguments are equal.

Let $(X, (<, >)_X)$ and $(Y, (<, >)_Y)$ be two inner product spaces. Then there is a natural inner product on the Cartesian product space

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

given by

$$((x_1, y_1), (x_2, y_2))_{X \times Y} = (x_1, x_2)_X + (y_1, y_2)_Y$$

The associated norm on $X \times Y$ is

$$\|x, y\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

Unless stated otherwise, we will use this inner product and norm of the Cartesian product of two inner product spaces. $\square$

Theorem 1.2.3. *Let X be an inner product space. The inner product is a continuous map from $X \times X \rightarrow \mathbb{C}$*

Proof. For all $x_1, x_2, y_1, y_2 \in X$, the Cauchy-Schwarz inequality implies that

$$\begin{aligned} |(x_1, y_1) - (x_2, y_2)| &= |(x_1 - x_2, y_1) + (x_2, y_1 - y_2)| \\ &\leq \|x_1 - x_2\| \|y_1\| + \|x_2\| \|y_1 - y_2\|. \end{aligned}$$

This estimate implies the continuity of the inner product. $\square$

1.3 Operators on Hilbert spaces

1.3.1 Bounded linear Operators

In this paragraph, X and Y are two normed linear spaces defined on the same scalar field $\mathbb{K}$.

Definition 1.3.1. *A linear operator is a mapping $T : D(T) \subset X \rightarrow Y$ that satisfies:*

$$T(\lambda x + y) = \lambda T(x) + T(y), \forall \lambda \in \mathbb{K}, \forall x, y \in D(T),$$

where $D(T)$ is a linear subspace of X , called the domain of T .

The image of x is denoted Tx and

$$R(T) := \{y \in Y; \exists x \in D(T), y = Tx\}.$$

is called the range or the image of T .

The space of all linear operators from X into Y is denoted by $L(X, Y)$, it is a linear space over $\mathbb{K}$ with respect to the addition and multiplication by scalars,

$$(T + S)x = Tx + Sx, \forall x, y \in D(T) \cap D(S),$$

$$(Tx), \forall x \in D(T).$$

The identity operator is denoted I , $I(x) = x, \forall x \in X$.

Definition 1.3.2. The operator $T : D(T) \subset X \rightarrow Y$ is said to be continuous or bounded if it satisfies the property

$$\forall x \in D(T), \forall \epsilon > 0, \exists \delta > 0 : \forall y \in D(T); \|x - y\| < \delta \Rightarrow \|Tx - Ty\| < \epsilon.$$

the space of all bounded operators from X into Y is denoted by $\mathcal{L}(X, Y)$ or $\mathcal{B}(X, Y)$. In particular, if $Y = X$ the space is denoted by $L(X)$ and if $Y = \mathbb{K}$, the space $\mathcal{L}(X, \mathbb{K})$ is written X' and called the dual space of X . The elements of X' are called linear forms or functionals.

Proposition 1.3.1. [7] The operator $T : D(T) \subset X \rightarrow Y$ is said to be continuous or bounded if it satisfies the property

1. T is uniformly continuous on $D(T)$
2. T is continuous on $D(T)$,
3. T is continuous in 0
4. $\exists C > 0 : \forall x \in D(T), \|x\| \leq 1 \Rightarrow \|Tx\| \leq C,$
5. $\exists C > 0, \|Tx\| \leq C \|x\|, \forall x \in D(T).$

Proof. It is clear that $1) \Rightarrow 2) \Rightarrow 3)$.

3) $\Rightarrow$ 4) Assume that T is continuous at 0, then for $\varepsilon = 1$ we have

$$\exists \delta > 0 : \forall x \in E, \|x\| < \delta \Rightarrow \|Tx\| < 1,$$

then, for any $x \in E$ such that $\|x\| < 1$ we have $\|\frac{\delta x}{2}\| < \delta$, consequently

$$\|T(\frac{\delta x}{2})\| < 1$$

therefore,

$$\|Tx\| < \frac{2}{\delta} = C.$$

To prove that 4) $\Rightarrow$ 5), let $x \in E$ if $x \neq 0$, we have $\|\frac{x}{\|x\|}\| = 1$, hence, by 4) we deduce

$$\|T(\frac{x}{\|x\|})\| \leq C$$

which gives $\|Tx\| \leq C \|x\|$. If $x = 0$ clearly $\|T(0)\| = C \|0\|$. Thus 5) is satisfied.

Finally, let us show that 5) $\Rightarrow$ 1). Let $x, y \in E$ and take $\epsilon > 0$, since,

$$\|Tx - Ty\| = \|T(x - y)\| \leq C \|x - y\|$$

then, so that $\|Tx - Ty\| < \epsilon$, it suffices that $C \|x - y\| < \epsilon$, which is satisfied if $\|x - y\| < \frac{\epsilon}{C}$

Thus, $\delta = \frac{\epsilon}{C}$ guaranties the result. $\square$

1.3.2 Norm of an operator

Lemma 1.3.1. Let X and Y be two normed linear spaces, then, $\|\cdot\|: \mathcal{L}(Y, X) \Rightarrow \mathbb{R}_+$ define by

$$\|T\|_{\mathcal{L}(Y, X)} = \sup\{\|Tx\|; x \in D(T), \|x\| \leq 1\}.$$

is a norm on $\mathcal{L}(Y, X)$.

Proof. i) Suppose that $T = 0$, that is $Tx = 0$ for all $x \in D(T)$, then, $\sup_{x \in D(T), \|x\| \leq 1} \|Tx\| = 0$. Thus, $\|T\| = 0$. Reciprocally, suppose that $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = 0$, then, $Tx = 0$, $\forall x \in D(T)$, $\|x\| \leq 1$. Suppose that $x \in D(T)$, $\|x\| > 1$, then $z = \frac{x}{\|x\|}$ is such that $\|z\| = 1$ consequently $Tz = 0$, which implies that $Tx = 0$ and therefore, $Tx = 0, \forall x \in D(T)$.

ii) Let $\lambda \in C$, then,

$$\| \lambda T \| = \sup_{\|x\| \leq 1} \| \lambda Tx \| = |\lambda| \sup_{\|x\| \leq 1} \| Tx \| = |\lambda| \| T \| .$$

iii) Let $S, T \in \mathcal{L}(X, Y)$

$$\begin{aligned} \| T + S \| &= \sup \| Tx + Sx \|; x \in D(T) \cap D(S), \| x \| \leq 1 \\ &\leq \sup \| Tx \| + \| Sx \|; x \in D(T) \cap D(S), \| x \| \leq 1 \\ &\leq \sup \| Tx \|; x \in D(T), \| x \| \leq 1 + \sup \| Sx \|; x \in D(S), \| x \| \leq 1 = \| T \| + \| S \| . \end{aligned}$$

which completes the proof of the lemma. $\square$

Example 1.3.1. Let $T : C_{\mathbb{R}}[0, 1] \rightarrow \mathbb{R}$, be defined by $Tf = f(0)$, and equipped $C_{\mathbb{R}}[0, 1]$ by the usual norm $\| f \| = \sup_{0 \leq x \leq 1} |f(x)|$. We have

$$\| T \| = \sup_{\|f\| \leq 1} |f(0)| \leq \sup_{f \leq x \leq 1} \sup_{\|f\| \leq 1} |f(x)| = \sup_{\|f\| \leq 1} \| f \| = 1.$$

On the other hand, let $g : [0, 1] \rightarrow \mathbb{R}$ with $g(x) = 1$, for all $x \in [0, 1]$. Then, $g \in C[0, 1]$ and $\| g \| = 1$. Moreover, $|Tg| = |g(0)| = 1$, hence, $\| T \| \geq |Tg| = 1$, therefore $\| T \| = 1$.

Proposition 1.3.2. The norm defined above is also given by

$$\| T \| = \inf \{ C > 0 : \| Tx \| \leq C \| x \|, \forall x \in D(T) \}.$$

Demonstration 1.3.1. Since T is continuous, then from proposition (1.3.1), there exists $C > 0$, such that $\| Tx \| \leq C \| x \|$, $\forall x \in D(T)$.

For all $C \in \{ C > 0 : \| Tx \| \leq C \| x \|, \forall x \in D(T) \}$, we have

$$\| T \| = \sup_{x \in D(T), \|x\| \leq 1} \| Tx \| \leq \sup_{x \in D(T)} \| Tx \| \leq C$$

Thus,

$$\| T \| \leq \inf \{ C > 0 : \| Tx \| \leq C \| x \|, \forall x \in D(T) \}$$

onversely, denote $\inf \{ C > 0 : \| Tx \| \leq C \| x \|, \forall x \in D(T) \} = \overline{C}$, then, for all $\epsilon > 0$, there exists $x_{\epsilon} \in D(T)$, such that

$$\| Tx_{\epsilon} \| > (\overline{C} - \epsilon) \| x_{\epsilon} \|^2$$

Clearly, $x_\epsilon \neq 0$, then,

$$\forall \epsilon > 0, \exists y_\epsilon = \frac{x_\epsilon}{\|x_\epsilon\|}; \|Ty_\epsilon\| > (\overline{C} - \epsilon)$$

, therefore

$$\forall \epsilon > 0, \exists y_\epsilon = \frac{x_\epsilon}{\|x_\epsilon\|}; \|T\| = \sup_{\|y_\epsilon\| \leq 1} \|Ty_\epsilon\| > (\overline{C} - \epsilon)$$

, Passing to the limit $\epsilon \rightarrow 0$ we infer that

$$\|T\| \geq \overline{C} = \inf\{C > 0 : \|Tx\| \leq C\|x\|, \forall x \in D(T)\}.$$

hence the equality $\|T\| = \overline{C}$.

1.3.3 Invertible operators

Definition 1.3.3. An operator $T \in \mathcal{L}(X, Y)$ is said to be invertible if there exists an operator $S \in \mathcal{L}(Y, X)$ and $ST = TS = I$. In this case S is denoted T^{-1} .

Example 1.3.2. For $f \in C[0, 1]$ and defined $T_f \in \mathcal{L}(L^2[0, 1])$ by

$$(T_f u)(x) = f(x)u(x), \quad u \in L^2[0, 1].$$

Clearly, $T_f \in \mathcal{L}(L^2[0, 1])$. Let f be the function defined by $f(x) = 1 + x$. Then, T_f is invertible. Indeed, for $g(x) = \frac{1}{x+1}$, we have $T_g \in \mathcal{B}(L^2[0, 1])$. Moreover,

$$(T_f T_g u)(x) = f(x)g(x)u(x) = u(x)$$

and

$$(T_g T_f u)(x) = g(x)f(x)u(x) = u(x).$$

which shows that T_f is invertible and $T_f^{-1} = T_g$.

Theorem 1.3.2. Let X be a Banach space and $T \in \mathcal{L}(X)$ with $\|I - T\| < 1$. Then, T is invertible with

$$T^{-1} = \sum_{n \geq 0} (I - T)^n.$$

Proof. Since $\|I - T\| < 1$, the series $\sum_{n \geq 0} \|I - T\|^n$ converges. On the other hand,

$$\|(I - T)^n\| \leq \|I - T\|^n,$$

then the series $\sum_{n \geq 0} \|(I - T)^n\|$ converges, and $\sum_{n \geq 0} (I - T)^n$ is an absolutely convergent series. Let S be its limit and $S_k = \sum_{n=0}^k (I - T)^n$. Then we have

$$\|TS_k - I\| = \|(I - (I - T))S_k - I\| = \|(I - T)^{k+1}\| \leq \|I - T\|^{k+1}.$$

Thus,

$$0 \leq \lim_{k \rightarrow \infty} \|TS_k - I\| \leq \lim_{k \rightarrow \infty} \|I - T\|^{k+1} = 0.$$

Therefore, $TS - I = \lim_{k \rightarrow \infty} (TS_k - I) = 0$. Similarly $ST - I = \lim_{k \rightarrow \infty} (S_k T - I) = 0$, which completes the proof. $\square$

Theorem 1.3.3. *Let T be a linear operator from normed linear space X into normed linear space Y . Then, T^{-1} exists and is continuous if and only if there exists $m > 0$ such that*

$$\|Tx\| \geq m\|x\| \quad , \forall x \in X.$$

Definition 1.3.4. *Let X, Y be normed linear spaces. If an invertible operator $T \in \mathcal{L}(X, Y)$ exists, then X and Y are isomorphic, and T is an isomorphism (between X and Y).*

Lemma 1.3.4. *If the normed linear spaces X and Y are isomorphic, then:*

- a) $\dim X < \infty$ if and only if $\dim Y < \infty$, in which case $\dim X = \dim Y$.*
- b) X is separable if and only if Y is separable.*
- c) X is complete (i.e., Banach) if and only if Y is complete (i.e., Banach).*

Theorem 1.3.5. *Let X et Y two Banach spaces. if $T \in \mathcal{L}(X, Y)$ is bijective, then it is invertible.*

1.3.4 The adjoint of an operator

Let $A : D(A) \subset X \rightarrow Y$ be an unbounded operator with dense domain $\overline{D(A)} = X$.

Demonstration 1.3.2. $D(A^*)$ is a linear subspace of Y' : Let $f : D(A) \rightarrow Y$ be define by $f(x) = \langle \psi, Ax \rangle$, it is clear that f is linear and

$$|f(x)| = |\langle \psi, Ax \rangle| \leq c\|x\|_X.$$

From Hahn-Banach Theorem, we deduce that f can be prolonged linearly by a unique $\varphi : X \rightarrow \mathbb{R}$ such that

$$|\varphi(x)| \leq c\|x\|, \forall x \in X,$$

consequently, $\varphi \in X'$ and since φ is the prolongement of f , we get

$$\langle \psi, Ax \rangle = \langle \varphi, x \rangle, \forall x \in D(A).$$

Proposition 1.3.3. Define the set

$$D(A^*) : \{\psi \in Y' : \exists c > 0 \text{ such that } |\langle \psi, Ax \rangle| \leq c\|x\|_X, \forall x \in D(A)\}$$

Then, for all $\psi \in D(A^*)$ there exists a unique $\varphi \in X'$ such that

$$\langle \psi, Ax \rangle = \langle \varphi, x \rangle, \forall x \in D(A).$$

Definition 1.3.5. The mapping

$$A^* : D(A^*) \subset Y' \rightarrow X'$$

$$: \psi \rightarrow \varphi = A^*\psi,$$

is called the adjoint operator of A and denoted A^* , it is a bounded operator that satisfies

$$\langle A^*\psi, x \rangle = \langle \psi, Ax \rangle, \forall \psi \in D(A^*), \forall x \in D(A).$$

Remark 1.3.1. It is necessary that $D(A)$ be dense in X , to define A^* correctly. Indeed, suppose that there exist $\varphi_1, \varphi_2 \in X'$ such that $A^*\psi = \varphi_1$ et $A^*\psi = \varphi_2$, then,

$$\langle A^*\psi, x \rangle = \langle \varphi_1, x \rangle = \langle \varphi_2, x \rangle, \quad \forall x \in D(A)$$

therefore

$$\langle \varphi_1 - \varphi_2, x \rangle = 0, \quad \forall x \in D(A)$$

which implies that $\varphi_1 - \varphi_2 = 0$, if and only if $D(A)$ is dense in X .

Remark 1.3.2. $D(A^*)$ can not be dense in Y' .

Theorem 1.3.6. [6] Let H and K be two Hilbert spaces over the same scalar field $\mathbb{K}$, and let $A \in \mathcal{L}(H, K)$ be a linear and bounded operator. Then, there exists a unique operator from $A^* \in \mathcal{L}(K, H)$ such that

$$(Ax, y) = (x, A^*y), \quad \forall x \in H, \forall y \in K.$$

Definition 1.3.6. *The operator A^* just constructed is called the adjoint operator of A*

Example 1.3.3. *Endowed $\mathbb{R}^2$ by the canonical basis $\{e_1, e_2\} = \{(1, 0), (0, 1)\}$ and let $A \in \mathcal{L}(\mathbb{R}^2)$ given by*

$$A(x_1, x_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Assume that A^ is given by a matrix $B = (b_{ij})$, that is*

$$A^*(y_1, y_2) = B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Recall that

$$(Ax, y) = (x, A^*y),$$

that is,

$$\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} b_{11}y_1 + b_{12}y_2 \\ b_{21}y_1 + b_{22}y_2 \end{pmatrix}$$

then,

$$a_{11}x_1y_1 + a_{12}x_2y_1 + a_{21}x_1y_2 + a_{22}x_2y_2 = b_{11}x_1y_1 + b_{12}x_1y_2 + b_{21}x_2y_1 + b_{22}x_2y_2,$$

which gives,

$$b_{11} = a_{11}, \quad b_{12} = a_{21}, \quad b_{21} = a_{12}, \quad b_{22} = a_{22}.$$

Thus,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

Remark 1.3.3. *Note that if $A \in \mathcal{L}(\mathbb{C}^2)$, then the adjoint of A is given by*

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}.$$

Example 1.3.4. *Let $k \in C([0, 1])$ and $A \in \mathcal{L}(L^2[0, 1])$ defined by*

$$(Af)(x) = k(x)f(x).$$

We have

$$\langle Af, g \rangle = \langle f, A^*g \rangle, \quad \forall f, g \in L^2[0, 1],$$

that is

$$\int_0^1 k(x)f(x)g(x)dx = \int_0^1 f(x)A^*g(x)dx, \quad \forall f, g \in L^2[0, 1],$$

therefore, $A^*g(x) = k(x)g(x)$, thus, $A^* = A$.

Example 1.3.5. Let $T : \ell^2(\mathbb{R}) \longrightarrow \ell^2(\mathbb{R})$

$$T(x_n) = (0, x_2, x_3, \dots)$$

$$\langle T(x_n), (y_n) \rangle = \sum_{n=2}^{\infty} x_n y_n$$

Now, suppose that $T^*(y_n) = (z_n)$, then,

$$\langle (x_n), T^*(y_n) \rangle = \langle (x_n), (z_n) \rangle = \sum_{n=1}^{\infty} x_n z_n = \sum_{n=2}^{\infty} x_n y_n$$

Definition 1.3.7. If $A^* = A$ we say that the operator A is self adjoint.

Remark 1.3.4. Clearly, $I^* = I$.

Lemma 1.3.7. [12] Let H, K and N be three Hilbert spaces over $\mathbb{C}$ and let $\lambda, \mu \in \mathbb{C}$, $A, B \in \mathcal{L}(H, K)$ and $T \in \mathcal{L}(K, N)$. Therefore,

1. $(\lambda A + \mu B)^* = \bar{\lambda}A^* + \bar{\mu}B^*$,
2. $(AT)^* = T^*A^*$.

Theorem 1.3.8. [12] Let H, K be two Hilbert spaces over $\mathbb{C}$ and $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, then,

1. $(A^*)^* = A$,
2. $\|A^*\| = \|A\|$,
3. the function $F : \mathcal{L}(H, K) \rightarrow \mathcal{L}(K, H)$ defined by $F(A) = A^*$ is continuous,
4. $\|A^*A\| = \|A^*\| \cdot \|A\| = \|A\|^2$.

Lemma 1.3.9. [6] Let $A \in \mathcal{L}(\mathcal{H})$ be a linear and continuous operator on the Hilbert space H . Then, A is invertible if and only if A^* is invertible.

Proof. Suppose that A is invertible, then A^{-1} exists and we have $AA^{-1} = A^{-1}A = I$, then

$$(AA^{-1})^* = (A^{-1})^*A^* = I, \text{ consequently, } (A^{-1})^*A^* = I.$$

Thus, A^* is invertible and $(A^{-1})^* = (A^*)^{-1}$.

Reciprocally, if A^* is invertible, then, using the above argument, we deduce that $A = (A^*)^{-1}$ is invertible and

$$A^{-1} = ((A^*)^{-1})^* = (A^*)^{-1}.$$

Consequently, $(A^{-1})^* = (A^*)^{-1}$. □

1.3.5 Self adjoint operator

Definition 1.3.8. Let H be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. The operator A is said to be self adjoint if $A^* = A$

Example 1.3.6. Let $k \in C[0, 1]$ and consider the complex Hilbert space $L^2[0, 1]$. Let $A \in \mathcal{L}(L^2[0, 1])$ be defined by

$$(Af)(x) = k(x)f(x),$$

then

$$(A^*f)(x) = \overline{k(x)}f(x)$$

and

$$(A^*Af)(x) = (A^*kf)(x) = \overline{k(x)}k(x)f(x) = k(x)\overline{k(x)}f(x) = (AA^*f)(x).$$

Thus,

$$(A^*Af) = (AA^*f),$$

which shows that A is normal .

Lemma 1.3.10. Let $S(\mathcal{H})$ be the set of all self-adjoint operators over H . Then, for all $\lambda, \mu \in \mathbb{R}$, $\lambda S + \mu T \in S(\mathcal{H})$ and $S(\mathcal{H})$ is a closed linear subspace of $\mathcal{L}(\mathcal{H})$.

Proof. 1) Let $\lambda, \mu \in \mathbb{R}$ and $S, T \in S(\mathcal{H})$. We have

$$(\lambda S + \mu T)^* = \lambda S^* + \mu T^* = \lambda S + \mu T.$$

2) Moreover, let $\{T_n\}$ be a sequence of self-adjoint operators which converges to $T \in \mathcal{L}(\mathcal{H})$. Since the function $T \rightarrow T^*$ is continuous, then $\{T_n\}$ converges to T^* . But $T_n^* = T_n$, then $\{T_n\}$ converges to T . By the uniqueness of the limit, we have $T^* = T$. $\square$

Lemma 1.3.11. [12] *Let H be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Then,*

1. TT^* and T^*T are self-adjoint operators,
2. there exist two self-adjoint operators R, S such that $T = R + iS$.

Proof. 1) $(TT^*)^* = (T^*)^*T^* = TT^*$ and $(T^*T)^* = T^*(T^*)^* = T^*T$.

2) Set $R = \frac{T+T^*}{2}$ and $S = \frac{T-T^*}{2i}$. Then $T = R + iS$ and

$$R^* = \left(\frac{T+T^*}{2} \right)^* = \frac{T^*+T}{2} = R$$

and

$$S^* = \left(\frac{T-T^*}{2i} \right)^* = \frac{T^*-T}{-2i} = S.$$

$\square$

Lemma 1.3.12. *Let H be a complex Hilbert space, $A \in \mathcal{L}(\mathcal{H})$, and $\lambda \in \mathbb{C}$. Then, A is normal if and only if $(A - \lambda I)$ is normal.*

Proof. If A is normal, then

$$\begin{aligned} (A - \lambda I)^*(A - \lambda I) &= (A^* - \bar{\lambda}I)(A - \lambda I) \\ &= A^*A - \lambda A^* - \bar{\lambda}A + \bar{\lambda}\lambda I \\ &= AA^* - \bar{\lambda}A - \lambda A^* + \bar{\lambda}\lambda I \\ &= A(A^* - \bar{\lambda}I) - \lambda I(A^* - \bar{\lambda}I) \\ &= (A - \lambda I)(A^* - \bar{\lambda}I) \\ &= (A - \lambda I)(A - \lambda I)^*. \end{aligned}$$

Conversely, if $(A - \lambda I)$ is normal, then $A = (A - \lambda I) - (-\lambda)I$ is normal. $\square$

Lemma 1.3.13. *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator. Then,*

1. $\|Ax\| = \|A^*x\|$ for all $x \in H$.
2. If there exists $\alpha > 0$ such that $\|Ax\| \geq \alpha\|x\|$ for all $x \in H$, then $\ker A^* = \{0\}$.

Proof. 1) Let $x \in \mathcal{H}$. Since $A^*A = AA^*$, we have $A^*Ax = AA^*x$, and hence

$$\langle A^*Ax, x \rangle = \langle AA^*x, x \rangle,$$

which implies

$$\langle Ax, Ax \rangle = \langle A^*x, A^*x \rangle \iff \|Ax\|^2 = \|A^*x\|^2.$$

2) Let $y \in \ker A^*$. Then, $A^*y = 0$, and by virtue of part 1), we have

$$0 = \|A^*y\| = \|Ay\| \geq \alpha\|y\|.$$

Therefore, $\|y\| = 0$, which implies $y = 0$. Thus, $\ker A^* = \{0\}$. □

Corollary 1.3.1. *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator. The following statements are equivalent:*

1. A is invertible.
2. There exists $\alpha > 0$ such that $\|Ax\| \geq \alpha\|x\|$ for all $x \in H$.

Definition 1.3.9. *An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be unitary if $A^*A = AA^* = I$. An isometry is an operator $A \in \mathcal{L}(\mathcal{H})$ that satisfies $\|Ax\| = \|x\|$ for all $x \in H$. The norm of an isometry is equal to 1, that is, $\|A\| = 1$. We denote by $U(\mathcal{H})$ the set of all unitary operators over H .*

Remark 1.3.5. *A unitary operator is invertible, and its inverse is its adjoint.*

Basics of C^* -algebras

In this chapter we will discuss the most important definitions related to C^* -algebras, including its main properties.

2.1 Banach Spaces and Algebras

Definition 2.1.1. A normed space E is **complete** if every Cauchy sequence in E converges. A complete normed real vector space is called a **Banach space**.

Example 2.1.1. Every finite-dimensional normed real vector space is a Banach space.

Definition 2.1.2. An algebra is a vector space $\mathcal{A}$ together with a bilinear map

$$\begin{aligned}\mathcal{A} \times \mathcal{A} &\rightarrow \mathcal{A} \\ (a, b) &\mapsto a \cdot b\end{aligned}$$

which verifies, $\forall a, b, c \in \mathcal{A}$:

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. $a \cdot (b + c) = a \cdot b + a \cdot c,$
 $(a + b) \cdot c = a \cdot c + b \cdot c$

Remark 2.1.1. •

- If $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{A}$, the algebra is commutative.
- If there exists $1_{\mathcal{A}} \in \mathcal{A}$ such that $a \cdot 1_{\mathcal{A}} = 1_{\mathcal{A}} \cdot a = a$ for all $a \in \mathcal{A}$, the algebra is unital and $1_{\mathcal{A}}$ is called the unit element.

Definition 2.1.3. A normed algebra is an algebra $\mathcal{A}$ over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) equipped with a norm $\|\cdot\|$ satisfying the submultiplicativity condition:

$$\|x \cdot y\| \leq \|x\| \cdot \|y\| \quad \text{for all } x, y \in \mathcal{A}.$$

2.2 Banach Algebras

A **Banach algebra** over the complex field $\mathbb{C}$ is a complete normed $\mathbb{C}$ -algebra. This means it is a vector space equipped with:

- A bilinear vector multiplication (satisfying distributivity and associativity).
- A norm under which the space is complete (i.e., a Banach space).

Definition 2.2.1. A complex Banach space $\mathcal{A}$ with multiplication $(a, b) \mapsto a \cdot b$ is a **Banach algebra** if for all $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ satisfying the following conditions :

1. **Associativity:** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. **Distributivity:** $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$
3. **Scalar compatibility:** $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$
4. **Submultiplicativity:** $\|a \cdot b\| \leq \|a\| \cdot \|b\|$
5. **Commutativity (optional):** If $a \cdot b = b \cdot a$ for all a, b , $\mathcal{A}$ is commutative.
6. **Unity element:** If there exists $1_{\mathcal{A}} \in \mathcal{A}$ with $\|1_{\mathcal{A}}\| = 1_{\mathcal{A}}$ such that $a \cdot 1_{\mathcal{A}} = 1_{\mathcal{A}} \cdot a = a$ for all $a \in \mathcal{A}$ is unital.

Remark 2.2.1. 1.

2. **Power inequality:** $\|a^n\| \leq \|a\|^n$ for all $a \in \mathcal{A}$, $n \in \mathbb{N}$.

3. Unital extension:

- The unity element $1_{\mathcal{A}}$ is unique if it exists.
- Any Banach algebra $\mathcal{A}$ can be isometrically embedded into a unital Banach algebra $\mathcal{A}^{\#} = \mathcal{A} \times \mathbb{C}$ with:

$$\text{Operations: } (x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$

$$\text{Multiplication: } (x, \alpha) \cdot (y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$$

$$\text{Norm: } \|(x, \alpha)\| = \|x\| + |\alpha|$$

Example 2.2.1. 1. Banach algebra:

$(C[a, b], \|\cdot\|_{\infty})$ with pointwise operations and norm $\|f\|_{\infty} = \sup_{t \in [a, b]} |f(t)|$ is a commutative Banach algebra.

2. Not a Banach algebra:

$(C[a, b], \|\cdot\|_1)$ fails submultiplicativity. For $f(t) = t$, $g(t) = t$ on $[0, 1]$:

$$\begin{aligned} \|f \cdot g\|_1 &= \int_0^1 t^2 dt = \left[\frac{t^3}{3} \right]_0^1 = \frac{1}{3}, \\ \|f\|_1 \cdot \|g\|_1 &= \left(\int_0^1 t dt \right)^2 = \left(\left[\frac{t^2}{2} \right]_0^1 \right)^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4} \end{aligned}$$

Since $\frac{1}{3} > \frac{1}{4}$, $\|f \cdot g\|_1 \not\leq \|f\|_1 \|g\|_1$.

2.3 Invertibility and spectrum

2.3.1 Invertibility

Definition 2.3.1. Let $\mathcal{A}$ be a unital Banach algebra. An element $x \in \mathcal{A}$ is called invertible if there exists an inverse $x^{-1} \in \mathcal{A}$ satisfying:

$$x^{-1}x = xx^{-1} = 1_{\mathcal{A}}.$$

where $1_{\mathcal{A}}$ is the multiplicative identity of $\mathcal{A}$.

The set of invertible elements in $\mathcal{A}$ is denoted by $\text{Inv}(\mathcal{A})$.

1. **Uniqueness:** If an inverse exists, it is unique.
2. **Group Structure:** $\text{Inv}(\mathcal{A})$ forms a group under multiplication.
3. **Product Rule:** For all $a, b \in \text{Inv}(\mathcal{A})$, the product $a \cdot b$ is invertible with

$$(a \cdot b)^{-1} = b^{-1}a^{-1}.$$

Example 2.3.1. 1. In the Banach algebra $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$):

a is invertible $\iff a \neq 0$, so

$$\text{Inv}(\mathbb{K}) = \mathbb{K}^* = \mathbb{K} \setminus \{0\}.$$

2. In the Banach algebra $M_n(\mathbb{K})$ ($n \times n$ matrices):

A is invertible $\iff \det(A) \neq 0$.

Proposition 2.3.1. Let $\mathcal{A}$ be a unital Banach algebra and $x \in \mathcal{A}$ such that $\|x\| < 1$. Then

1. $1_{\mathcal{A}} - x \in \text{Inv}(\mathcal{A})$,
2. $\|(1_{\mathcal{A}} - x)^{-1}\| \leq \frac{1}{1 - \|x\|}$,
3. $\|(1_{\mathcal{A}} - x)^{-1} - 1_{\mathcal{A}} - x\| \leq \frac{\|x\|^2}{1 - \|x\|}$.

Proof. Since $\|x^n\| \leq \|x\|^n$ and $\|x\| < 1$, the sequence $(s_n)_{n \geq 0}$, defined by

$$s_n = 1_{\mathcal{A}} + x + x^2 + \cdots + x^n,$$

forms a Cauchy sequence in $\mathcal{A}$. Because $\mathcal{A}$ is complete, there exists $s \in \mathcal{A}$ such that $s_n \rightarrow s$. A straightforward calculation shows that

$$s_n(1_{\mathcal{A}} - x) = 1_{\mathcal{A}} - x^{n+1} = (1_{\mathcal{A}} - x)s_n$$

and since $x^n \rightarrow 0$, the continuity of multiplication implies

$$s(1_{\mathcal{A}} - x) = 1_{\mathcal{A}} = (1_{\mathcal{A}} - x)s.$$

In other words, $1_{\mathcal{A}} - x$ is invertible, and its inverse is

$$(1_{\mathcal{A}} - x)^{-1} = s = \sum_{n=0}^{\infty} x^n.$$

We have

$$\|(1_{\mathcal{A}} - x)^{-1}\| = \left\| \sum_{n=0}^{\infty} x^n \right\| \leq \sum_{n=0}^{\infty} \|x^n\| \leq \sum_{n=0}^{\infty} \|x\|^n = \frac{1}{1 - \|x\|}.$$

We have

$$(1_{\mathcal{A}} - x)^{-1} - 1_{\mathcal{A}} - x = \sum_{n=2}^{\infty} x^n,$$

which gives

$$\|(1_{\mathcal{A}} - x)^{-1} - 1_{\mathcal{A}} - x\| \leq \sum_{n=2}^{\infty} \|x\|^n = \frac{\|x\|^2}{1 - \|x\|}.$$

□

Corollary 2.3.1. *Let $a \in \mathcal{A}$ such that $\|1_{\mathcal{A}} - a\| < 1$. Then a is invertible, and*

$$\|a^{-1}\| \leq \frac{1}{1 - \|1_{\mathcal{A}} - a\|}.$$

.

Proof. Suppose $\|1_{\mathcal{A}} - a\| < 1$ for $a \in \mathcal{A}$.

We have $a = 1_{\mathcal{A}} - (1_{\mathcal{A}} - a)$, and thus

$$\|a^{-1}\| = \|(1_{\mathcal{A}} - (1_{\mathcal{A}} - a))^{-1}\| \leq \frac{1}{1 - \|1_{\mathcal{A}} - a\|}.$$

□

Proposition 2.3.2. *The set $\text{Inv}(\mathcal{A})$ is open in $\mathcal{A}$, and the map $a \mapsto a^{-1}$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.*

Proof. Let $a \in \text{Inv}(\mathcal{A})$. We show that $B(a, r) \subset \text{Inv}(\mathcal{A})$, where $r = \frac{1}{\|a^{-1}\|} = \|a^{-1}\|^{-1}$. Let $b \in B(a, r)$, so $\|b - a\| < r$. We have $b = a - (a - b) = a[1_{\mathcal{A}} - a^{-1}(a - b)]$, and $\|a^{-1}(a - b)\| \leq \|a\|^{-1}\|a - b\| < \frac{1}{r}, r = 1$.

Therefore, by (2.3.1), $1_{\mathcal{A}} - a^{-1}(a - b)$ is invertible, and thus $b = a(1_{\mathcal{A}} - a^{-1}(a - b))$ is invertible with

$$b^{-1} = [1_{\mathcal{A}} - a^{-1}(a - b)]^{-1}a^{-1} = \sum_{k=0}^{\infty} (a^{-1}(a - b))^k a^{-1}.$$

We show that:

$$\|b^{-1} - a^{-1}\| \leq \frac{\|a^{-1}\|^2 \|b - a\|}{1 - \|a^{-1}\| \|b - a\|}.$$

Let (a_n) be a sequence of invertible operators such that $a_n \rightarrow a$, where a is invertible. Then

$$\exists N : n \geq N \quad \|a_n - a\| < \|a^{-1}\|^{-1},$$

and

$$\|a_n^{-1} - a^{-1}\| \leq \frac{\|a^{-1}\|^2 \|a_n - a\|}{1 - \|a^{-1}\| \|a_n - a\|}.$$

As $n \rightarrow \infty$, we obtain

$$\|a_n^{-1} - a^{-1}\| \rightarrow 0,$$

so $a_n^{-1} \rightarrow a^{-1}$. Thus, the map $a \rightarrow a^{-1}$ in $\text{Inv}(\mathcal{A})$ is continuous.

□

2.3.2 Spectrum

Definition 2.3.2. For $a \in \mathcal{A}$, the spectrum of a , denoted $\sigma(a)$ is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible}\}.$$

The complement of $\sigma(a)$ is called the resolvent of a and is denoted $\rho(a)$ such that

$$\rho(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is invertible}\}.$$

The spectral radius of a is the following number

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Proposition 2.3.3. [14] *Let $a \in \mathcal{A}$. Then*

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

Proof. • Define

$$r = \lim_{n \rightarrow \infty} \inf \|a^n\|^{1/n}$$

- Show $r(a) \leq r$.
- Use the Neumann series to analyze $(1_{\mathcal{A}} - za)^{-1}$:

$$(1_{\mathcal{A}} - za)^{-1} = \sum_{n=0}^{\infty} z^n a^n \quad \text{for } |z| < \frac{1}{r(a)}.$$

- Conclude that the radius of convergence is $\frac{1}{r(a)}$, hence $r(a) = r$.

□

Example 2.3.2. Let $a = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ be a 2×2 matrix.

1. **Compute the eigenvalues:**

- Solve the equation: $\det(\lambda A - a) = 0$

$$\det \begin{pmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix} = (3 - \lambda)(2 - \lambda) = 0.$$

- The eigenvalues are: $\lambda_1 = 3, \lambda_2 = 2$.

2. **Spectrum $\sigma(a)$:**

- In finite dimensions: the spectrum is the set of eigenvalues. So

$$\sigma(a) = \{2, 3\}.$$

3. **Explanation:**

- Why is $\lambda = 2$ in the spectrum? Because $a - 21_{\mathcal{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not invertible (the matrix is not full rank).
- Same thing for $\lambda = 3$.

Proposition 2.3.4. [14] *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$. Then:*

- The resolvent set $\rho(a)$ is a non-empty open subset of $\mathbb{K}$.
- The spectrum $\sigma(a)$ is a non-empty, closed and bounded subset of $\mathbb{C}$, contained in the closed disk centered at 0 with radius $\|a\|$:

$$\sigma(a) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}.$$

- If $a \in \text{Inv}(\mathcal{A})$, then

$$\sigma(a^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(a) \right\}.$$

Proof.

To show $\rho(a) \neq \emptyset$, pick $\lambda \in \mathbb{K}$ with $|\lambda| > \|a\|$, then $a - \lambda 1_{\mathcal{A}}$ is invertible. To prove openness, use the Neumann series: if $a - \lambda_0 1_{\mathcal{A}}$ is invertible, then for λ close enough to λ_0 , $a - \lambda 1_{\mathcal{A}}$ is also invertible.

For $|\lambda| > \|a\|$, one can write:

$$a - \lambda 1_{\mathcal{A}} = -\lambda \left(1_{\mathcal{A}} - \frac{1}{\lambda} a \right).$$

Since $\left\| \frac{1}{\lambda} a \right\| < 1$, the term in parentheses is invertible, so $\lambda \notin \sigma(a)$. Thus, $\sigma(a)$ is bounded. Being the complement of the open set $\rho(a)$, it is also closed.

Let $\lambda \in \sigma(a^{-1})$. Then $a^{-1} - \lambda 1_{\mathcal{A}}$ is not invertible. We can write:

$$a^{-1} - \lambda 1_{\mathcal{A}} = -\lambda a^{-1} (a - \lambda^{-1} 1_{\mathcal{A}}).$$

So $a - \lambda^{-1} 1_{\mathcal{A}}$ is not invertible, i.e., $\lambda^{-1} \in \sigma(a)$. Hence:

$$\sigma(a^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(a) \right\}.$$

□

Theorem 2.3.1 (Gelfand-Mazur). [17] *Let $\mathcal{A}$ be a unital Banach algebra. If every nonzero element of $\mathcal{A}$ is invertible, then $\mathcal{A}$ is 1-dimensional, i.e. $1_{\mathcal{A}}$ is a linear basis for $\mathcal{A}$.*

Proof. Let $a \in \mathcal{A}$ be given.

Choose $\lambda \in \sigma(a)$. Then, $a - \lambda 1_{\mathcal{A}}$ is not invertible.

Hence, $a - \lambda 1_{\mathcal{A}} = 0$, i.e. $a = \lambda 1_{\mathcal{A}}$. □

Proposition 2.3.5. *Let $\mathcal{A}$ be a unital Banach algebra, $a \in \mathcal{A}$, and $p \in \mathbb{C}[X]$ a complex polynomial. Then*

$$\sigma(p(a)) = p(\sigma(a)).$$

Proof. We prove the equality by showing both inclusions. **(1) First**, we show that:

$$p(\sigma(a)) \subset \sigma(p(a)).$$

Let $\lambda \in \sigma(a)$. Then $a - \lambda 1_{\mathcal{A}}$ is not invertible. We want to show that $p(\lambda) \in \sigma(p(a))$, i.e., that $p(a) - p(\lambda)1_{\mathcal{A}}$ is not invertible. Since $p(X) - p(\lambda)$ vanishes at $X = \lambda$, we can write:

$$p(X) - p(\lambda) = (X - \lambda)q(X)$$

for some $q \in \mathbb{C}[X]$. Evaluating at a , we get:

$$p(a) - p(\lambda)1_{\mathcal{A}} = (a - \lambda 1_{\mathcal{A}})q(a).$$

If $p(a) - p(\lambda)1_{\mathcal{A}}$ were invertible, then so would be $a - \lambda 1_{\mathcal{A}}$ (since $q(a)$ is an element of $\mathcal{A}$), which contradicts $\lambda \in \sigma(a)$. Hence, $p(\lambda) \in \sigma(p(a))$.

(2) Conversely, we show that:

$$\sigma(p(a)) \subset p(\sigma(a)).$$

If p is the zero polynomial, the result is trivial. Otherwise, let $\lambda \in \sigma(p(a))$. Consider the factorization:

$$p(X) - \lambda = \alpha(X - \alpha_1)(X - \alpha_2) \cdots (X - \alpha_n)$$

with $\alpha \neq 0$, and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ the roots of $p(X) - \lambda$.

Evaluating at a , we obtain:

$$p(a) - \lambda 1_{\mathcal{A}} = \alpha(a - \alpha_1 1_{\mathcal{A}})(a - \alpha_2 1_{\mathcal{A}}) \cdots (a - \alpha_n 1_{\mathcal{A}}).$$

If each $a - \alpha_i 1_{\mathcal{A}}$ were invertible, then so would $p(a) - \lambda 1_{\mathcal{A}}$, contradicting the assumption that $\lambda \in \sigma(p(a))$.

Therefore, there exists some i such that $a - \alpha_i 1_{\mathcal{A}}$ is not invertible, i.e., $\alpha_i \in \sigma(a)$, and thus:

$$\lambda = p(\alpha_i) \in p(\sigma(a)).$$

Combining both directions, we conclude:

$$\sigma(p(a)) = p(\sigma(a)).$$

□

2.4 C^* -algebras

2.4.1 $*$ -algebra

Definition 2.4.1. If $\mathcal{A}$ is a Banach algebra, an involution is a map $a \mapsto a^*$ on $\mathcal{A}$ such that:

- Has a unary operator $*$ is the star operator .
- $*$ is an involution : $(a^*)^* = a$.
- $*$ is conjugate-linear : $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$, for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.
- $*$ is an antihomomorphism : $(ab)^* = b^* a^*$.

The pair $(\mathcal{A}, *)$ is called an involutive algebra, or $*$ -algebra .

Remark 2.4.1. If $\mathcal{A}$ is a unital involutive algebra (with an involution), then

$$a = (a^*)^* = (a^* 1_{\mathcal{A}})^* = 1_{\mathcal{A}}^* (a^*)^* = 1_{\mathcal{A}}^* a.$$

Similarly, $a = a 1_{\mathcal{A}}^*$. Hence, for every $a \in \mathcal{A}$,

$$a = 1_{\mathcal{A}}^* a = a 1_{\mathcal{A}}^*,$$

which, by the uniqueness of the neutral element, implies that $1_{\mathcal{A}}^* = 1_{\mathcal{A}}$.

2.4.2 C^* -algebra

Definition 2.4.2. A C^* -algebra is a Banach $*$ -algebra whose norm satisfies the C^* condition:

$$\|a^*a\| = \|a\|^2.$$

for all a in $\mathcal{A}$.

Example 2.4.1. $\mathbb{C}$ is a C^* -algebra over itself. Indeed,

$\mathbb{C}$ is a vector space over itself (with the usual operations of addition and multiplication).

Involution ($$):* For any complex number $z = a + ib$ (where $a, b \in \mathbb{R}$), the involution $*$ is defined as complex conjugation:

$$z^* = \bar{z} = a - ib.$$

Norm:

The absolute value is used as the norm:

$$\|z\| = |z| = \sqrt{a^2 + b^2}.$$

Involutive Property:

$$(z^*)^* = \bar{\bar{z}} = z.$$

Antilinearity: For any $\lambda \in \mathbb{C}$ and $z, w \in \mathbb{C}$:

$$(\lambda z + w)^* = \bar{\lambda} z^* + w^*.$$

Product Rule:

$$(zw)^* = w^* z^* \quad (\text{analogous to the conjugate of a product of two complex numbers}).$$

C^ -Identity:*

$$\|z^*z\| = \|\bar{z}z\| = \| |z|^2 \| = |z|^2 = \|z\|^2.$$

(since $\bar{z}z = a^2 + b^2 = |z|^2$).

Generally speaking, C^* -algebra can be seen as a way to generalize the set of complex numbers.

Example 2.4.2. X : locally compact, hausdorff.

$C_0(X)$: continuous functions on X that vanish at infinity, $f : X \rightarrow \mathbb{C}$ with $\lim_{|x| \rightarrow \infty} f(x) = 0$.

- *Multiplication:* $fg(x) = f(x)g(x)$ (commutative).
- *$*$ operator:* $f^*(x) = \overline{f(x)}$.
- $\|\cdot\|$ is the supremum norm: $\|f\| = \sup\{|f(x)|, x \in X\}$.

$C_0(X)$ is a C^* -algebra .

Example 2.4.3. Let $\mathcal{B}(H)$ denote the set of bounded operators on a Hilbert space H .

- *Multiplication is composition of operators,* $A, B \in \mathcal{B}(H)$ is their composition $A \circ B$.
- *The $*$ operator is given by the inner product $\langle \cdot, \cdot \rangle$ on H : $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x, y \in H$.*
- $\|\cdot\|$ is the operator norm: $\|f\| = \sup_{\|x\| \leq 1} \|Ax\|$, where the norm on H is the induced norm from $\langle \cdot, \cdot \rangle$.

Then $\mathcal{B}(H)$ is a C^* -algebra .

Proposition 2.4.1. [9] If $\mathcal{A}$ is a C^* -algebra and $a \in \mathcal{A}$, then

$$\|a^*\| = \|a\|.$$

Proof. We have $\mathcal{A}$, a C^* -algebra. By definition, a C^* -algebra satisfies the key condition $\|a^*a\| = \|a\|^2$, implies the following:

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\| \Rightarrow \|a\| \leq \|a^*\|.$$

$$\|a^*\|^2 = \|(a^*)^*a^*\| \leq \|a\|\|a^*\| \Rightarrow \|a^*\| \leq \|a\|.$$

Which means: $\|a^*\| = \|a\|$. □

Proposition 2.4.2. [13] If $\mathcal{A}$ is a C^* -algebra and $a \in \mathcal{A}$, then

$$\begin{aligned} \|a\| &= \sup\{\|ax\| : x \in \mathcal{A}, \|x\| \leq 1\} \\ &= \sup\{\|xa\| : x \in \mathcal{A}, \|x\| \leq 1\}. \end{aligned}$$

Proof. Proving $\|a\| = \sup \|ax\|$

Let $\alpha = \sup\{\|ax\| : x \in \mathcal{A}, \|x\| \leq 1\}$. This means α is the least upper bound of all values $\|ax\|$ where $\|x\| \leq 1$.

From the properties of the norm in any C^* -algebra, we know:

$$\|ax\| \leq \|a\| \cdot \|x\|.$$

If $\|x\| \leq 1$, then

$$\|ax\| \leq \|a\|.$$

Thus, $\alpha \leq \|a\|$ because α is the upper bound of $\|ax\|$.

Proving equality $\alpha = \|a\|$,

Choose the specific element $x = \frac{a^*}{\|a\|}$. we know $\|a^*\| = \|a\|$, so:

$$\|x\| = \left\| \frac{a^*}{\|a\|} \right\| = \frac{\|a^*\|}{\|a\|} = 1.$$

Now compute $\|ax\|$,

$$\|ax\| = \left\| a \cdot \frac{a^*}{\|a\|} \right\| = \frac{\|aa^*\|}{\|a\|}.$$

From the C^* -condition $\|aa^*\| = \|a\|^2$ (since $\|a\| = \|a^*\|$), we get:

$$\|ax\| = \frac{\|a\|^2}{\|a\|} = \|a\|.$$

Since α is the supremum of $\|ax\|$, and we found x such that $\|ax\| = \|a\|$, it follows that:

$$\alpha \geq \|a\|.$$

But we already proved $\alpha \leq \|a\|$, so

$$\alpha = \|a\|.$$

Proving $\|a\| = \sup \|xa\|$ The proof here is entirely analogous to Part 1, but using left multiplication instead of right multiplication. Define $\beta = \sup\{\|xa\| : x \in \mathcal{A}, \|x\| \leq 1\}$.

By the same logic, $\beta \leq \|a\|$ because $\|xa\| \leq \|x\|\|a\| \leq \|a\|$.

Again, choose $x = \frac{a^*}{\|a\|}$ and compute:

$$\|xa\| = \left\| \frac{a^*}{\|a\|} a \right\| = \frac{\|a^*a\|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|.$$

(Here we used $\|a^*a\| = \|a\|^2$ from the C^* -condition.)

Thus $\beta \geq \|a\|$, and therefore $\beta = \|a\|$. □

Definition 2.4.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^* -algebras. A map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is called a $*$ -algebra homomorphism if φ is an algebra homomorphism satisfying:

$$\varphi(a^*) = \varphi(a)^*, \quad \forall a \in \mathcal{A}.$$

A $*$ -isomorphism between C^* -algebras is a bijective $*$ -homomorphism.

Theorem 2.4.1. [14] A $*$ -homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ from a Banach $*$ -algebra $\mathcal{A}$ to C^* -algebra $\mathcal{B}$ is necessarily norm-decreasing, such that:

$$\|\varphi(a)\| \leq \|a\|.$$

for all $a \in \mathcal{A}$.

Proof. Without loss of generality, we may assume that $\mathcal{A}$, $\mathcal{B}$, and φ are unital (otherwise, we can pass to their unitizations $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, and $\tilde{\varphi}$). For any $a \in \mathcal{A}$, the spectrum satisfies $\sigma(\varphi(a)) \subseteq \sigma(a)$. Then

$$\|\varphi(a)\|^2 = \|\varphi(a)^* \varphi(a)\| = \|\varphi(a^* a)\| = r(\varphi(a^* a)) \leq r(a^* a) \leq \|a^* a\| \leq \|a\|^2.$$

Here, $r(\cdot)$ denotes the spectral radius. Taking square roots yields $\|\varphi(a)\| \leq \|a\|$, as required. $\square$

Studing some classes of operators in C^* -algebras

In this chapter, we focus on analyzing important categories of these operators such as normal, positive and self-adjoint operators which possess distinct algebraic and spectral properties. We will explore the relationships between these classes, their membership conditions in C^* -algebras .

3.1 Self-adjoint elements

Definition 3.1.1. *An element a of a C^* -algebra $\mathcal{A}$ is self-adjoint or hermitian if $a = a^*$.*

For each $a \in \mathcal{A}$ there exist unique hermitian elements $b, c \in \mathcal{A}$ such that $a = b + ic$ ($b = \frac{1}{2}(a + a^)$ and $c = \frac{1}{2i}(a - a^*)$).*

The set of hermitian elements of $\mathcal{A}$ is denoted by $\mathcal{A}_{sa}$ such that :

$$\mathcal{A}_{sa} = \{a \in \mathcal{A} \mid a = a^*\}.$$

Example 3.1.1. $M_n(\mathbb{C})$ is the C^* -algebra of $n \times n$ complex matrices with the adjoint operation $a^* = \bar{a}^T$ (conjugate transpose). We take

$$a = \begin{pmatrix} 1 & 2+i \\ 2-i & 3 \end{pmatrix}$$

$$a^* = \bar{a}^T = \begin{pmatrix} \bar{1} & \overline{2-i} \\ \overline{2+i} & \bar{3} \end{pmatrix} = \begin{pmatrix} 1 & 2+i \\ 2-i & 3 \end{pmatrix} = a.$$

Thus, a is self-adjoint.

Example 3.1.2. If $\mathcal{A} = C(X)$, the C^* -algebra of continuous complex-valued functions on a compact Hausdorff space X , then a function $f \in C(X)$ is self-adjoint if $\overline{f(x)} = f(x)$ for all $x \in X$ (i.e., f is real-valued).

We take $f(x) = x^2$ on $X = [0, 1]$ is self-adjoint.

Theorem 3.1.1. If a is self-adjoint element of a C^* -algebra $\mathcal{A}$ ($a \in \mathcal{A}_{sa}$), then

$$\|a\| = r(a).$$

Proof. Since $a^* = a$,

$$\|a^2\| = \|a^*a\| = \|a\|^2; \text{ by induction, } \|a^{2^n}\| = \|a\|^{2^n} \text{ for } n \geq 1.$$

$$\text{That is, } \|a^{2^n}\|^{1/2^n} = \|a\| \text{ for } n \geq 1.$$

Hence

$$r(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{1/2^n} = \|a\|$$

□

Theorem 3.1.2. If a is a hermitian element of a C^* -algebra $\mathcal{A}$, then

$$\sigma(a) \subseteq \mathbb{R}$$

Proof. By previous observation, $e^{ia} \in \text{Inv}(\mathcal{A})$ with

$$e^{-ia} = e^{(ia)^*} = (e^{ia})^* \text{ as its inverse}$$

$$\Rightarrow e^{ia} \text{ is a unitary hence } \sigma(e^{ia}) \subseteq \mathbb{T}$$

$$\text{Let } \lambda \in \sigma(a), \text{ and set } b = \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^{n-1}}{n!}$$

Then a, b commute, and moreover

$$e^{ia} - e^{i\lambda} = (e^{i(a-\lambda)} - 1_{\mathcal{A}})e^{i\lambda} = (a - \lambda)be^{i\lambda}.$$

$$\Rightarrow e^{ia} \in \sigma(e^{ia}) \text{ so } |e^{ia}| = 1_{\mathcal{A}} \text{ which means we must have } \lambda \in \mathbb{R}. \text{ Thus, } \sigma(a) \subseteq \mathbb{R}.$$

□

Corollary 3.1.1. If $a \in \mathcal{A}$, then $a^*a \in \mathcal{A}_{sa}$ so

$$\sigma(a^*a) \subseteq \mathbb{R}$$

Theorem 3.1.3. [13] *Let $\mathcal{B}$ be a C^* -subalgebra of a unital C^* -algebra $\mathcal{A}$ containing the unit of $\mathcal{A}$. Then*

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b) \quad (b \in \mathcal{B}).$$

Proof. First suppose that b is a hermitian element of $\mathcal{B}$. Since in this case $\sigma_{\mathcal{A}}(b)$ is contained in $\mathbb{R}$, it has no holes, and therefore by Theorem (3.1.2), $\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b)$.

Therefore, b is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{A}$.

Now suppose that b is an arbitrary element of $\mathcal{B}$, that is invertible in $\mathcal{A}$, so there is an element $a \in \mathcal{A}$ such that $ba = ab = 1_{\mathcal{A}}$.

Then $a^*b^* = b^*a^* = 1_{\mathcal{A}}$, so $bb^*a^*a = 1_{\mathcal{A}} \Rightarrow bb^*$ is invertible in $\mathcal{A}$ and therefore in $\mathcal{B}$.

Hence, there is an element $c \in \mathcal{B}$ such that $bb^*c = 1_{\mathcal{A}}$. Consequently, $b^*c = a$, so $a \in \mathcal{B}$, which implies that b is invertible in $\mathcal{B}$.

Thus, for any element of $\mathcal{B}$, its invertibility in $\mathcal{A}$ is equivalent to its invertibility in $\mathcal{B}$. $\square$

Proposition 3.1.1. [1] *If x and y are self-adjoint elements of $\mathcal{A}$ with $\sigma(x) \in \mathbb{R}^+$ and $\sigma(y) \in \mathbb{R}^+$, then $\sigma(x + y) \in \mathbb{R}^+$.*

Proof. By multiplying the sum $x + y$ by a sufficiently small positive scalar, if necessary, we may assume $\|x\| \leq 1$ and $\|y\| \leq 1$. Now since the spectral radius does not exceed the norm, the spectrum of x must be contained in the real interval $[0, +1]$. Thus, $\sigma(1_{\mathcal{A}} - x) \subseteq [0, +1]$. Since $1_{\mathcal{A}} - x$ is self-adjoint, its norm agrees with its spectral radius, and we have $\|1_{\mathcal{A}} - x\| = r(1_{\mathcal{A}} - x) \leq 1$. Similarly, $\|1_{\mathcal{A}} - y\| \leq 1$. It follows from the triangle inequality that $1_{\mathcal{A}} - \frac{1}{2}(x + y) = \frac{1}{2}(1_{\mathcal{A}} - x) + \frac{1}{2}(1_{\mathcal{A}} - y)$ has norm at most 1, and so the spectrum of the self-adjoint element $\frac{1}{2}(x + y)$ must lie in the interval $[0, +2]$. The desired conclusion is now immediate. $\square$

Proposition 3.1.2. [8] *Let $a = a^*$ in a C^* -algebra $\mathcal{A}$. Then there is a unique self-adjoint $b \in \mathcal{A}$ with $b^3 = a$.*

Proof. We can take $b = f(a)$, where $f(t) = t^{1/3}$. For uniqueness, suppose $c = c^*$ and $c^3 = a$. Then c commutes with a , and hence with b which is a limit of polynomials in a . So $C^*(b, c)$ is commutative, and contains a . But a self-adjoint element of a commutative C^* -algebra obviously has a unique self-adjoint cube root.

The same is true for n 'th roots, for any odd n . For an analogous statement for even roots. $\square$

Theorem 3.1.4. [1] *Let $\mathcal{A}$ be a self-adjoint algebra of operators, and let $\mathcal{A}_s$ be the closure of $\mathcal{A}$ in the strong operator topology. Then every self-adjoint element in the unit ball of $\mathcal{A}_s$ can be strongly approximated by self-adjoint elements in the unit ball of $\mathcal{A}$.*

Proof. First, observe that every self-adjoint element in the unit ball of the norm closure of $\mathcal{A}$ can be norm-approximated by self-adjoint elements in the unit ball of $\mathcal{A}$. Thus, we may assume $\mathcal{A}$ is norm closed.

The $*$ -operation is not strongly continuous, so we cannot directly conclude that the strong closure of the convex set $\mathcal{P}$ of self-adjoint elements of $\mathcal{A}$ contains $\{a \in \mathcal{A}_s : a = a^*\}$. However, its weak closure does (since if a net b_n converges strongly to $a = a^*$, the real parts of b_n converge weakly to a). Moreover, because the weak and strong operator topologies have the same continuous linear functionals, they must also have the same closed convex sets. This shows that the strong closure of $\mathcal{P}$ contains the self-adjoint elements of $\mathcal{A}_s$.

Now, consider the continuous functions $f : \mathbb{R} \rightarrow [-1, +1]$ and $g : [-1, +1] \rightarrow \mathbb{R}$ defined by $f(x) = 2x(1 + x^2)^{-1}$ and $g(y) = y(1 + \sqrt{1 - y^2})^{-1}$. These satisfy $f \circ g(y) = y$ for all $y \in [-1, +1]$, and $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. We claim that the map $b \mapsto f(b)$ is strongly continuous on the set of all self-adjoint operators on $\mathcal{H}$. Assuming this for the moment. Indeed, if $a = a^* \in \mathcal{A}_s$ with $a \leq 1$, then $b_0 = g(a)$ is a self-adjoint element of $\mathcal{A}_s$. By the preceding argument, there exists a net b_n of self-adjoint elements of $\mathcal{A}$ converging strongly to b_0 . Hence, $f(b_n) \rightarrow f(b_0)$ strongly. Each $f(b_n)$ is self-adjoint, belongs to $\mathcal{A}$ (since $\mathcal{A}$ is norm closed), and has norm ≤ 1 because $|f| \leq 1$ on $\mathbb{R}$. Since $f \circ g(y) = y$ on $[-1, 1]$, we have $f(b_0) = f \circ g(a) = a$ (as $a \leq 1$), proving that a is the strong limit of self-adjoint elements in the unit ball of $\mathcal{A}$.

The strong continuity of $f(b) = 2b(I + b^2)^{-1}$ follows from the operator identity:

$$2[f(b) - f(b_0)] = 4(1 + b^2)^{-1}(b - b_0)(1 + b_0^2)^{-1} - f(b)(b - b_0)f(b_0),$$

by considering b tending strongly to b_0 . $\square$

Corollary 3.1.2. [1] *Let $\mathcal{A}$ be a self-adjoint algebra of operators on a separable Hilbert space*

$\mathcal{H}$. Then, for every operator a in the strong closure of $\mathcal{A}$, there exists a sequence $a_n \in \mathcal{A}$ such that $a_n \rightarrow a$ in the strong operator topology.

Proof. We may assume $a \leq 1$. By considering the real and imaginary parts of a separately, we may also assume $a = a^*$. Let $\xi_1, \xi_2, \dots$ be a countable dense set in $\mathcal{H}$. For each $n \geq 1$, there exists a self-adjoint element $a_n \in \mathcal{A}$ with $a_n \leq 1$ such that $a_n \xi_k - a \xi_k < 1/n$ for $k = 1, 2, \dots, n$. Thus, $a_n \rightarrow a$ strongly on the dense set $\{\xi_1, \xi_2, \dots\}$, and since $a_n \leq 1$, the corollary follows. $\square$

3.2 Positive elements

Definition 3.2.1. An element a of a C^* -algebra $\mathcal{A}$ is positive if $a = b^*b$, for some b in $\mathcal{A}$.

We write $a \geq 0$ to mean that a is positive

The set of positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$

Definition 3.2.2. [17] If $\mathcal{A}$ is a C^* -algebra and $a \in \mathcal{A}$, then a is positive if a is hermitian and $\sigma(a) \subseteq [0, \infty)$. For two self-adjoint elements $x, y \in \mathcal{A}$, we say $y \geq x$ if $y - x \geq 0$.

Remark 3.2.1. We have already proved that if a is self-adjoint, then $\sigma(a) \subset \mathbb{R}$. Thus, for a self-adjoint element $a \in \mathcal{A}$, a^2 is positive and we set, $|a| = (a^2)^{1/2}$, $a^+ = \frac{1}{2}(|a| + a)$ and $a^- = \frac{1}{2}(|a| - a)$.

Example 3.2.1. If $\mathcal{A} = C(X)$, then f is positive in $\mathcal{A}$ if and only if $f(x) \geq 0$ for all $x \in X$.

Example 3.2.2. An operator $T \in \mathcal{B}(\mathcal{H})$ is positive if: $T^* = T$,

$\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$.

Let $\mathcal{H} = \mathbb{C}^2$, and define:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$T^* = T$,

For any $\xi = (x, y)$, $\langle T\xi, \xi \rangle = |x|^2 + 2|y|^2 \geq 0$.

Proposition 3.2.1. *Let $\mathcal{A}$ be a C^* -algebra*

Then every $a \in \mathcal{A}^+$ has a unique positive square-root such that

$$a = b^2, b \in \mathcal{A}^+$$

Proof. Given that $a \geq 0$, we wish to prove that there is a unique positive square-root of a in $\mathcal{A}$

Let $f(x) = \sqrt{x}$, $x \in \sigma(a)$; since a is positive, we have that f is well-defined, continuous function.

Thus, $f(a) = \sqrt{a}$ is self-adjoint because f is real-valued and $\sigma(\sqrt{a}) = f(\sigma(a)) \subseteq [0, \infty)$. Thus, $\sqrt{a}$ is positive.

$$\text{Moreover, } (\sqrt{a})^2 = f^2(a) = 1_{\mathcal{A}}(a) = a.$$

Suppose that c is another positive square-root of a . Then the functional calculus implies that

$$c = f(c^2) = f(a) = \sqrt{a}.$$

□

Proposition 3.2.2. *If a is self-adjoint element, then there are unique positive elements a^+ and a^- in $\mathcal{A}$ such that $a = a^+ - a^-$ and $a^+a^- = 0$.*

Proof. Given $a = a^* \in \mathcal{A}$, we wish to prove that there are $a^+, a^- \geq 0$ such that $a = a^+ - a^-$ and $a^+a^- = 0$.

Let

$$f(x) = \frac{x + |x|}{2} \quad \text{and} \quad g(x) = \frac{-x + |x|}{2}, \quad x \in \sigma(a).$$

Then it's clear that $f, g \geq 0$, $fg = 0$ and that $f - g = id$. Using the functional calculus, we set $a^+ = f(a)$ and $a^- = g(a)$; then

$$a^+a^- = f(a)g(a) = (fg)(a) = 0.$$

□

Lemma 3.2.1. [8] *For a self-adjoint element $a \in \mathcal{A}$, the following are equivalent:*

$$1 \ a \geq 0;$$

2 $a = b^2$ for some $b = b^*$;

3 $\|c1_{\mathcal{A}} - a\| \leq c$ for all $c \geq \|a\|$;

4 $\|c1_{\mathcal{A}} - a\| \leq c$ for some $c \geq \|a\|$.

Proof. Note that the implication $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$

- $1 \Rightarrow 2$ has already been demonstrated(3.2.1) .

- $2 \Rightarrow 3$

Assuming 2, i.e, that $a = b^2$, then $a = f(b)$ where $f(x) = x^2, x \in \sigma(b)$. As a result,

$$\|f\|_{\sigma(b)} = \|a\|.$$

implying that $0 \leq f \leq \|a\| \leq c$. Therefore $0 \leq c - f \leq c$. Thus,

$$\|c1_{\mathcal{A}} - a\| = \|(c - f)(b)\| = \|c - f\|_{\sigma(b)} \leq c.$$

Thus, $2 \Rightarrow 3 \Rightarrow 4$, where the last implication is trivial.

- $4 \Rightarrow 1$

Assuming that 4 holds for a praticular c , we have that

$$c \geq \|c1_{\mathcal{A}} - a\| = \|(c - 1_{\mathcal{A}})a\| = \|c - 1_{\mathcal{A}}\|_{\sigma(a)}.$$

This ipmlies that the identity function is non-negative on $\sigma(a)$, wich is equivalent to stating that $\sigma(a) \subseteq [0, \infty)$. Thus, $4 \Rightarrow 1$ and the proof copmlete.

□

Lemma 3.2.2. *Let $a, b \in \mathcal{A}$ be positive. Then, $a + b$ is positive.*

Proof. We can assume that $\mathcal{A}$ is unital. Clearly, $a + b$ is self-adjoint and hence $\sigma(a + b) \subset \mathbb{R}$.

Note that c is positive if and only if tc is positive for some (and every) $t > 0$.

Thus, by scaling, we can assume that $\|a\|, \|b\| \leq 1$. This implies that $\sigma(a)$ and $\sigma(b)$ are contained in $[0, 1]$.

Consequently, $\sigma(1_{\mathcal{A}} - a)$ and $\sigma(1_{\mathcal{A}} - b)$ are contained in $[0, 1]$. Therefore, $\|1_{\mathcal{A}} - a\| \leq 1$ and $\|1_{\mathcal{A}} - b\| \leq 1$. Calculate as follows to observe that

$$\begin{aligned} \left\|1_{\mathcal{A}} - \frac{a+b}{2}\right\| &= \frac{1}{2}\|(1_{\mathcal{A}} - a) + (1_{\mathcal{A}} - b)\| \\ &\leq \frac{1}{2}(\|1_{\mathcal{A}} - a\| + \|1_{\mathcal{A}} - b\|) \\ &\leq 1. \end{aligned}$$

Thus, if $\lambda \in \sigma(\frac{a+b}{2})$, then $|1_{\mathcal{A}} - \lambda| \leq 1$, i.e. $\lambda \in [0, 2]$. Hence, $\lambda \in \sigma(\frac{a+b}{2}) \subset [0, \infty)$. This implies $a + b$ is positive. $\square$

Corollary 3.2.1. [17] *The order $\leq$ is a partial order on the set of self-adjoint elements of $\mathcal{A}$.*

Lemma 3.2.3. [3] *Let $x, y \in \mathcal{A}$ be given. Then, $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$*

Proof. We will prove that $1_{\mathcal{A}} - xy$ is invertible if and only if $1_{\mathcal{A}} - yx$ is invertible.

Suppose that $1_{\mathcal{A}} - xy$ is invertible with inverse v . Set $u = 1_{\mathcal{A}} + yvx$. Calculate as follows to observe that

$$\begin{aligned} (1_{\mathcal{A}} - yx)u &= 1_{\mathcal{A}} + yvx - yx - y(xy)vx \\ &= 1_{\mathcal{A}} + yvx - yx + y(1_{\mathcal{A}} - xy - 1_{\mathcal{A}})vx \\ &= 1_{\mathcal{A}} + y(1_{\mathcal{A}} - xy)vx - yx \\ &= 1_{\mathcal{A}}. \end{aligned}$$

A similar calculation yields $u(1_{\mathcal{A}} - yx) = 1_{\mathcal{A}}$. Hence, u is the inverse of $1_{\mathcal{A}} - yx$. We can interchange the roles of x and y to complete the proof. $\square$

Lemma 3.2.4. [8] *Let $a \in \mathcal{A}$ be such that $\sigma(a^*a) \subset (-\infty, 0]$. Then, $a = 0$.*

Proof. Write $a = b + ic$ with b, c self-adjoint. Then, $a^*a + aa^* = b^2 + c^2$, we have $aa^* \leq 0$. and we have $a^*a + aa^* = b^2 + c^2 \leq 0$. But $b^2, c^2 \geq 0$. Consequently, $b^2 + c^2 \geq 0$ and we deduce $b^2 + c^2 = 0$, i.e $b^2 = c^2$. Now, $c^2 \leq 0$, thus $0 \geq b^2 \geq 0$. Hence, $b^2 = c^2 = 0$. As b and c are self-adjoint, we have $b = c = 0$ which implies $a = 0$. $\square$

Theorem 3.2.5. *Let $a \in \mathcal{A}$. Then $a^*a \geq 0$.*

Proof. Let $b = a^*a$. Then b is self-adjoint and we can use the (Hahn) decomposition to write $b = b^+ - b^-$ where $b^+, b^- \geq 0$ and $b^+b^- = 0$.

Let $c = \sqrt{b^-}$ and set $t = ac$. Note that since c is a limit of polynomials in b^- with zero constant coefficient, we have that $cb^+ = 0$. Then

$$-t^*t = -ca^*ac = -c(b^+ - b^-)c = cb^-c = (b^-)^2.$$

In particular, $-t^*t$ is positive, and thus $\sigma(-t^*t) \subseteq [0, \infty)$. The next phase of the proof involves showing that t^*t is also positive.

Write $t = x + iy$, where x and y are the real and imaginary parts of t , i.e.,

$$x = \frac{t + t^*}{2} \text{ and } y = \frac{t - t^*}{2i}.$$

Then

$$t^*t + tt^* = (x + iy)^*(x + iy) + (x + iy)(x + iy)^* = 2(x^2 + y^2).$$

This is a sum of positive elements and thus it is itself positive.

$$tt^* = (t^*t + tt^*) - t^*t = (t^*t + tt^*) + (b^-)^2$$

is a sum of positive elements, and hence is positive. And since $\sigma(tt^*) \cup \{0\} = \sigma(t^*t) \cup \{0\}$, this implies that $t^*t \geq 0$.

We have just proven that t^*t and $-t^*t$ are positive, which implies that $\sigma(t^*t) = 0$. Since t^*t is self-adjoint, its norm equals its spectral radius; hence $(b^-)^2 = -t^*t = 0$. As b^- is self-adjoint, it follows that $b^- = 0$. Thus, $b = a^*a = b^+ - b^- = b^+ \geq 0$. $\square$

Proposition 3.2.3. [9] *If $\mathcal{A}$ is a C^* -algebra, then $\mathcal{A}^+$ is a closed cone.*

Proof. Let $\{a_n\}$ be a sequence of positive elements and suppose $\|a_n - a\| \rightarrow 0$. Clearly $a \in \mathcal{A}_{sa}$. By Lemma(3.2.1) $\|a_n - \|a_n\|\| \leq \|a_n\|$ for all $n \geq 1$. Taking limits shows that $\|a - \|a\|\| \leq \|a\|$, $a \geq 0$

$\alpha a \geq 0$ when $a \in \mathcal{A}^+$ and $\alpha \in \mathbb{R}^+$. Now let $a, b \in \mathcal{A}^+$. To show that $a + b \in \mathcal{A}^+$. It suffices to assume that $\|a\|, \|b\| \leq 1$. But $\|1_{\mathcal{A}} - \frac{1}{2}(a + b)\| = \frac{1}{2}\|(1_{\mathcal{A}} - a) + (1_{\mathcal{A}} - b)\| \leq 1$. By (3.2.1), $\frac{1}{2}(a + b)$, so that $a + b \in \mathcal{A}^+$. $\square$

Proposition 3.2.4. [9] *If $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$, then A is positive if and only if $\langle Ah, h \rangle \geq 0$ for every vector h .*

Proof. if $A \geq 0$, then $A = T^*T$ for some T in $\mathcal{B}(\mathcal{H})$. Hence $\langle Ah, h \rangle = \|Th\|^2 \geq 0$. Conversely, suppose $\langle Ah, h \rangle \geq 0$ for all h in $\mathcal{H}$. $A = A^*$. If $\lambda < 0$ and $h \in \mathcal{H}$, the fact that $\langle Ah, h \rangle \geq 0$ implies

$$\begin{aligned} \|(A - \lambda)h\|^2 &= \|Ah\|^2 - 2\lambda\langle Ah, h \rangle + \lambda^2\|h\|^2 \\ &\geq -2\lambda\langle Ah, h \rangle + \lambda^2\|h\|^2 \geq \lambda^2\|h\|^2 \end{aligned}$$

Thus λ not in $\sigma_{ap}(A)$, the approximate point spectrum of A . This says that $A - \lambda$ is left invertible. Since $A - \lambda$ is hermitian, it must also be right invertible. That is, λ not in $\sigma(A)$ and so $A \geq 0$. $\square$

Theorem 3.2.6. [13] *If a, b are positive elements of a C^* -algebra $\mathcal{A}$, then the inequality $a \leq b$ implies the inequality $a^{1/2} \leq b^{1/2}$.*

Proof. We show that $a^2 \leq b^2$ implies $a \leq b$, which will prove the theorem. We may assume that $\mathcal{A}$ is unital. Let $t > 0$, and let c, d be the real and imaginary Hermitian parts of the element $(t + b + a)(t + b - a)$. Then:

$$\begin{aligned} c &= \frac{1}{2} ((t + b + a)(t + b - a) + (t + b - a)(t + b + a)) \\ &= t^2 + 2tb + b^2 - a^2 \\ &\geq t^2. \end{aligned}$$

Consequently, c is both invertible and positive. Since $1 + ic^{-1/2}dc^{-1/2} = c^{-1/2}(c + id)c^{-1/2}$ is invertible, it follows that $c + id$ is invertible. Therefore, $t + b - a$ is left invertible and hence invertible because it is Hermitian. Thus, $-t \notin \sigma(b - a)$. This implies $\sigma(b - a) \subseteq \mathbb{R}^+$, so $b - a$ is positive, meaning $a \leq b$. $\square$

Lemma 3.2.7. *Assume $\mathcal{A}$ has a unit and let x, y be invertible positive elements of $\mathcal{A}$ satisfying $x \leq y$. Then $y^{-1} \leq x^{-1}$.*

Proof. Notice first that for every $z \in \mathcal{A}$ the condition $z^*z \leq 1_{\mathcal{A}}$ ($1_{\mathcal{A}}$ denoting the unit of $\mathcal{A}$) is equivalent to $\|z\| \leq 1$. For example, if $\|z\| \leq 1$ then $\|z^*z\| \leq 1$, and we see that $\sigma(z^*z)$ is contained in the unit interval $[0, 1]$. Hence $\sigma(1_{\mathcal{A}} - z^*z) \geq 0$. The converse is similar, and we omit the proof.

Since both x and y are positive, we may find positive square roots for each of them (using the functional calculus), which are denoted by $x^{1/2}$ and $y^{1/2}$. Now multiply $x \leq y$ on left and right by $y^{-1/2}$ to obtain $y^{-1/2}xy^{-1/2} \leq 1_{\mathcal{A}}$. The preceding paragraph, applied to $z = x^{1/2}y^{-1/2}$, implies $\|x^{1/2}y^{-1/2}\| \leq 1$, hence $\|y^{-1/2}x^{1/2}\| \leq 1$ (by taking the adjoint), and so $zz^* \leq 1_{\mathcal{A}}$. That is simply $x^{1/2}y^{-1}x^{1/2} \leq 1_{\mathcal{A}}$, and now the conclusion $y^{-1} \leq x^{-1}$ follows after another double multiplication by $x^{-1/2}$. $\square$

3.3 Normal elements

Definition 3.3.1. Let $\mathcal{A}$ is a C^* -algebra and $a \in \mathcal{A}$ then

a is normal if $a^*a = aa^*$.

Remark 3.3.1. Note that nontrivial normal elements always exist in $\mathcal{A}$, as $b = a^*a$ is normal for any $a \in \mathcal{A}$.

The element $b = a^*a$ is not just a normal element. One quickly notes that b is self-adjoint. In general, every self-adjoint element is normal, but not every normal element is self-adjoint.

Example 3.3.1. In $\mathbb{C}^2$, take the matrix:

$$a = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$$

- Its Hermitian adjoint is:

$$a^* = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}^* = \begin{pmatrix} 2 & -i \\ i & 2 \end{pmatrix}^T = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$$

Note that $a = a^*$ in this case,

$$a^*a = aa = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4i \\ -4i & 5 \end{pmatrix}$$

$$aa^* = aa = \text{Same result as above}$$

Thus, a is normal.

Example 3.3.2. Let $\mathcal{H} = \ell^2(\mathbb{N})$ and define D by

$$D(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots),$$

where $\{\lambda_n\}$ is a bounded sequence. Then D^* is given by

$$D^*(x_1, x_2, \dots) = (\overline{\lambda_1} x_1, \overline{\lambda_2} x_2, \dots).$$

We have $DD^* = D^*D$ because

$$(DD^*)(x_n) = (|\lambda_n|^2 x_n) = (D^*D)(x_n).$$

Thus, D is normal.

Theorem 3.3.1. If a is a normal element of a C^* -algebra, then $r(a) = \|a\|$.

Proof.

$$\begin{aligned} r(a) &= \lim_{n \rightarrow \infty} \|a^{2^n}\|^{2^{-n}} = \lim_{n \rightarrow \infty} \|(a^*)^{2^n} a^{2^n}\|^{2^{-n-1}} \\ &= \lim_{n \rightarrow \infty} \|(a^* a)^{2^n}\|^{2^{-n-1}} = [r(a^* a)]^{1/2} = \|a^* a\|^{1/2} = \|a\|. \end{aligned}$$

□

Lemma 3.3.2. [13] A normal element a in a unital C^* -algebra $\mathcal{A}$ is self-adjoint if and only if spectrum is entirely real, i.e., $\sigma(a) \subseteq \mathbb{R}$.

Proof. By the isomorphisme, the function $f(b) = \bar{b}$ restricted to $\sigma(a)$ corresponds to a . The element a is self-adjoint if and only if $f_{\sigma(a)} = \bar{f}_{\sigma(a)}$, which is equivalent to $\sigma(a)$ being a subset of the real numbers.

$$\sigma(a) \subseteq \mathbb{R}.$$

□

Remark 3.3.2. For the normal element $a \in \mathcal{A}$, we denoted by $C^*(a)$ the C^* -subalgebra of $\mathcal{A}$ generated by a and a^* . Note that, since a is normal, $C^*(a)$ is a commutative C^* -algebra, although it needs not be unital.

If $\mathcal{A}$ is a C^* -algebra and a is a normal element of $\mathcal{A}$, then for every f in $C(\sigma(a))$,

$$\sigma(f(a)) = f(\sigma(a)).$$

Theorem 3.3.3 (Spectral Mapping). [13] *If $\mathcal{A}$ is a C^* -algebra and a is a normal element of $\mathcal{A}$, then for every f in $C(\sigma(a))$,*

$$\sigma(f(a)) = f(\sigma(a)).$$

Proof. The map $f \rightarrow f(a)$ is a $*$ -isomorphism from $C(\sigma(a))$ onto $C^*(a)$. Thus $\sigma(f(a)) = \sigma_{C(\sigma(a))}(f) = f(\sigma(a))$. $\square$

Theorem 3.3.4. *Let b and a be normal operators. Then b and a are algebraically equivalent if, and only if, they have the same spectrum.*

Proof. Assume first that $\sigma(b) = \sigma(a)$. Then by the above we have

$$f(b) = \sup\{|f(z)| : z \in \sigma(b)\} = f(a)$$

, for every continuous function f on $\sigma(b)$. This shows that the map $\phi : f(b) \rightarrow f(a)$, $f \in C(\sigma(b))$, is an isometric $*$ -isomorphism of $C^*(b)$ onto $C^*(a)$ which carries b to a .

Conversely, if such a ϕ exists, then the spectrum of b relative to $C^*(b)$ must equal the spectrum of $\phi(b) = a$ relative to $C^*(a)$. By the preceding remarks, this implies $\sigma(b) = \sigma(a)$. $\square$

Corollary 3.3.1. *Let $\mathcal{A}$ be a C^* -algebra, and x a normal element of $\mathcal{A}$. Then $C^*(x)$ is isometrically isomorphic to $C_0(\sigma_{\mathcal{A}}(x))$ under an isomorphism which sends x to the function $f(t) = t$.*

Proposition 3.3.1. [3] *Let $\mathcal{A}$ be a C^* -algebra and $x \in \mathcal{A}$ a normal element. Then*

1. *For any $f \in C_0(\sigma_{\mathcal{A}}(x))$, $\sigma_{\mathcal{A}}(f(x)) = \{f(\lambda) : \lambda \in \sigma_{\mathcal{A}}(x)\}$.*
2. *If $f \in C_0(\sigma_{\mathcal{A}}(x))$ and $g \in C_0(f(\sigma_{\mathcal{A}}(x))) = C_0(f(\sigma_{\mathcal{A}}(x)))$, so $g \circ f \in C_0(\sigma_{\mathcal{A}}(x))$, then $(g \circ f)(x) = g(f(x))$.*
3. *If $f_n, f \in C_0(\sigma_{\mathcal{A}}(x))$ and $f_n \rightarrow f$ uniformly on $\sigma_{\mathcal{A}}(x)$, then $f_n(x) \rightarrow f(x)$.*

4. If $\mathcal{B}$ is a C^* -algebra and $\phi : A \rightarrow B$ a $*$ -homomorphism, then $\phi(f(x)) = f(\phi(x))$ for any $f \in C_0(\sigma_{\mathcal{A}}(x))$.

Proposition 3.3.2. [3] *Let Y be a compact subset of $\mathbb{C}$, and (f_n) a sequence of elements of $C_0(Y)$ converging uniformly on Y to f . Let $\mathcal{A}$ be a C^* -algebra, (x_n) a sequence of normal elements of $\mathcal{A}$ with $x_n \rightarrow x \in \mathcal{A}$ and $\sigma_{\mathcal{A}}(x_n) \subseteq Y$ (so $\sigma_{\mathcal{A}}(x) \subseteq Y$). Then $f_n(x_n) \rightarrow f(x)$.*

Proof. let $\epsilon > 0$, and approximate f uniformly on Y within $\epsilon/4$ by a polynomial p ; then

$$\|f_n(x_n) - f(x)\| \leq$$

$$\|f_n(x_n) - f(x_n)\| + \|f(x_n) - p(x_n)\| + \|p(x_n) - p(x)\| + \|p(x) - f(x)\|$$

. If n is large enough that $|f_n - f|$ is uniformly less than $\epsilon/4$ on Y , the first term is less than $\epsilon/4$; the second and fourth terms are $< \epsilon/4$ by choice of p , and the third term goes to 0 as $n \rightarrow \infty$ by continuity of addition, multiplication, and involution. $\square$

Definition 3.3.2. *Let $\mathcal{A}$ be a C^* -algebra.*

1. *If $p \in \mathcal{A}$ st. $p = p^* = p^2$, we call p a projection.*
2. *If $u \in \mathcal{A}$ st. $u^*u = uu^* = 1_{\mathcal{A}}$, we call u a unitary.*

$$u(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = 1_{\mathcal{A}}\}.$$

Example 3.3.3. *The Function $f(z) = z$ in $C(\mathbb{T})$*

$$f : \mathbb{T} \rightarrow \mathbb{C}, \quad \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

The adjoint f^ is the complex conjugate:*

$$f^*(z) = \bar{z} = \frac{1}{z} \quad (\text{since } |z| = 1).$$

$$(f^*f)(z) = \bar{z}z = |z|^2 = 1, \quad (ff^*)(z) = z \left(\frac{1}{z} \right) = 1.$$

Example 3.3.4. *Matrix in $M_2(\mathbb{C})$*

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Self-adjointness: $P^* = P$.

Idempotence: $P^2 = P$:

$$P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P.$$

Thus, p is a projection.

Proposition 3.3.3. *If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$), then $\langle ux, uy \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$.*

Proof.

$$\langle ux, uy \rangle = \langle x, u^*uy \rangle = \langle x, y \rangle \quad \text{since } u^*u = 1_{\mathcal{A}}.$$

□

Theorem 3.3.5. *Let u be a unitary in a unital C^* -algebra $\mathcal{A}$. If $\sigma(u) \neq \mathbb{T}$, then there exists $a \in \mathcal{A}_{sa}$ such that $u = e^{ia}$.*

(If $\|1_{\mathcal{A}} - u\| < 2$, then $\sigma(u) \neq \mathbb{T}$.)

Proof. By replacing u by λu for some $\lambda \in \mathbb{T}$ if necessary, we may suppose that $-1 \notin \sigma(u)$. Since u is normal, we may also suppose that $\mathcal{A}$ is abelian (replacing $\mathcal{A}$ by the C^* -subalgebra generated by 1 and u if need be). Let $\varphi : \mathcal{A} \rightarrow C(\Omega)$ be the Gelfand representation, let $f = \varphi(u)$, and as usual denote by $\ln : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ the principal branch of the logarithm function. Then $g = \ln(f)$ is a well-defined element of $C_0(\Omega)$, and $e^g = f$. Since $|f(\omega)| = 1$ for all $\omega \in \Omega$, the real part of g vanishes, so $g = ih$ where $h = \bar{h} \in C_0(\Omega)$. Let $a = \varphi^{-1}(h)$. Then $a \in \mathcal{A}_{sa}$ and $u = e^{ia}$ because $\varphi(u) = e^{ih} = e^{\varphi(ia)} = \varphi(e^{ia})$.

The parenthetical observation in the statement of the theorem follows from the equations

$$\|A - u\| = r(1 - u) = \sup\{|1 - \lambda| \mid \lambda \in \sigma(u)\},$$

which imply that $-1 \notin \sigma(u)$ when $\|A - u\| < 2$. $\square$

Proposition 3.3.4. *Recall that when $\mathcal{A}$ is a unital Banach algebra we required that $\|1_{\mathcal{A}}\| = 1$.*

For a unital C^ -algebra, this is automatic.*

More generally if $p \in \mathcal{A}$ a non-zero projection or if $\mathcal{A}$ is unital and $u \in \mathcal{A}$ is a unitary.

$$\|p\| = \|p^2\| = \|p^*p\| = \|p\|^2 \Rightarrow \|p\| = 1_{\mathcal{A}}$$

$$1_{\mathcal{A}} = \|1_{\mathcal{A}}\| = \|u^*u\| = \|u\|^2 \Rightarrow \|u\| = 1_{\mathcal{A}}$$

Proposition 3.3.5. *$u \in \mathcal{A}$ a unitary*

If $\lambda \in \sigma(u) \Rightarrow |\lambda| \leq 1$

$u^ = u^{-1}$, is also a unitary*

$$\begin{aligned} \lambda^{-1} \in \sigma(u^*) &\Rightarrow \frac{1}{|\lambda|} \leq 1, |\lambda| = 1, \\ &\Rightarrow \sigma(u) \subseteq \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}. \end{aligned}$$

Proposition 3.3.6. *If $p \in \mathcal{A}$ is a projection, then $\sigma(p) \subseteq \{0, 1\}$.*

Proof. We know $p^2 = p$, so p satisfies the equation $\lambda^2 = \lambda$.

The solutions to this equation are $\lambda = 0$ or $\lambda = 1$.

If $\lambda \notin \{0, 1\}$, then $(p - \lambda 1_{\mathcal{A}})^{-1} = \frac{1}{\lambda(1-\lambda)}p - \frac{1}{\lambda}1_{\mathcal{A}}$.

Hence, $\sigma(p) \subseteq \{0, 1\}$. $\square$

Proposition 3.3.7. *If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then every projection p corresponds to a closed subspace $\mathcal{K} = \text{Im}(p) \subseteq \mathcal{H}$.*

Proof. $\text{Im}(p)$ is closed because p is bounded.

p is an orthogonal projection since $p = p^*$.

For any $x \in \mathcal{H}$, $x = px + (1 - p)x$, where $px \in \text{Im}(p)$ and $(1 - p)x \in \text{Ker}(p)$. $\square$

Corollary 3.3.2. [3] *Let $\mathcal{A}$ be a C^* -algebra, and x a normal element of $\mathcal{A}$. Then*

x is self-adjoint if and only if $\sigma_{\mathcal{A}}(x) \subseteq \mathbb{R}$.

x is unitary if and only if $\sigma_{\mathcal{A}}(x) \subseteq \{\lambda : |\lambda| = 1\}$.

x is a projection if and only if $\sigma_{\mathcal{A}}(x) \subseteq \{0, 1\}$.

Conclusion

In conclusion, this memory highlights the importance of studying special classes of operators in C^* -algebras, such as self-adjoint, positive and normal operators, in understanding the algebraic and spectral structure of these algebras. Through this work, we have reviewed the fundamental properties of these operators and their applications in various fields, including quantum mechanics and non-commutative geometry. We have also emphasized the significance of spectral theory in analyzing these operators and how it can be utilized to achieve profound results in both theory and practical applications.

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المخلص

تهدف هذه المذكرة الى دراسة بعض الاصناف الخاصة من المؤثرات في C^* -Algebras مع التركيز على المؤثرات الذاتية المترافقة، الموجبة والمؤثرات الطبيعية وإعطاء لمحة عامة حول المفاهيم الأساسية لدراسة هذه الأصناف مثل: الفضاءات البنائية والهيلبرتية، بما في ذلك تحليل مفصلا لـ C^* -Algebras كما تم تقديم نتائج مهمة في نظرية التمثيل الطيفي للمؤثرات الطبيعية وعلاقة المؤثرات الذاتية المترافقة بالخواص الطيفية مع ابراز التطبيقات المحتملة في مجالات مثل: ميكانيك الكم والهندسة اللاتبادلية.

الكلمات المفتاحية: C^* -Algebras، الذاتية المترافقة، المؤثرات الموجبة، المؤثرات الطبيعية، النظرية الطيفية، نظرية المؤثرات، فضاء هيلبرت، فضاء بناخ.

Abstract

This memory aims to study some special classes of operators in C^* -algebras, with emphasis on self-adjoint, positive and normal operators. It gives an overview of the basic concepts for studying these classes, such as Structural and Hilbertian spaces including a detailed analysis of C^* -algebras. Important results are also presented in the theory of spectral representation of natural operators and the relationship of conjugate eigenoperators to spectral properties, highlighting potential applications in fields such as quantum mechanics and non-commutative geometry.

Keywords : C^* -Algebras, Self-adjoint, Positive operators, Normal operators, Spectral theory, Operators theory, Hilbert space, Banach space.

Résumé

Cette mémoire vise à étudier certaines classes spéciales d'opérateurs dans les C^* -algèbres en se concentrant sur les opérateurs auto-adjoints, positives les opérateurs normaux, et à donne un aperçu général des concepts fondamentaux pour l'étude de ces catégories, tels que les espaces de Banach et de Hilbert, cela inclut une analyse détaillée des C^* -algèbres. De plus, des résultats importants de la théorie de la représentation spectrale des opérateurs normaux sont présentés, ainsi que la relation entre les opérateurs auto-adjoints et les propriétés spectrales, tout en mettant en lumière les les applications potentielles dans des domaines tels que la mécanique quantique et la géométrie non commutative.

Mots clés: C^* -Algèbres, auto-adjoints, Opérateurs positives, Opérateurs normaux, Théorie spectrale, Théorie des Opérateurs, Espace de Hilbert, Espace de Banach.