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Atangana-Baleanu Fractional Operator and Applications

Submitted by : Hassan Meriem , Maamra Chaima
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Before the jury:

Dr.GHENDIR AOUN ABDELLATIF	**CMA**	**President**	**Univ. of El-Oued**
Dr.BEKKAR MENECEUR	**CMB**	**Reporter**	**Univ. of El-Oued**
Dr.BELOUL SAID	**CMA**	**Examiner**	**Univ. of El-Oued**

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Contents

General Notations

$\mathbb{N}$	Set of natural numbers .
$\mathbb{R}$	Set of real numbers.
$\mathbb{C}$	Set of complex numbers .
J	Interval of $\mathbb{R}$
$\Re e$	Real part.
Γ	Gamma function.
β	Beta function.
$E_(\alpha, \beta)$	Two-parameter Mittag-Leffler function.
E_α	one-parameter Mittag-Leffler function.
$\pounds$	Laplace Transforms.
$\pounds^{-}1$	Inverse Laplace Transforms.
s	Parameter in the Laplace transformation.
${}^{ABR}_{\alpha}D^{\alpha}_{x}$	Atangana-Baleanu Riemann-Liouville fractional derivative.
${}^{ABC}_{\alpha}D^{\alpha}_{x}$	Atangana-Baleanu Caputo fractional derivative.
${}^{AB}_{\alpha}I^{\alpha}_{t}$	Atangana-Baleanu fractional integral.
${}^{AB}_{\alpha}D^{\alpha}_{x}$	Atangana-Baleanu fractional derivative.

Introduction

Fractional derivatives calculus has gained much interest by the many researcher in the last decades and it has strong mathematical background and many papers are attributed to the development of it.Among them , we can cite some for example, [17 ,22].Fractional calculus has been also used for modeling physical phenomena including control systems, mechanics and viscoelasticity. Many researchers have introduced new definitions of the concept of derivative with fractions Request. These definitions pass from Riemann-Liouville ,Caputo and Caputo-Fabrizio to the newly proposed definition of Atangana-Baleanu . Older versions of the designed definition of the fractional derivative are a product From the convolution is derived from the function $f(t)$ with kernel $k(t,s)=\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}$ in the case of Caputo old edition between $f(t)$ and the kernel in the case of Riemann-Liouville .Both the Riemann-Liouville and Caputo old version are designed with singular kernel.
Recently , Atangana-Baleanu suggested a much better version of a derivative with non singular kernel that satisfy the issues pointed .

This work is divided into five chapter :
■ In the First chapter, we will provide some definitions and theorems that we will use in this note.
■ In the Second chapter, we will present some definitions and basic properties of fractional derivatives, Riemann-Liouville fractional derivative, Caputo fractional derivative and Caputo-Fabrizio fractional derivative .
■ In the Third chapter, we will mainly introduces definitions and basic properties of Atangana–Baleanu fractional derivative and Atangana-Baleanu fractional integral . The subject of this chapter is taken from the article [1 ,2].
■ In the Fourth chapter, we will review the existence and and uniqueness of solutions of some fractional differential equations by applying the Laplace Effect to a definition. The result presented in this chapter are from the article [4, 5, 10]
■ In the Fifth chapter, we will study the existence and uniqueness of solutions to a nonlocal implicit problem with A-B fractional derivative of the form : [14]

$$^{ABC}D^{\alpha}_{0,t}u(t)=f(t,u(t),{}^{ABC}D^{\alpha}_{0,t}u(t)),t\in[0,\chi],0<\alpha\leq 1$$

$$\sum_{k=1}^{m} \beta_k u(\tau_k) = u_0, \tau_k \in (0, \chi),$$

The subject of this chapter is taken from the article [15].

Symbols

1) AB : Atangana-Baleanu

2) ABC : Atangana-Baleanu in Caputo sense

3) ABR : Atangana-Baleanu in Riemann-Liouville sense

4) ML : Mittag-Leffler function

5) FD : Fractional Derivative

6) CF : Caputo-Fabrizio

Chapter 1

Preliminaries

1.1 Some Results from functional analysis

1.1.1 Spaces Absolutely Continuous and Continuous Functions

Definition 1.1. [8]
Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R}; f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p d\mu \right]^{\frac{1}{p}}$$

Definition 1.2. [8]
We set

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \begin{array}{l} \text{f is measurable and there is a constant } C \\ \text{ suth that } |f| \leq C \text{ a.e. on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C|f(x)| \leq C \text{ a.e. on } \Omega\}.$$

Definition 1.3. [17]
Let [a,b] a finite interval. We denote by AC[a,b] the space of primitive functions of integrable
functions in the sense of Lebesgue

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt, \varphi(t) \in L[a,b].$$

and we call AC[a,b] the space of absolutely continuous functions on [a,b].

1.1.2 Sopolev spaces

Consider an open subset Ω of $\mathbb{R}^{\mathbb{N}}$, for all $\varphi \in D(\Omega)$. In that case it is well known that f is unique.

Definition 1.4. [26]
Let $m \in \mathbb{N}$ and let $p \in [1, \infty]$. Define

$$W^{m,p}(\Omega) = \left\{ f \in L^p(\Omega) \backslash D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^N \text{ such that } |\alpha| \leq m \right\}.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm

$$\|f\|_{W_{m_1 P(\Omega)}} = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p}$$

If $p = 2$ one sets $W^{m,2}(\Omega) = H^m(\Omega), W^{m,2}(\Omega) = H_0^m$, then $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u . D^\alpha v dx.$$

And it is equipped with the following notm :

$$\|f\|_{H^m} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Remark 1.1.

for all $u, v \in H^m(\Omega)$ it may be more convenient to equip $H_0^1(\Omega)$ with the following scalar product:

$$\langle u, v \rangle = \int_\Omega \nabla u . \nabla v dx$$

which defines an equivalent nomm to $\|.\|_{H^1}$ on the closed space $H_0^1(\Omega)$.

1.2 Some real analysis properties

Definition 1.5. (The continuity) [25]

Let $f : \mathbb{R} \to \mathbb{R}$ be a application. We say that f is continuous if it is continuous at any point of $\mathbb{R}$ In other words $f : \mathbb{R} \to \mathbb{R}$ is continuous in a if:

$$\forall \varepsilon \in \mathbb{R}_+^*, \exists \alpha \in \mathbb{R}_+^*, \forall x \in \mathbb{R}, |x - a| < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Definition 1.6. (Uniformly continuous applications)
Let (X, d) and (X', d') metric spaces. A map $f : X \to X'$ is said to be uniformly continue if $\forall \varepsilon \in \mathbb{R}_+^*$, there exists $\alpha \in \mathbb{R}_+^*$ such that

$$\forall (x, y) \in X * X', d(x, y) < \alpha \Longrightarrow d'(f(x), f(y)) < \varepsilon$$

Definition 1.7. (Lipschtzian) [7]
Let G be a part of $\mathbb{R}^2, f : G \to \mathbb{R}$ a application and K a positive real number. We say that f is K-Lipschitzian according to y if:

$$\forall (t, y) \in G, \left| f\left(t, y_1\right) - f\left(t, y_2\right) \right| \leq K \left| y_1 - y_2 \right|.$$

Definition 1.8. (Bounded function)
A function $f : G \subset \mathbb{R} \to \mathbb{R}$ is bounded if:

$$\exists M > 0, \forall t \in G : |f(t)| \leq M.$$

Definition 1.9. (Convex function)
The map f is convex if and only if, $\forall x, y, z \in I \subset \mathbb{R}$ with $x \leqslant y \leqslant z$ for $y = tx + (1-t)z$, we have

$$f(y) \leq t f(x) + (1-t) f(z).$$

Definition 1.10. (Convolution product)
The convolution product of two real or complex functions f and g are integrable is:

$$f(x) * g(x) = \int_0^x f(x-t) g(t) dt = \int_0^x g(x-t) f(t) dt.$$

Theorem 1.1. (The derivation under the symbol of integration) [21]
We think that:

1. $f : I * [a, b] \to \mathbb{R}$ is continuous,
2. f admpits a partial derivative $\frac{\partial f}{\partial x}$ continue on I,
3. Applications $u : I \to [a, b]$ and $v : I \to [a, b]$ are derivable, then the function

$$\begin{aligned} \varphi : I &\to \mathbb{R} \\ x &\to \int_{u(x)}^{v(x)} f(x, t) dt \end{aligned}$$

is derivable, of derivative

$$\varphi'(x) = \int_{u(x)}^{v(x)} \frac{\partial f(x, t)}{\partial x} dt + v'(x) f(x, v(x)) - u'(x) f(x, u(x)).$$

Definition 1.11. (Lebesgue's dominated convergence theorem) [26]
Let E be a measurable set in $\mathbb{R}$ and let $\{f_n\}$ be a sequence of measurable functions such than

- $\lim_{n\to\infty} f_n(x) = f(x)$ a.e on E.

- For each $n \in \mathbb{N}_1$ $|f_n(x)| \leq g(x)$ a.e on E where g is integrable in the sense of Lebesgue on E. So

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$$

Theorem 1.2. (Fubini) [18]
Let $f(x,y)$ be a summable function over the product of measurable spaces (X,μ) and (Y,ν). We then have the following assertions:

1) For μ-almost all $x \in X$, the function $f(x,y)$ is summable over Y and its integral over Y is a summable function over X.

2) For ν - almost all $y \in Y$, the function $f(x,y)$ is summable over X and its integral over X is a summable function over Y.

3) We have:

$$\int_{XY} f(x,y)d(\mu\nu)(x,y) = \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y).$$

1.3 Some elements of topology

Definition 1.12. (Norm) Let E be a vector space on $\mathbb{R}$. We call a notm on E any application $\|.\| : E \to \mathbb{R}$ verify

- $\forall x \in \mathbb{E} : \|x\| = 0 \Longleftrightarrow x = 0$.

- $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{E} \quad \|\lambda x\| = |\lambda|\|x\|$.

- $\forall x, y \in \mathbb{E} : \|x+y\| \leq \|x\| + \|y\|$ "triangular inequality ".
Example 1.1. $J \subset \mathbb{R}$ Space C(J,R) provided with the norm

$$|y|_\infty = \sup\{|y(t)|, t \in J\}$$

Definition 1.13. (Banach space) [24]
We call Banach space any space full normalized vector on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
Example 1.2. C($J, \mathbb{R}$) space of continuous functions on J and with values in $\mathbb{R}$ is Banach.

Definition 1.14. (Closed parties) We call closed part of E any part of E whose complement is open.
Exercise 1.1. Any closed ball is a closed part

Definition 1.15. **(Compact parts)** We say that $J \subset \mathbb{R}$ is compact if for any overlap of C by openings we can extract a finite undercoverage.This translates as follows: if $(U_i)_{i\in I}$ is an open family such that $C \subset (U_i)_{i\in I}$ then there exists a finite subset $J \subset 1, C \subset \bigcup_{i\in J} U_i$.

Definition 1.16. **(Relatively compact parts)** We say that A is a relatively compact part of a metric space X if its adhesion is a part compact of X.

Definition 1.17. **(Convex parts)** Let C be a part of E. We say that C is convex in E if, $\forall x, y \in C$ and all $t \in [0, 1]$, we have $(1 - t)x + ty \in C$.

Definition 1.18. **(Operator)** [6]
Let E be a normalized vector space , an application linear A of E in itself is called a linear operator in E. We call domain of A and we denote it by D$_A$, *where*

$$D_A = \{x \in E, Ax \in E\}$$

Definition 1.19. **(Continuous operator)** [6]
The operator A is continuous, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$(x', x'' \in D_A) : \|x' - x''\| < \delta \Rightarrow \|Ax' - Ax''\| < \varepsilon.$$

Definition 1.20. **(Linear Bounded Operators)** [6]
Let E be a vector space standard , we call a bounded linear operator any continuous linear application of E in E.
- If A is a bounded linear operator, then

$$(\forall x \in D_A) : \|Ax\| \leq \|A\| \cdot \|x\|.$$

where the norm of A being defined by:

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{x\in D_A - \{0\}} \frac{\|Ax\|}{\|x\|}.$$

Definition 1.21. **(Compact operator)** Operator A is said to be compact if the image of the set $X \subset R$ by A that is to say the set A (X) is relatively compact.

Definition 1.22. norm infinity
$\|h\|_\propto = max|h(t)|$.

1.4 Special functions

1.4.1 The Gamma function

Definition 1.23. [22]
The gamma function $\Gamma(z)$ is defined by:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, (\Re e(z) > 0)$$

where $t^{z-1} = e^{(z-1)\log(t)}$, which converges in the right half of the complex plane $\Re e(z) > 0$. Indeed, we have

$$\begin{aligned}\Gamma(x + \mathrm{i}y) &= \int_0^{\infty} e^{-t} t^{x-1+iy} dt \\ &= \int_0^{\infty} e^{-t} t^{x-1} + e^{\mathrm{i}y\log(t)} dt \\ &= \int_0^{\infty} e^{-t} t^{x-1} [\cos(y\log(t)) + \mathrm{i}\sin(y\log(t))] dt\end{aligned}$$

The expression in the square brackets is bounded for all t, convergence at infinity is provided by e^{-t}, and for the convergence at $t = 0$ we must have $z = \Re e(z) > 0$.

Proposition 1.1. [22]

1. $\Gamma(z+1) = z\Gamma(z)(\Re e(z) > 0)$
2. $\Gamma(n+1) = n!, \forall n \in \mathbb{N}$.
3. $\Gamma(1) = 1$.
4. $\Gamma(-m) = \mp\infty, \forall m \in \mathbb{N}$.
5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
6. $\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)...(z+n)}, \Re e(z) > 0$.

1.4.2 The Beta function

Definition 1.24. [22]
The Beta function is a type of Euler integral defined by:

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt, \quad (\Re e\ u > 0, \Re e v > 0).$$

Proposition 1.2. [22] The relationship between Euler Beta function and Euler Gamma is given though:

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

1.4.3 Mittag-Leffler function

In this section we introduce the one and two-parameter Mittag-Leffler functions, denoted as $E_{\alpha}(.)$ and $E_{(\alpha,\beta)}(.)$ respectively.

Definition 1.25. [22]
The one-parameter Mittag-Leffler function $E_\alpha(.)$ is defined as:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad \Re e(\alpha)>0$$

The two-parameter Mittag-Leffler function $E_{(\alpha,\beta)}(.)$ is defined as:

$$E_{(\alpha,\beta)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\Re\operatorname{Re}(\alpha)>0,\Re e(\beta)>0,\beta\in\mathbb{C}$$

Example 1.3.

1. $E_{(1,1)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.$
2. $E_{(1,2)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{e^z-1}{z}.$
3. $E_{(1,3)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!}=\frac{e^z-1-z}{z^2}.$

1.5 Fixed Point Theorems

Definition 1.26. (Fixed Point)
Let T be a map of a set S in itself. We call a fixed point of T any point $s\in S$ such that $T(s)=s$.
Theorem 1.3. (Banach's principle) [21]
Let X be a Banach space and K be a nonempty closed subset of X. If $B:K\to K$ is a contraction, then there exists a unique fixed point of B
Theorem 1.4. (Banach's principle of contraction) [7]
Let S be a medial space complete link and let $T:S\to S$ be a contracting application, that is to say there exists $0<k<1$ such that

$$d(Tx,Ty)\leqslant kd(x,y),\forall x,y\in S.$$

Then T admits a unique fixed point $s\in S$. We have

$$\lim_{n\to\infty}T^n(s)=s,$$

with

$$d\left(T^n(s),s\right)\leq\frac{k^n}{1-k}d(s,T(s))$$

Demonstration. See [7]
Theorem 1.5. (Arzela-Ascoli) [18]
Let $C(X)$ be the normalized space of real functions continuous on a compact metric space X with norm :

$$\|f\|=\sup_{x\in X}|f(x)|.$$

For a family $A\subset A(C)$ is relatively compact, it is necessary and sufficient that it is:

- Uniformly bounded:

$$\exists C : |f(x)| \leq C, \forall f \in A, \forall x \in X.$$

- Equicontinuous:

$$\forall \varepsilon > 0, \exists \delta > 0, |x-y| < \delta \Longrightarrow |f(x)-f(y)| < \varepsilon, \forall f \in A$$

Demonstration. See [18]

1.6 Laplace transform

Let us recall some basic tools of the Laplace transform

Definition 1.27 (18)**.** The Laplace transform is a practical method for solving equations and differential systems, let f be a function defined for all the variable. $x > 0$ - Laplace transform defined by:

$$F(s) = \mathcal{L}[f(x)](s) = \int_0^{+\infty} f(x) e^{-sx} dx, \quad s \in \mathbb{C}.$$

- The original $f(x)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transom

$$f(x) = \mathcal{L}^{-1}[F(s)](x) = \int_{c-i\infty}^{c+i\infty} F(s) e^{sx} ds, \quad c = \Re e(s) > c_0$$

- We say that the convolution product of f by g is defined by:

$$f(x) * g(x) = \int_0^x f(x-t) g(t) dt = \int_0^x g(x-t) f(t) dt.$$

- Laplace transform of the convolution

$$\mathcal{L}[f(x) * g(x)](s) = F(s) \cdot G(s).$$

We assume that $F(s)$ and $G(s)$ exist. Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order n of the function $f(x)$:

$$\mathcal{L}\left[f^n(x)\right](s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^k f^{(n-k-1)}(0).$$

Table summarizes some Laplace transformations of some functions and some properties of Laplace transforms. [12]

The function	Transforme	The function	Transforme
$x^{m-1}e^{ax}$	$\frac{\Gamma(m)}{(s-a)^m}(m>0)$	$af(x)+bg(x)$	$aF(s)+bG(s)$
$\cos\beta x$	$\frac{s}{s^2+\beta^2}$	$\int_0^x dt ... \int_0^t f(t')dt'$	$s^{-n}F(s)$
$\sin\beta x$	$\frac{\beta}{s^2+\beta^2}$	$f^n(x)$	$s^nF(s)-\sum_{j=0}^{n-1} s^{n-1-j}f^j(0)$
$x^m(m>-1)$	$\frac{\Gamma(m+1)}{s^{m+1}}, \quad \Re s>0$	$f(cx)$	$\frac{1}{c}F(s/c)$
$\delta(x-a)$	e^{-as}	$xf(x)$	$-\frac{dF(s)}{df}$
$H(x-a)$	$\frac{1}{s}e^{-as}$	$\frac{f(x)}{x}$	$\int_s^\infty F(s)(s')ds'$
$(\pi x)^{\frac{1}{2}}e^{-a^2/4x}$	$\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}$	$\int_0^x g(x-t)f(t)dt$	$F(s)G(s)$

Chapter 2

Fractional Calculus

2.1 Riemann-Liouville fractional derivative and Caputo fractional derivative

2.1.1 Riemann-Liouville fractional derivative

The corresponding derivative is calculated using Lagrange's rule for differential operators. Computing nth order derivative over the integral of $(n-\alpha)$, the order derivative is obtained. It is important to remark that n is the smallest integer greater α that is $n = [\alpha] + 1$ [23]
${}^R_aD^\alpha_x f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n \int_a^x (x-t)^{n-\alpha-1}dt = \frac{d^n}{dt^n}{}_aI_x^{n-\alpha} f(x)$

$(n-1 \leq [Re(\alpha)] < n)$,and $x > a$
This operator has the following important properties:

- ${}_aD^\alpha_t\, {}_aD^\beta_t f(x) =_a D^{\alpha+\beta}_t f(x)$.

- In particular $\alpha = 0 \in \mathrm{N}$

$$({}^RD^0_a f)(x) = \frac{1}{\Gamma(1)}(\frac{d}{dx})\int_a^x f(t)dt = f(x)$$

$({}^RD^m_a f)(x) = \frac{1}{\Gamma(1)}(\frac{d^{m+1}}{dx^{m+1}})\int_a^x f(t)dt = (\frac{d^m}{dx^m})f(x)$,
consequently the fractional derivative in the sense of Riemann-Liouville coincides with the derivative classic by $\alpha \in N$.

2.1.2 Caputo fractional derivative

Another option for computing fractional derivatives is the Caputo fractional derivative. It was introduced by Michele Caputo in his 1967 paper. [22] In contrast to the Riemann-Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows, where again $n = \alpha$:
Definition 2.1. [22]

For a function of class $C^n([a,b])$, *the* α order Caputo left fractional derivative is defined by:

$$^{C}D_x^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt, n-1<\alpha<n\in N$$

.

let n-1$\leq\alpha<n, n\in\mathbb{N}, \alpha\in\mathbb{R}$ and $f(x)$ be a function such that $^{C}D_x^{\alpha}$ exists. Then the following for the Caputo fractional derivative hold:

$$\lim_{\alpha\to n}{}^{C}D_x^{\alpha}(f(x))=f^{n}(x). \tag{2.1}$$

$$\lim_{\alpha\to n-1}{}^{C}D_x^{\alpha}(f(x))=f^{n-1}(x)-f^{n-1}(0). \tag{2.2}$$

proof:

The proof is achieved by mean of integration by parts as follows:

$$^{C}D_x^{\alpha}f(x)=\int_0^x\frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt=\frac{1}{n-\alpha+1}(-f^{n}(\tau)\frac{(x-\tau)^{n-\alpha}}{n-\alpha})+f^{n-1}(\tau)\frac{(x-\tau)^{n-\alpha}}{n-\alpha}$$

(2.3)

$=\frac{1}{n-\alpha+1}(f^{n}(0)t^{(n-\alpha)}+\int_0^x f^{(n+1)}(\tau)(x-\tau)^{n-\alpha}d\tau)$

Nevertheless, taking the limit for $\alpha\to n$ and $\alpha\to n-1$, respectively, we obtain

$$\lim_{\alpha\to n}{}^{C}D_x^{\alpha}(f(x))=f^{n}(x),$$

$\lim_{\alpha\to n-1}{}^{C}D_x^{\alpha}(f(x))=f^{n-1}(x)-f^{n-1}(0)(2.4)$

Theorem 2.1.3: [22]

Assuming that the Laplace transform $F(s)$ of the function f exists, then the Laplace transform of Caputo fractional derivative of$f(x)$ is given by:

$$L(^{C}D_x^{\alpha}(f(x)))(s)=s^{\alpha}F(s)-\sum_{k=0}^{n-1}s^{n-\alpha-1}f^{k}(0).$$

Theorem 2.1.4: [22]

Let $x>0, \alpha\in R, n-1<\alpha<n\in N$; if in addition the function $f(x)$ is n-times differentiable, then, the following relation between the Riemann–Liouville and Caputo derivatives of fractional order holds

$$^{C}D_x^{\alpha}(f(x))=D_x^{\alpha}(f(x))-\sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)} \tag{2.5}$$

proof:

Since the function is n-times differentiable, using the well-known Taylor series expansion around the point 0, we obtain

$$f(x)=\sum_{k=0}^{n-1}\frac{x^k}{k!}f^{k}(0)+R_{n-1}, \tag{2.6}$$

where

$$R_{n-1} = \frac{1}{\Gamma(n)} \int_0^x f^{(n)}(\tau)(x-\tau)^{n-1} d\tau. \tag{2.7}$$

However, employing the linearity property of the Riemann–Liouville fractional derivative and also the Riemann–Liouville fractional derivative of power function, then:

$$D^\alpha f(x) = D^\alpha (\sum_{k=0}^{n-1} \frac{x^k}{k!} f^k(0) + R_{n-1})$$

(2.8)

$$= \sum_{k=0}^{n-1} \frac{D^\alpha(x^k)}{k!} f^{(k)}(0) + D^\alpha R_{n-1}$$

$$= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} f^{(k)}(0) + I^{n-\alpha} f^{(n)}(x)$$

$$= \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} f^{(k)}(0) + {}^C D_x^\alpha(f(x)).$$

Thus,

$$D^\alpha f(x) = \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} f^{(k)}(0) + {}^C D_x^\alpha(f(x)). \tag{2.9}$$

there for

$${}^C D_x^\alpha(f(x)) = D_x^\alpha(f(x)) - \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)}$$

Example 2.1: Consider the function:

$$f(x) = x^\beta.$$

for $0 < n-1 < \alpha < n$,we have:

$${}^C D^\alpha(f(x)) = I^{n-\alpha}(D^n x^\beta),$$

or

$$D^n x^\beta = (\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n}).$$

As a result:

$$I^{n-\alpha}(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} t^{\beta-n} dt,$$

by performing the change of variable $t = yx$ so $dt = xdy$,we obtain:

$$\int_0^x (x-t)^{n-\alpha-1} t^{\beta-n} dt = \int_0^1 (x-xy)^{n-\alpha-1} (xy)^{\beta-n} x dy$$

$$
\begin{aligned}
&= \int_0^1 x^{n-\alpha-1}(1-y)^{n-\alpha-1}y^{\beta-n}x^{\beta-n+1}dy \\
&= \int_0^1 x^{\beta-\alpha}(1-y)^{n-\alpha-1}y^{\beta-n}dy \\
&= x^{\beta-\alpha}\int_0^1 (1-y)^{n-\alpha-1}y^{\beta-n}dy \\
&= x^{\beta-\alpha}B(n-\alpha,\beta-n+1) \\
&= x^{\beta-\alpha}\frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-n+1)}.
\end{aligned}
$$

So

$$I^{n-\alpha}(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}x^{\beta-n}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-n+1)}x^{\beta-\alpha},$$

finally,we obtain

$$^{C}D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}.$$

In particular , for $\beta = 0$, we have:

$$^{C}D^{\alpha}x^{0} = D^{\alpha}1 = 0.$$

unlike the Riemann-Liouville derivative , the fractional order derivative in the Caputo sense of a constant is zero .

2.1.3 Some Theoremes of Fractional derivatives

Theorem 2.1.5:
Leibniz rule for Caputo derivative: Let $t > 0$,$\alpha \in R$,$n-1 < \alpha < n \in R$. If both functions$f(t)$ and $g(t)$ are continuous together with their derivatives in [0,t] then the following relation is valid:

$$^{C}D_t^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty} C_{\alpha}^{k}(D^{\alpha-n}f(t))g^{(k)}(t) - \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}((f(t)g(t))^{k}(0)).$$

Theorem 2.1.6:(Linearity) [22]
Similarly to integer-order differentiation, fractional differentiation is a linear operator:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha}f(x) + \mu D^{\alpha}g(x).$$

2.2 Caputo-Fabrizio Fractional derivative :

Recently, a new derivative was launched by Caputo and Fabrizio [9] and it was followed by some related theoretical and applied results (see for example Refs. [19] and the references therein). We recall that the existing fractional derivatives have been used in many real world problems with great success (see for example Refs. [9] and the references therein) but still there are many thinks to be done in this direction.
Let us recall the usual Caputo fractional time derivative of order $\alpha, givenby$

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f^{'}(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{2.10}$$

Definition 2.2
with α ∈[0,1] *and* a∈ $[-\infty, t], f \in H^1(a,b), b > a$, By changing the kernel $(t-\tau)^{-\alpha}$ with the function $\exp(-\frac{\alpha}{1-\alpha}t)and\frac{1}{\Gamma((1-\alpha))}$with $\frac{M(\alpha)}{1-\alpha}$,we obtain the following new definition of fractional time derivative

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f^{'}(\tau) \exp[-\frac{\alpha(t-\tau)}{1-\alpha}] d\tau \tag{2.11}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.According to the definition (2.12),the FD , is zero when $f(t)$ is constant, as in the FD , but, contrary to the FD , the kernel does not have singularity for $t = \tau$. The FD can also be applied to functions that do not belong to $H^1(a,b)$. Indeed, the definition (2.12) can be formulated also for $f \in L^1(-\infty, b)$and for any $\alpha \in [0,1]$as

$$D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_{-\infty}^t (f(t) - f(\tau)) \exp[-\frac{\alpha(t-\tau)}{1-\alpha}] d\tau$$

Now, it is worth to observe that if we put

$$\sigma = \frac{1-\alpha}{\alpha} \in [0, \infty], \alpha = \frac{1}{1+\sigma} \in [0,1]$$

the definition (2.12) of FD assumes the form

$$D_t^{\sigma} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f^{'}(\tau) \exp[-\frac{(t-\tau)}{\sigma}] d\tau \tag{2.12}$$

where $\sigma[0,\infty]$ and $N(\sigma)$ is the corresponding normalization term of $M(\alpha)$, such that $N(0) = N(\infty) = 1$. Moreover,because

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp[-\frac{(t-\tau)}{\sigma}] = \delta(t-\tau) \tag{2.13}$$

and for $\alpha \to 1$, we have $\sigma \to 0$.

$$\begin{aligned} \lim_{\alpha \to 1} D_t^{\alpha} f(t) &= \lim_{\alpha \to 1} \frac{M(\alpha)}{1-\alpha} \int_a^t f^{'}(\tau) \exp[-\frac{\alpha(t-\tau)}{1-\alpha}] d\tau \\ &= \lim_{\sigma \to 0} \frac{N(\sigma)}{\sigma} \int_a^t f^{'}(\tau) \exp[-\frac{(t-\tau)}{\sigma}] d\tau = f(t). \end{aligned} \tag{2.14}$$

Otherwise, when $\alpha \to 0$, then $\sigma \to +\infty$. Hence ,

$$\lim_{\alpha\to 0} D_t^{\alpha} f(t) = \lim_{\alpha\to 0} \frac{M(\alpha)}{1-\alpha}\int_a^t f^{'}(\tau)\exp[-\frac{\alpha(t-\tau)}{1-\alpha}]d\tau$$

$$= \lim_{\sigma\to+\infty} \frac{N(\sigma)}{\sigma}\int_a^t f^{'}(\tau)\exp[-\frac{(t-\tau)}{\sigma}]d\tau = f(t)-f(a). \qquad (2.15)$$

Theorem 2.1.7
.For FD , if the function $f(t)$ is such that

$$f^{(s)}(a) = 0, s = 1, 2, ..., n$$

then ,we have

$$D_t^{(n)}(D_t^{(\alpha)} f(t)) = D_t^{(\alpha)}(D_t^{(n)} f(t)) \qquad (2.16)$$

proof :
We begin considering $n = 1$, then from definition (2.18)of $D_t^{\alpha+1} f(t)$,, we obtain

$$D_t^{\alpha}(D_t^{(1)} f(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t f^{'}(\tau)\exp[-\frac{\alpha(t-\tau)}{1-\alpha}]d\tau \qquad (2.17)$$

Hence, after an integration by parts and assuming $f^{'}(a) = 0$, we have

$$D_t^{\alpha}(D_t^{(1)} f(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t (\frac{d}{d\tau} f^{'}(\tau))\exp[-\frac{\alpha(t-\tau)}{1-\alpha}]d\tau. \qquad (2.18)$$

$$\frac{M(\alpha)}{1-\alpha}[\int_a^t \frac{d}{d\tau}(f^{'}(\tau))\exp(-\frac{\alpha(t-\tau)}{1-\alpha})d\tau - \frac{\alpha}{1-\alpha}\int_a^t f^{'}(\tau)\exp(-\frac{\alpha(t-\tau)}{1-\alpha})d\tau]$$

$$= \frac{M(\alpha)}{1-\alpha}[f^{'}(\tau) - \frac{\alpha}{1-\alpha}\int_a^t f^{'}(\tau)\exp(-\frac{\alpha(t-\tau)}{1-\alpha})d\tau]$$

.
otherwise

$$D_t^{(1)}(D_t^{(\alpha)}) f(t)) = \frac{d}{dt}(\frac{M(\alpha)}{1-\alpha}\int_a^t f^{'}(\tau)\exp(-\frac{\alpha(t-\tau)}{1-\alpha})d\tau) \qquad (2.19)$$

$$\frac{M(\alpha)}{1-\alpha}[f^{'}(t) - \frac{\alpha}{1-\alpha}\int_a^t f^{'}(\tau)\exp(-\frac{\alpha(t-\tau)}{1-\alpha})d\tau].$$

It is easy to generalize the proof for any$n > 1$.
In the following, we suppose the function $M(\alpha) = 1$.

2.2.1 Laplace transform of CF fractional derivative

In order to study the properties of the Caputo-Fabrizio fractional derivative , defined in equation (2.3) with $a = 0$, has priority the computation of its Laplace transform ($\mathcal{L}$) given with p variable

$$\mathcal{L}\{D_t^{\alpha} f(t)\} = \frac{1}{(1-\alpha)}\int_0^{\infty}\exp(-pt)\int_0^t f^{'}(\tau)\exp(-\frac{\alpha(t-\tau)}{1-\alpha})d\tau dt$$

Hence, from the property of Laplace transform of a convolution, we have

$$\mathcal{L}\{D_t^{\alpha}f(t)\} = \frac{1}{(1-\alpha)}\mathcal{L}\{f^{'}(t)\}\mathcal{L}\{\exp(-\frac{\alpha t}{1-\alpha})\} = \frac{(p\mathcal{L}\{f(t)-f(0)\}}{p+\alpha(1-p)}$$

Similarly ,

$$\mathcal{L}\{D_t^{(\alpha+1)}f(t)\} = \frac{1}{1-\alpha}\mathcal{L}\{f^{''}(t)\}\mathcal{L}\{\exp(-\frac{\alpha t}{1-\alpha})\}$$

=(p$^2\mathcal{L}\{f(t)\} - pf(0) - f^{'}(0)\overline{\scriptstyle p+\alpha(1-p)}$

Finally

$$\mathcal{L}\{D_t^{(\alpha+n)}f(t)\} = \frac{1}{1-\alpha}\mathcal{L}\{f^{(n+1)}(t)\}\mathcal{L}\{\exp(-\frac{\alpha t}{1-\alpha})\}$$

=p$^{n+1}\mathcal{L}\{f(t) - p^n f(0) - p^{n-1}f^{'}(0)... - f^{(n)}(0)\overline{\scriptstyle p+\alpha(1-p)}$

2.2.2 The associated fractional integral

After the notion of fractional derivative of order $0 < \alpha < 1$, that of fractional integral of order $0 < \alpha < 1$ becomes a natural requirement. In this section we obtain the fractional integral associated to the Caputo-Fabrizio fractional derivative previously introduced. Let $0 < \alpha < 1$. Consider now the following fractional differential equation,

$$^{CF}D^{\alpha}f(t) = u(t), t \geq 0 \tag{2.20}$$

using Laplace transform, we obtain:

$$\mathcal{L}\{^{CF}D^{\alpha}f(t)\}(s) = \mathcal{L}\{u(t)\}(s), s > 0.$$

we have that

$$\frac{(2-\alpha)M(\alpha)}{2(s\mathcal{L}\{f(t)}(s) - f(0)) = \mathcal{L}\{u(t)\}(s), s > 0$$

or equivalently

$$\mathcal{L}\{f(t)\}(s)\frac{1}{s}f(0) + \frac{2\alpha}{s(2-\alpha)M(\alpha)}\mathcal{L}\{u(t)\}(s) + \frac{2(2-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{L}\{u(t)\}(s), s > 0$$

Hence, using now well known properties of inverse Laplace transform, we deduce tha

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds + f(0), t \geq 0 \tag{2.21}$$

In other words, the function defined as

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds + c, t \geq 0.$$

where $c \in R$ is a constant, is also a solution of (2.20)
We can also rewrite fractional differential equation (2.20) as

$$\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t \exp(-\frac{\alpha(t-s)}{1-\alpha})f^{'}(s)ds = u(t), t \geq 0,$$

or equivalently

$$\int_0^t \exp(\frac{\alpha}{1-\alpha}s)f^{'}(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\exp(\frac{\alpha}{1-\alpha}t)u(t), t \geq 0$$

.
Differentiating both sides of the latter equation, we obtain that,

$$f^{'}(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}(u^{'}(t) + \frac{\alpha}{1-\alpha}u(t)), t \geq 0.$$

Hence, integrating now from 0 to t, we deduce as in (2.21), that

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[u(t)-u(0)] + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds + f(0), t \geq 0$$

Thus, as consequence, we expect that the fractional integral of Caputo-Fabrizio type must be defined as follows.
Definition 2.3 [7]
. Let $0 < \alpha < 1$. The fractional integral of order α of a function f is defined by:

$$^{CF}I_{\alpha}^{t}(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds, t \geq 0. \tag{2.22}$$

We have the following relation.
Remark 2.1
Note that, according to the above definition, the fractional integral of Caputo type of function of order $0 < \alpha < 1$ is an average between function f and its integral of order one [13]. This therefore imposes

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1. \tag{2.23}$$

The above expressions yields an explicit formula for

$$M(\alpha) = \frac{2}{(2-\alpha)}, 0 \leq \alpha \leq 1.$$

Because of the above, Nieto and Losada proposed that the new Caputo derivative of order $0 < \alpha < 1$ can be reformulated as
equation $^{CF}D^{\alpha}f(t) = \frac{1}{1-\alpha}\int_0^t \exp(-\frac{\alpha}{1-\alpha}(t-s))f^{'}(s)ds, t \geq 0.$

Chapter 3

Atangana - Baleanu Fractional derivative

In 2016, Atangana and Baleanu suggested differential operators based on the generalized Mittag-Leffler function. The aim was to introduce fractional differential operators with non-singular nonlocal kernel. Their fractional differential operators are given below in Riemann-Liouville sense and Caputo sense respectively.
We recall that the Mittag-Leffler function is the solution of the following fractional ordinary differential equation [1]

$$\frac{d^\alpha y}{dx^\alpha} = ay, 0 < \alpha < 1.$$

The Mittag-Leffler function and its generalized versions are therefore considered as nonlocal functions. Let us consider the following generalized Mittag-Leffler function [2]:

$$E_\alpha(-t^\alpha) = \sum_{k=0}^{\infty} \frac{(-t)^{\alpha k}}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{(-t)^{\alpha k}}{k!} = \exp(-\alpha t) \tag{3.1}$$

The Taylor series ofexp$(-(t-y))$ at the point t is given by:

$$\exp(-a(t-y)) = \sum_{k=0}^{\infty} \frac{(-a(t-y))^k}{k!} \tag{3.2}$$

If we chose $a = \frac{\alpha}{1-\alpha}$ and replace the above expression into Caputo-Fabrizio derivative we conclude that

$$D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(y) \sum_{k=0}^{\infty} \frac{(-a(t-y))^k}{k!} dy = \frac{M(\alpha)}{1-\alpha} \sum_{k=0}^{\infty} \frac{-(a)^k}{k!} \int_a^t f'(y)(t-y)^k dy. \tag{3.3}$$

To solve the problem of non-locality, we derive the following expression. In equation (3.4), we $k!$ by also $\Gamma(\alpha k + 1)$ also $(t-y)^k$ is replaced by $(t-y)^{\alpha k}$ to obtain:

$$D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(-a)^k}{\Gamma(\alpha k+1)}\int_b^t \frac{df(y)}{dy}(t-y)^{\alpha k}dy. \tag{3.4}$$

This, the following derivative is proposed

3.1 Atangana-Baleanu Fractional derivative in caputo sense

Definition 3.1. Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$then , the definition of the the Atangana-Baleanu derivative in caputo sense is given by: [2]

$$ {}_a^{ABC}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\int_b^t f'(x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx. \tag{3.5}$$

3.2 Atangana-Baleanu Fractional Derivative in Riemann-Liouville sense

Definition 3.2. Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$then , the definition of the Atangana-Baleanu derivative Riemann-Liouville in the left sense is given by [2]

$$ {}_a^{ABR}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_b^t f(x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx. \tag{3.6}$$

Definition 3.3. Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$then , the definition of the Atangana-Baleanu derivative Riemann-Liouville in the right sense is given by:

$$ {}_a^{ABR}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_t^b f(x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx. \tag{3.7}$$

3.3 Properties of A-B fractional derivative:

3.3.1 Laplace transform of the ABR

the Laplace transform of the Atangana–Baleanu fractional derivative in Riemann–Liouville sense is given as

$$\begin{aligned}
\mathcal{L}\{{}_0^{ABR}D_t^\alpha f(t)\}(p) &= \frac{M(\alpha)}{1-\alpha}\mathcal{L}\{\frac{d}{dt}\int_b^t f(x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx\}(p)\\
&= \frac{M(\alpha)}{1-\alpha}p\mathcal{L}\{\int_b^t f(x)E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]dx\}(p)\\
&= \frac{M(\alpha)}{1-\alpha}\mathcal{L}\{f(t)\}(p)\mathcal{L}\{E_\alpha[-\alpha\frac{(t-x)^\alpha}{1-\alpha}]\}(p)
\end{aligned}$$

$$= \frac{M(\alpha)}{1-\alpha}\mathcal{L}\{f(t)\}(p)\frac{p^{\alpha}}{p^{\alpha}+\frac{\alpha}{1-\alpha}}. \tag{3.8}$$

3.3.2 Laplace transform of the ABC

The Laplace transform of Atangana–Baleanu fractional derivative in Caputo sense is given as

$$\begin{aligned}\mathcal{L}\{{}_{0}^{ABC}D_{t}^{\alpha}f(t)\}(p) &= \frac{M(\alpha)}{1-\alpha}\mathcal{L}\{\int_{b}^{t} f^{'}(x)E_{\alpha}[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}]dx\} \\ &= \frac{M(\alpha)}{1-\alpha}\frac{1}{p}(p\mathcal{L}\{f(t)\}-f(0))\frac{p^{\alpha}}{p^{\alpha}+\frac{\alpha}{1-\alpha}} \\ &= \frac{M(\alpha)}{1-\alpha}\frac{p^{\alpha}\mathcal{L}\{f(t)\}(p)-p^{\alpha-1}f(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}} \end{aligned} \tag{3.9}$$

3.3.3 Relation between ABCFD and ABRFD

Theorem 3.1
Let $f \in H^{1}(a,b), b > a, \alpha \in [0,1]$then,the following relation is obtained

$$ {}_{0}^{ABC}D_{t}^{\alpha}f(t) = {}_{0}^{ABR}D_{t}^{\alpha}f(t) + H(t) \tag{3.10}$$

proof :
By using the definition (3.10) and the Laplace transform applied on both sides we obtain easily the following result:

$$\mathcal{L}\{{}_{0}^{ABC}D_{t}^{\alpha}f(t)\}(p) = \frac{M(\alpha)}{1-\alpha}\frac{p^{\alpha}\mathcal{L}\{f(t)\}(p)}{p^{\alpha}+\frac{\alpha}{1-\alpha}} - \frac{p^{\alpha-1}f(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{M(\alpha)}{1-\alpha} \tag{3.11}$$

Following equation (3.8) we have:

$$\mathcal{L}\{{}_{0}^{ABC}D_{t}^{\alpha}f(t)\}(p) = \mathcal{L}\{{}_{0}^{ABR}D_{t}^{\alpha}f(t)\}(p) - \frac{p^{\alpha-1}f(0)}{p^{\alpha}+\frac{\alpha}{1-\alpha}}\frac{M(\alpha)}{1-\alpha} \tag{3.12}$$

Applying the inverse Laplace on both sides of equation (3.12) we obtain:

$${}_{0}^{ABC}D_{t}^{\alpha}f(t) = {}_{0}^{ABR}D_{t}^{\alpha}f(t) - \frac{M(\alpha)}{1-\alpha}f(0)E_{\alpha}(-\frac{\alpha}{1-\alpha}t^{\alpha}). \tag{3.13}$$

Then taking $H(t)$to be $\frac{M(\alpha)}{1-\alpha}f(0)E_{\alpha}(-\frac{\alpha}{1-\alpha}t^{\alpha})$

3.3.4 Some Estimates AB fractional derivative

Theorem 3.2
Let f be a continuous function on a closed interval $[a, b]$. Then the following inequality is obtained on $[a, b]$

$$\|{}_0^{ABR}D_t^{\alpha} f(t)\| < \frac{M(\alpha)}{1-\alpha} K, \tag{3.14}$$

proof :

$\|{}_0^{ABR}D_t^{\alpha} f(t)\| = \|\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t f(x)E_{\alpha}[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}]dx\| < \frac{M(\alpha)}{1-\alpha}\|\frac{d}{dt}\int_0^t f(x)dx\| = \frac{M(\alpha)}{1-\alpha}\|\|f(x)\|.(3.15)$
Then taking K to be $\|f(x)\|$ the proof is completed.

Theorem 3.3

The A-B. derivative in Riemann and Caputo sense possess the Lipschitz condition, that is to say, for a given couple function and , the following inequalities can be established:

$$\|{}_0^{ABR}D_t^{\alpha}(f(t)) - {}_0^{ABR}D_t^{\alpha}(h(t))\| \leq H\|f(t) - h(t)\| \tag{3.16}$$

and also

$$\|{}_0^{ABC}D_t^{\alpha}(f(t)) - {}_0^{ABC}D_t^{\alpha}(h(t))\| \leq H\|f(t) - h(t)\|. \tag{3.17}$$

We present the proof of (3.16) as the proof of (3.17) can be obtained similarly.
proof :

$\|{}_0^{ABR}D_t^{\alpha}(f(t)) - {}_0^{ABR}D_t^{\alpha}(h(t))\| =$
$\|\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t f(x)E_{\alpha}[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}]dx - \frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t h(x)E_{\alpha}[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}]dx\|(3.18)$
Using the Lipschitz condition of the first order derivative, we can find a small positive constant such that:

$$\|{}_0^{ABR}D_t^{\alpha}(f(t)) - {}_0^{ABR}D_t^{\alpha}(h(t))\| < \frac{M(\alpha)\Theta_1}{1-\alpha}E_{\alpha}[-\alpha\frac{t^{\alpha}}{1-\alpha}]\|\int_0^t f(x)dx - \int_0^t h(x)dx\|$$

$andthenthefollowingresultisobtained$:

$$\|{}_0^{ABR}D_t^{\alpha}(f(t)) - {}_0^{ABR}D_t^{\alpha}(h(t))\| < \frac{M(\alpha)\Theta_1}{1-\alpha}E_{\alpha}[-\alpha\frac{t^{\alpha}}{1-\alpha}]\|f(x) - h(x)\|t$$

$$= H\|f(x) - h(x)\|,$$

which produces the requested result.
Let$f$ be an n-times differentiable with natural number and $f^k(0) = 0, k = 1, 2, 3, ..., n,$ then by inspection we obtain

$$ {}_0^{ABC}D_t^\alpha\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^n}{dt^n}\left({}_0^{ABR}D_t^\alpha(f(t))\right). \tag{3.19}$$

Now, we can easily prove by taking the inverse Laplace transform and using the convolution theorem that the following time fractional ordinary differential equation:

$$ {}_0^{ABC}D_t^\alpha(f(t)) = u(t) \tag{3.20}$$

has a unique solution, namely

$$ f(t) = \frac{1-\alpha}{M(\alpha)}u(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t u(y)(t-y)^{\alpha-1}dy. $$

3.4 Atangana-Baleanu fractional integral

Definition 3.3 :
The fractional integral associate to A-B fractional derivative with non-local kernel is defined as:

$$ {}_a^{AB}I_t^\alpha(f(t)) = \frac{1-\alpha}{M(\alpha)}f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy. \tag{3.21}$$

When α =0 we recover the initial function and if also $\alpha = 1$, we obtain the ordinary integral.

3.4.1 Fitting the integral with the derivative

Theorem 3.4 :
The functions $({}_a^{ABR}D^\alpha f)(t)$ and $({}^{ABR}D_b^\alpha f)(t)$ satisfy the equations

$$ ({}_a^{AB}I^\alpha g)(t) = f(t), ({}^{AB}I_b^\alpha g)(t) = f(t), $$

proof :
We just prove the left case. The right case can be proved by means of the Q-operator. From the definition, the first equation is equivalent to

$$ \frac{1-\alpha}{M(\alpha)}g(t) + \frac{\alpha}{M(\alpha)}({}_aI^\alpha g)(t) = f(t) $$

Apply the Laplace transform to see that

$$ \frac{1-\alpha}{M(\alpha)}\mathcal{L}\{g(t)\}(p) + \frac{\alpha}{M(\alpha)}p^{-\alpha}L\{g(t)\}(p) = L\{f(t)\}(p) $$

From which it follows that

$$ L\{g(t)\}(p) = \frac{M(\alpha)}{1-\alpha}\frac{L\{f(t)\}(p)p^\alpha}{p^\alpha + \frac{\alpha}{1-\alpha}} $$

Remark : the Laplace inverse will lead to that $g(t) = ({}_a^{ABR}D^\alpha f)(t)$

$$({}_a^{ABR}D^\alpha {}_a^{AB}I^\alpha f)(t) = f(t)$$

and above we have shown that

$$({}^{ABR}D_b^\alpha {}^{AB}I_b^\alpha f)(t) = f(t)$$

.On the other hand we next prove that

$$({}_a^{AB}I^\alpha {}_a^{ABR}D^\alpha f)(t) = f(t)$$

and

$$({}^{AB}I_b^\alpha {}^{ABR}D_b^\alpha f)(t) = f(t)$$

and hence the function spaces ${}_a^{ABR}I^\alpha(L_p)$ and ${}^{AB}I_b^\alpha(L_p)$ are nonempty .

3.4.2 Integration by parts

Theorem 3.5 :

Let $\alpha > 0, p \geqslant 1, q \geqslant 1$, and $\frac{1}{p} + \frac{1}{q} \leqslant 1 + \alpha \Big(p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha\Big)$.
Then - If $\varphi(\mathrm{x}) \in \mathrm{L}_p(\mathrm{a}, \mathrm{b})$ and $\psi(\mathrm{x}) \in \mathrm{L_q}(\mathrm{a}, \mathrm{b})$, then

$$\begin{aligned} \int_a^b \varphi(x)\left({}_a^{AB}I^\alpha\psi\right)(x)dx &= \frac{1-\alpha}{M(\alpha)}\int_a^b \psi(x)\varphi(x)dx + \frac{\alpha}{M(\alpha)}\int_a^b \left(I_b^\alpha\varphi\right)(x)\psi(x)dx \\ &= \int_a^b \psi(x)\left({}^{AB}I_b^\alpha\varphi\right)(x)dx \end{aligned} \tag{3.22}$$

and similarly,

$$\int_a^b \varphi(x)\left({}^{AB}I_b^\alpha\psi\right)(x)dx$$

$$= \frac{1-\alpha}{M(\alpha)}\int_a^b \psi(x)\varphi(x)dx + \frac{\alpha}{M(\alpha)}\int_a^b \left({}_aI^\alpha\varphi\right)(x)\psi(x)dx$$

$$= \int_a^b \psi(x)\left({}_a^{AB}I^\alpha\varphi\right)(x)dx$$

- If $f(x) \in^{AB} \mathrm{I}_{\mathrm{b}}^\alpha(\mathrm{L}_p)$ and $g(x) \in {}_{\mathrm{a}}^{\mathrm{AB}^\beta}\mathrm{I}^\alpha(\mathrm{L_q})$,
then

$$\int_a^b f(x)\left({}_a^{ABR}D^\alpha g\right)(x)dx = \int_a^b \left({}^{ABR}D^\alpha f\right)(x)g(x)dx.$$

Proof :

-From the definition and the integration by parts for (classical) Riemann-Liouville fractional integrals we have

$$\begin{aligned}\int_a^b \varphi(x)\left({}_a^{AB}I^\alpha\psi\right)(x)dx &= \int_a^b \varphi(x)\left[\frac{1-\alpha}{M(\alpha)}\psi(x)+\frac{\alpha}{M(\alpha)}{}_aI^\alpha\psi(x)\right]dx\\ &= \frac{1-\alpha}{M(\alpha)}\int_a^b \varphi(x)\psi(x)dx+\frac{\alpha}{M(\alpha)}\int_a^b \psi(x)I_b^\alpha\varphi(x)dx\\ &= \int_a^b \psi(x)\left[\frac{1-\alpha}{M(\alpha)}\varphi(x)+\frac{\alpha}{M(\alpha)}I_b^\alpha\varphi(x)\right]dx=\int_a^b \psi(x)\left({}^{AB}I_b^\alpha\varphi(x)\right)dx\end{aligned}$$

The other case follows similarly by Definition (3.3) and the integration by parts for (classical) RiemannLiouville fractional integrals.
-From definition and the first part we have

$\int_a^b f(x)\left({}_a^{ABR}D^\alpha g\right)(x)dx=\int_a^b\left({}^{AB}I_b^\alpha\phi\right)(x)\cdot\left({}_a^{ABR}D^\alpha\circ{}_a^{ABR}I^\alpha\varphi\right)(x)dx$

$$\begin{aligned}&=\int_a^b\left({}^{AB}I_b^\alpha\phi\right)(x)\cdot\varphi(x)dx\\ &=\frac{1-\alpha}{M(\alpha)}\int_a^b\phi(x)\varphi(x)dx+\frac{\alpha}{M(\alpha)}\int_a^b\phi(x)\left(a_aI^\alpha\varphi\right)(x)dx\\ &=\frac{1-\alpha}{M(\alpha)}\int_a^b\left({}^{ABR}D_b^\alpha f\right)(x)\left({}_a^{ABR}D^\alpha g\right)dx\\ &\quad+\frac{\alpha}{M(\alpha)}\int_a^b\left({}^{ABR}D_b^\alpha f\right)(x)\left[\frac{M(\alpha)}{\alpha}g(x)-\frac{1-\alpha}{\alpha}\left({}_a^{ABR}D^\alpha g\right)\right]dx\\ &=\int_a^b\left({}^{ABR}D_b^\alpha f\right)(x)g(x)dx.\end{aligned}$$

In the proof, the identity $\left({}_aI^\alpha\varphi\right)(x)=\frac{M(\alpha)}{\alpha}\left({}_a^{AB}I^\alpha\varphi\right)(x)-\frac{1-\alpha}{\alpha}\varphi(x)$ deriven from (3.21) has used.

Example : This example is a numerical application of Theorem 3.5

- To verify Theorem 3.5, let $\psi(x)=x,\varphi(x)=1-x,\alpha=\frac{1}{2},[a,b]=[0,1]$, and $B(\alpha)=1$. Then,

$$_0^{\mathrm{AB}}\mathrm{I}^{1/2}x=\frac{x}{2}+\frac{1}{2}\frac{\Gamma(2)x^{3/2}}{\Gamma(5/2)}=\frac{x}{2}+\frac{2x^{3/2}}{3\sqrt{\pi}},$$

and
${}^{AB}I_1^{1/2}(1-x)=\frac{1-x}{2}+\frac{2(1-x)^{3/2}}{3\sqrt{\pi}}$.
Hence, the left hand side of Theorem 3.5 results in
$\int_a^b\varphi(x)\left({}^{AB}{}_aI^\alpha\psi\right)(x)dx=\int_0^1(1-x){}^{AB}{}_0I^{1/2}x=\int_0^1(1-x)\left[\frac{x}{2}+\frac{2x^{3/2}}{3\sqrt{\pi}}\right]dx=\frac{1}{12}+\frac{8}{105\sqrt{\pi}}{}'$
and
$\int_a^b\psi(x)\Big({}^{\mathrm{AB}}\mathrm{I}_{\mathrm{b}}^\alpha\varphi(x)\mathrm{d}x=\int_0^1x\Big({}^{\mathrm{AB}}\mathrm{I}_1^{1/2}(1-x)\mathrm{d}x=\int_0^1x\left[\frac{1-x}{2}+\frac{2(1-x)^{3/2}}{3\sqrt{\pi}}\right]\mathrm{d}x=\frac{1}{12}+\frac{8}{105\sqrt{\pi}}.$
- To verify the second part of Theorem 3.5, let

$f(x)=\frac{1-x}{2}+\frac{2(1-x)^{3/2}}{3\sqrt{\pi}}$ and $g(x)=\frac{x}{2}+\frac{2x^{3/2}}{3\sqrt{\pi}}$ with $\alpha=\frac{1}{2}$, $[a,b]=[0,1]$, and $\mathrm{M}(\alpha)=1$.
Then,
$\int_a^b f(x)\left({}_a^{ABR}D^{\alpha}g\right)(x)dx=\int_0^1\left[\frac{1-x}{2}+\frac{2(1-x)^{3/2}}{3\sqrt{\pi}}\right]xdx=\frac{1}{12}+\frac{8}{105\sqrt{\pi}}$ and $\int_a^b\left({}^{ABR}D_b^{\alpha}f\right)(x)g(x)dx=\int_0^1(1-x)\left[\frac{x}{2}+\frac{2x^{3/2}}{3\sqrt{\pi}}\right]dx=\frac{1}{12}+\frac{8}{105\sqrt{\pi}}$.

Chapter 4

Some partielle linear AB Fractional Differential Equations

In this section we study some simple but useful fractional equations. [4, 5]

4.1 Existence and Uniqueness Solution

Theorem 4.1 : Let $0 < \alpha < 1$ Then the unique solution of the following initial value problem

$$^{ABC}D^{\alpha}f(t) = u(t), t \geq 0 \tag{4.1}$$

$$f(0) = f_0 \in R \tag{4.2}$$

is given by

$$f(t) = \frac{1-\alpha}{M(\alpha)}u(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t u(s)(t-s)^{\alpha-1}ds. \tag{4.3}$$

proof :
Suppose that the initial value problem (4.3) and (4.4) has two solutions, f_1 and f_2. In that case, we have that

$$\begin{gathered} ^{ABC}D^{\alpha}f_1(t) - {}^{ABC}D^{\alpha}f_2(t) \\ = [^{ABC}D^{\alpha}f_1 - f_2](t) = 0 \end{gathered}$$

and

$$(f_1 - f_2)(0) = 0.$$

we have that $f_1 - f_2 = 0$ That is $f_1(t) = f_2(t)$ for all $t \geq 0$
it is clear that the function defined by (4.3) is a solution of the fractional derivative equation (4.1).
and, on using the Laplace transform ,net on the solution (4.3)

$$\mathcal{L}\{^{ABC}D^{\alpha}f(t)\}(p) = U(p), p > 0$$

$$\frac{M(\alpha)}{1-\alpha}\frac{p^{\alpha}F(p)-p^{\alpha-1}f_0}{p^{\alpha}+\frac{\alpha}{1-\alpha}}=U(p)$$

$$\frac{M(\alpha)p^{\alpha}F(p)}{p^{\alpha}(1-\alpha)+\alpha}=U(p)+\frac{p^{\alpha-1}f_0}{p^{\alpha}(1-\alpha)+\alpha}$$

$$M(\alpha)p^{\alpha}F(p)=U(p)p^{\alpha}(1-\alpha)+\alpha+p^{\alpha-1}f_0$$

$$F(p)=\frac{1-\alpha}{M(\alpha)}U(p)+\frac{\alpha}{M(\alpha)p^{\alpha}}U(p)+\frac{1}{pM(\alpha)}f_0$$

Hence, using now well khown propereties of inverse Laplace transform ,we that

$$\mathcal{L}^{-1}\{F(P)\}=\mathcal{L}^{-1}\{\frac{1}{pM(\alpha)}f_0\}+\frac{1-\alpha}{M(\alpha)}\mathcal{L}^{-1}\{U(P)\}+\frac{\alpha}{M(\alpha)}\mathcal{L}^{-1}\{\frac{1}{P^{\alpha}}U(p)\}$$

From the properties of the inverse Laplace transform, we get the following expressions

$$f(0)L^{-1}\{\frac{1}{p}\}=f(0),L^{-1}\{\frac{1}{P^{\alpha}}U(p)\}=\frac{1}{\Gamma(\alpha)}\int_0^t u(s)(t-s)^{\alpha-1}ds,L^{-1}\{U(P)\}=u(t)$$

$$f(t)=\frac{f(0)}{M(\alpha)}+\frac{1-\alpha}{M(\alpha)}u(t)+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t u(s)(t-s)^{\alpha-1}ds.$$

4.1.1 Initial value problem

Now, we consider the following linear fractional differential equation:

$$^{ABC}D^{\alpha}f(t)=\lambda f(t)+u(t),t\geq 0 \tag{4.4}$$

$$f(0)=f_0 \tag{4.5}$$

where $\lambda\in R,\lambda\neq 0$ ($\lambda=0$ corresponds to the case previously studied).
The solution of this initial value problem is formulated in the following theorem :
Theorem 4.2 :
If $\lambda\neq\frac{M(\alpha)}{1-\alpha}$, $u(t)\in C(0,\infty),u(0)=-\lambda f_0,$

then the solution of the initial value problem (4.6)-(4.7) is given by

$$f(t)=\frac{M(\alpha)f_0}{M(\alpha)-\lambda(1-\alpha)}E_{\alpha}[\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}t^{\alpha}]+\frac{1-\alpha}{M(\alpha)-\lambda(1-\alpha)}u(t)$$

$$+\frac{\alpha M(\alpha)}{(M(\alpha)-\lambda(1-\alpha))^2}\int_0^t u(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}[\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}(t-s)^{\alpha}]ds. \tag{4.6}$$

If $\lambda=\frac{M(\alpha)}{1-\alpha}, u(t)\in AC^1[0,\infty), u(0)=-\lambda f_0$ then the solution of the initial value problem is given by

$$f(t)=-\frac{(1-\alpha)^2}{\alpha M(\alpha)}{}^CD^{\alpha}u(t)-\frac{(1-\alpha)}{M(\alpha)}u(t), \tag{4.7}$$

where the space $AC^1[0,\infty)$ is defined as

$$AC^1=\left\{f(t)=f(0)+\int_0^t g(z)dz, g\in L_1[0,\infty)\right\}$$

proof :
Applying the Laplace transform to both sides of equation (4.6) and using (4.5), we have

$$\frac{M(\alpha)}{1-\alpha}\frac{P^{\alpha}F(p)-p^{\alpha-1}f_0}{p^{\alpha}+\frac{\alpha}{1-\alpha}}-\lambda F(p)=U(p), \tag{4.8}$$

where $F(p)=\mathcal{L}\{f(t)\}(p)$

Simplifying and solving for $F(p)$ we get

$$F(p)=\frac{M(\alpha)p^{\alpha-1}f_0}{p^{\alpha}(M(\alpha)-\lambda(1-\alpha))-\lambda\alpha}+\frac{(1-\alpha)p^{\alpha}+\alpha}{p^{\alpha}(M(\alpha)-\lambda(1-\alpha))-\lambda\alpha}U(p),$$

for $\lambda\neq\frac{M(\alpha)}{1-\alpha}$,
which can be rewritten as

$$F(p)=\frac{M(\alpha)p^{\alpha-1}f_0}{(M(\alpha)-\lambda(1-\alpha))[p^{\alpha}-\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}]}$$

$$+\frac{(1-\alpha)p^{\alpha}+\alpha}{(M(\alpha)-\lambda(1-\alpha))[p^{\alpha}-\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}]}U(p)$$

$$=\frac{M(\alpha)p^{\alpha-1}f_0}{(M(\alpha)-\lambda(1-\alpha))[p^{\alpha}-\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}]}+\frac{(1-\alpha)}{(M(\alpha)-\lambda(1-\alpha))}U(p)$$

$$+\frac{\alpha M(\alpha)}{(M(\alpha)-\lambda(1-\alpha))^2[p^{\alpha}-\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}]}U(p)$$

applying the Laplace inverse transformtion will then give

$$f(t)=\frac{M(\alpha)f_0}{M(\alpha)-\lambda(1-\alpha)}E_{\alpha,1}(\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}t^{\alpha})+\frac{(1-\alpha)}{M(\alpha)-\lambda(1-\alpha)}f(t)$$

$$+\frac{\alpha M(\alpha)}{(M(\alpha)-\lambda(1-\alpha))^2}(t^{\alpha-1}E_{\alpha,\alpha}(\frac{\alpha\lambda}{M(\alpha)-\lambda(1-\alpha)}t^{\alpha})*f(t)),$$

which is equivalent to the desired result as given in Equation (4.8). Note that the condition $u(0)=-\lambda f_0$ is needed to ensure that $f(0)=f_0$.
for $\lambda=\frac{M(\alpha)}{1-\alpha}$, equation (4.10) gives,

$$F(p)=\frac{-(1-\alpha)}{\alpha}p^{\alpha-1}f_0-\frac{(1-\alpha)^2}{\alpha M(\alpha)}(p^{\alpha}+\frac{\alpha}{1-\alpha})U(s).$$

Applying the Laplace inverse , we obtain

$$f(t)=\frac{-(1-\alpha)}{\alpha\Gamma(1-\alpha)}t^{-\alpha}u_0-\frac{(1-\alpha)^2}{\alpha M(\alpha)}(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}*u^{'}(t))-\frac{(1-\alpha)^2}{\alpha M(\alpha)}\frac{t^{-\alpha}u(0)}{\Gamma(1-\alpha)}-\frac{1-\alpha}{M(\alpha)}u(t).$$

Using the condition $f(0)=-\lambda f_0$, one can obtain the desired solution (4.9)
The case $\lambda=0$ and $u(0)=0$, we get

$$f(t)=\frac{1-\alpha}{M(\alpha)}u(t)+\frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t u(s)(t-s)^{\alpha-1}ds,$$

4.1.2 Initial-boundary value problem

Now, we consider a direct problem of
determining $f(x,t)$ in a domain $\Omega=\{(x,t):0<x<1,0<t<T\}$,
such that $f(.,t)\in C^2(0,1), f(x,.)\in H^1(0,T)$ and satisfies the following
initial-boundary value problem:

$$^{ABC}D^{\alpha}f(x,t)-f_{xx}(x,t)=u(x,t),(x,t)\in\Omega \tag{4.9}$$

$$f(0,t)=0,f(1,t)=0,0\leq t\leq T, \tag{4.10}$$

$$f(x,0)=0,0\leq x\leq 1, \tag{4.11}$$

we use the method of separation of variables to solve the homogeneous equation corresponding to the equation along with the boundary conditions,thus we get the following spectral problem :

$$\begin{cases} X''+\lambda X=0,\\ X(0)=0,X(1)=0.\end{cases} \tag{4.12}$$

has the following eigenvalues

$$\lambda_k = (k\pi)^2, k = 1, 2, 3, ...$$

The corresponding eigenfunction are

$$X_k = sin(k\pi x), k = 1, 2, 3, ... \tag{4.13}$$

we can then write the solution $f(x,t)$ and the given function $u(x,t)$in the form of series expansions as follows :

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t)\sin(k\pi x), \tag{4.14}$$

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin(k\pi x), \tag{4.15}$$

where f_k are the unknown coefficients to be found , and $u_k(t)$ are given by

$$u_k(t) = 2\int_0^1 u(x,t)\sin(k\pi x)dx$$

Substituting (4.16) and (4.17) into (4.11) and (4.13) , we obtain the following fractional differential equation

$$^{ABC}D^{\alpha} f_k(t) + k^2\pi^2 f_k(t) = u_k(t),$$

along with the following condition

$$f_k(0) = 0.$$

the solution is given by

$$f_k(t) = \frac{1-\alpha}{M(\alpha)+k^2\pi^2(1-\alpha)}u_k(t) + \frac{\alpha M(\alpha)}{(M(\alpha)+k^2\pi^2(1-\alpha))^2}$$
$$\int_0^t u_k(t)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{M(\alpha)+K^2\pi^2(1-\alpha)}(t-s)^{\alpha}\right)ds,$$

with $f_k(0) = 0$, thus , the solution $f(x,t)$ can now be written as

$$f(x,t) = \sum_{k=1}^{\infty}\left(\frac{1-\alpha}{M(\alpha)+k^2\pi^2(1-\alpha)}u_k(t) + \frac{\alpha M(\alpha)}{(M(\alpha)+k^2\pi^2(1-\alpha))^2}\right.$$
$$\left.\int_0^t u_k(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{M(\alpha)+k^2\pi^2(1-\alpha)}(t-s)^{\alpha}\right)ds\sin(k\pi x).\right.$$

Now,the series representation of $f_k(x,t)$ is given by

$$f_{xx}(x,t) = -\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2(1-\alpha)}{M(\alpha)+k^2\pi^2(1-\alpha)}u_k(t) + \frac{k^2\pi^2\alpha M(\alpha)}{(M(\alpha)+k^2\pi^2(1-\alpha))^2}\right.$$

$$\left.\int_0^t u_k(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{M(\alpha)+k^2\pi^2(1-\alpha)}(t-s)^{\alpha}\right)ds\sin(k\pi x)\right)$$

$$= \sum_{k=1}^{\infty}u_k(t)\sin(k\pi x) + \sum_{k=1}^{\infty}\frac{M(\alpha)}{M(\alpha)+k^2\pi^2(1-\alpha)}\sin(k\pi x)$$

$$\int_0^t u_k^{'}(s)E_{\alpha,1}\left(\frac{-\alpha k^2\pi^2}{M(\alpha)+k^2\pi^2(1-\alpha)}(t-s)^{\alpha}\right)ds$$

.

For the convergence of the first term , we assume $u(0,t)=u(1,t)=0$ and use integration by parts to get

$$\left|\sum_{k=1}^{\infty}u_k(t)sin(k\pi x)\right| = \left|\sum_{k=1}^{\infty}\frac{1}{k\pi}u_{1k}(t)\sin(k\pi x)\right| \leq \sum_{k=1}^{\infty}\frac{1}{k\pi}|u_{1k}(t)|$$

where

$$u_{1k}(t) = 2\int_0^1 u_x(x,t)\cos(k\pi x)dx.$$

Using the inequality $ab \leq \frac{1}{2}(a^2+b^2)$ and the Bessel inequality for trigonometric series we then have the following estimate

$$\left|\sum_{k=1}^{\infty}u_k(t)\sin(k\pi x)\right| \leq \sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{k^2\pi^2}+|u_{1k}|^2\right)$$

$$\leq \sum_{k=1}^{\infty}\frac{1}{k^2\pi^2} + \frac{1}{2}\|u_x(x,t)\|^2_{L^2(0?1)}$$

The uniqueness of the solution can be obtained using the completeness properties of the system $\{\sin(k\pi x)\}$.
such that $u(x,0)=u(0,t)=u(1,t)=0$,$u_1(x,t)\in L^1(0,T)$, and $u_x(x,t)\in L^2(0,1)$, then the problem (4.11)-(4.13) has a unique solution $f(x,t)$ given by

$$f(x,t) = \sum_{k=1}^{\infty}\left(\frac{1-\alpha}{M(\alpha)+k^2\pi^2(1-\alpha)}u_k(t) + \frac{\alpha M(\alpha)}{(M(\alpha)k^2\pi^2(1-\alpha))^2}\right.$$

$$\left.\int_0^t u_k(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{M(\alpha)+k^2\pi^2(1-\alpha)}(t-s)^{\alpha}\right)ds\sin(k\pi x).\right)$$

where

$$u_k(t) = 2\int_0^1 u(x,t)\sin(k\pi x)dx.$$

4.2 Examples

In this section, we give some examples ,choosing $M(\alpha) = 1$.

Example 4.1

Let $0 < \alpha < 1$,$\lambda = 1$, then the following initial value problem of fractional differential equation :

$$\begin{cases} {}^{ABC}D^{\alpha} f(t) + f(t) = 1, \\ f(0) = 1. \end{cases}$$

has a unique solution as a constant function

$f(t) = 1$

In fact, by the Laplace transformation, the equation can be written as

$$\frac{1}{1-\alpha} \frac{p^{\alpha} F(P) - P^{\alpha-1} f_0}{p^{\alpha} + \frac{\alpha}{1-\alpha}} + F(p) = \frac{1}{p}$$

$$\frac{p^{\alpha} F(P) - P^{\alpha-1}}{P^{\alpha}(1-\alpha) + \alpha} + F(p) = \frac{1}{p}$$

$$F(p)(p^{\alpha} + p^{\alpha}(1-\alpha) + \alpha) - p^{\alpha-1} = \frac{p^{\alpha}(1-\alpha) + \alpha}{p}$$

$$F(p)(p^{\alpha} + p^{\alpha}(1-\alpha) + \alpha) = \frac{p^{\alpha}(1-\alpha) + \alpha + p^{\alpha}}{p}$$

$$F(p) = \frac{1}{p}$$

Now,the inverse Laplace transformation given as that

$$f(t) = 1$$

.

Example 4.2 Let $0 < \alpha < 1, \lambda = 1$ consider the initial value problem:

$$\begin{cases} {}^{ABC}D^{\alpha} f(t) + f(t) = t^{v-1}, \\ f(0) = 0. \end{cases}$$

Applying the Laplace transformation leads to have

$$\frac{1}{1-\alpha} \frac{p^{\alpha} F(p) - p^{\alpha-1} f_0}{p^{\alpha} + \frac{\alpha}{1-\alpha}} + F(p) = \mathcal{L}\{t^{v-1}\}(p)$$

$$\frac{p^{\alpha} F(p) - p^{\alpha-1}}{p^{\alpha}(1-\alpha) + \alpha} + F(p) = \mathcal{L}\{t^{v-1}\}(p)$$

$$\frac{F(p)[2p^{\alpha} + p^{\alpha}(1-\alpha) + \alpha]}{p^{\alpha}(1-\alpha) + \alpha} = \mathcal{L}\{t^{v-1}\}(p)$$

$$F(p)=\frac{[p^{\alpha}(1-\alpha)+\alpha]\mathcal{L}\{t^{v-1}\}(p)}{p^{\alpha}(1-\alpha)+\alpha+2p^{\alpha}}$$

Now, the inverse Laplace transformation gives the exact solution as follows

$$f(t)=\frac{1-\alpha}{\alpha}t^{v}+\frac{\Gamma(v)}{\alpha}t^{\alpha+v-1}E_{v,v+\alpha}(t^{\alpha}).$$

Example 4.3 Let $0<\alpha<1, \lambda=1$consider the initial value problem :

$$\begin{cases} {}^{ABC}D^{\alpha}f(t)+f(t)=E_{\alpha}(\gamma t^{\alpha}), \\ f(0)=1. \end{cases}$$

Applying the Laplace transformation leads to have

$$\frac{1}{1-\alpha}\frac{p^{\alpha}F(p)-p^{\alpha-1}}{p^{\alpha}+\frac{\alpha}{1-\alpha}}+F(p)=\mathcal{L}\{E_{\alpha}(\gamma t^{\alpha})\}(p)$$

$$\frac{p^{\alpha}F(p)-p^{\alpha-1}}{p^{\alpha}(1-\alpha)+\alpha}+F(p)=\frac{p^{\alpha-1}}{p^{\alpha}-\gamma}$$

$$F(p)(p^{\alpha}+p^{\alpha}(1-\alpha)+\alpha)=\frac{p^{\alpha-1}+p^{-1}-\gamma p^{\alpha-1}}{p^{\alpha}-\gamma}$$

$$F(p)=\frac{(1-\alpha)p^{\alpha-1}+p^{-1}}{(p^{\alpha}-\gamma)(2p^{\alpha}-\alpha p^{\alpha}+\alpha)}$$

$$F(p)=\frac{\alpha+\gamma(1-\alpha)}{\alpha+\gamma(2-\alpha)}\frac{p^{\alpha-1}}{p^{\alpha}-\gamma}+\frac{\gamma}{\alpha+\gamma(2-\alpha)}\frac{p^{\alpha-1}}{p^{\alpha}+\frac{\alpha}{2-\alpha}}$$

Applying the inverse Laplace transformation gives the exact solution as follows

$$f(t)=\frac{\alpha+\gamma(1-\alpha)}{\alpha+\gamma(2-\alpha)}E_{\alpha}(\gamma t^{\alpha})+\frac{\gamma}{\alpha+\gamma(2-\alpha)}E_{\alpha}(\frac{-\alpha}{2-\alpha}t^{\alpha}).$$

Example 4.4 consider the boundary value problem :

$$\begin{cases} {}^{ABC}D^{\alpha}f(x,t)-f_{xx}(x,t)=t\sin(\pi x), \\ f(0,t)=0, f(1,t)=0, 0\leq t\leq T \\ f(x,0)=0, 0\leq x\leq 1 \end{cases}$$

.

Applying , the Laplace transforme and inverse Laplace transformation ,the exact solution given by :

$$f(x,t)=\left(\frac{1-\alpha}{1+\pi^{2}(1-\alpha)}t+\frac{\alpha}{(1+\pi^{2}(1-\alpha))^{2}}t^{\alpha+1}E_{\alpha,\alpha+2}\left(-\frac{\alpha\pi^{2}}{1+\pi^{2}(1-\alpha)}t^{\alpha}\right)\right)\sin(\pi x).$$

Chapter 5

On a nonlocal implicit problem under A-B fractional derivative

we study a class of initial value problems for a nonlinear implicit fractional differential equation with nonlocal conditions involving the Atangana–Baleanu–Caputo fractional derivative. the following ABC-type nonlocal fractional problem: [15]

$$^{ABC}D^{\alpha}_{0,t}u(t) = f(t, u(t), {}^{ABC}D^{\alpha}_{0,t}u(t)), t \in [0, \chi] \tag{5.1}$$

$$\sum_{k=1}^{m} \beta_k u(\tau_k) = u_0, \tau_k \in (0, \chi), \tag{5.2}$$

where $0 < \alpha \leq 1$, $f : [0, \chi] * R * R \to R$ is a continuous function with
$f(0, u(0), {}^{ABC}D^{\alpha}_{0,t}u(0)) = 0, 0 < \tau_1 < \tau_2 < ... < \tau_m < \chi$, β_k are real numbers
$(k = 1, 2, ..., m)$,m) , and $u \in C[0, \chi]$

Definition 1 : Let $n < \alpha \leq n + 1$ and f be such that $f^{(n)} \in H^1(a, b)$.set $\beta = \alpha - n$. Then $\beta \in [0, 1]$ and we define

$$({}^{ABC}_{a}D^{\alpha}f)(t) = ({}^{ABC}_{a}D^{\beta}f^{(n)})(t) \tag{5.3}$$

and the left Riemann-Liouville sense it has the following form :

$$({}^{ABR}_{a}D^{\alpha}f)(t) = ({}^{ABR}_{a}D^{\beta}f^{(n)})(t) \tag{5.4}$$

Definition 2: Let $n < \alpha \leq n + 1$ and f be such that $f^{(n)} \in H^1(a, b)$.set $\beta = \alpha - n$. Then $\beta \in [0, 1]$ and we define

$$({}^{ABC}D^{\alpha}_{b}f)(t) = ({}^{ABC}D^{\beta}_{b}(-1)^n f^{(n)})(t) \tag{5.5}$$

and in the right Riemann-Liouville sense it has the following form:

$$({}^{ABR}D^{\alpha}_{b}f)(t) = ({}^{ABR}D^{\beta}_{b}(-1)^n f^{(n)})(t) \tag{5.6}$$

We have the associated fractional integral

$$({}^{AB}I_b^{\alpha} f)(t) = (I_b^{n}{}^{AB}I_b^{\beta} f)(t). \tag{5.7}$$

Lemma 1: Let $\alpha \in (0,1]$ and $f \in H^1(0,\chi)$, if an ABC fractional derivative exists, then we have

$${}^{ABC}_{a}D_t^{\alpha}{}^{AB}_{a}I_t^{\alpha} f(t) = f(t)$$

and

$${}^{AB}_{a}I_t^{\alpha}{}^{ABC}_{a}D_t f(t) = f(t) - f(a).$$

propositions : [3] For $n < \alpha \leq n+1$, for some $n \in N_0$ and $u(t)$ defined on $[0,\chi]$, we have

1) ${}^{ABC}_{a}D_t^{\alpha}{}^{AB}_{a}I_t^{\alpha} u(t) = u(t)$;

2) ${}^{AB}_{a}I_t^{\alpha}{}^{ABC}_{a}D_t^{\alpha} u(t) = u(t) - \sum_{k=0}^{n} \frac{u^{(k)}(a)}{k!}(t-a)^k$;

3) ${}^{AB}_{a}I_t^{\alpha}{}^{ABR}_{a}D_t^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t-a)^k$.

proof: By Definition (1) and Definition (2) we have

$$({}^{ABR}_{a}D^{\alpha}{}^{AB}_{a}I^{\alpha}u)(t) = ({}^{ABR}_{a}D^{\beta}\frac{d^n}{dt^n}{}_{a}I^{n}{}^{AB}_{a}I^{\beta}u)(t) = ({}^{ABR}_{a}I^{\beta}u)(t) = u(t) \tag{5.8}$$

where $\beta = \alpha - n$. By Definition (1) we have

$$({}^{AB}_{a}I^{\alpha}{}^{ABR}_{a}D^{\alpha}u)(t) = ({}_{a}I^{n}{}^{AB}_{a}I^{\beta}{}^{ABR}_{a}D^{\beta}u^{(n)})(t) = {}_{a}I^{n}u^{(n)}(t) = u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)(a)}}{k!}(t-a)^k. \tag{5.9}$$

By Lemma 1 applied to $f(t) = u^{(n)}(t)$ we have

$$({}^{AB}_{a}I^{\alpha}{}^{ABC}_{a}D^{\alpha}u)(t) = {}_{a}I^{n}{}_{a}I^{\beta}{}^{ABC}_{a}D^{\beta}u^{(n)}(t) = {}_{a}I^{n}[u^{(n)}(t) - u^{n}(a)] \tag{5.10}$$

$$= \text{u(t)-}\sum_{k=0}^{n-1}\frac{u^{(k)(a)}}{k!}(t-a)^k - u^{(n)}(a)\frac{(t-a)^n}{n!}$$

$$= \text{u(t)-}\sum_{k=0}^{n}\frac{u^{(k)(a)}}{k!}(t-a)^k.$$

lemma 3 [1] For $n < \alpha \leq n+1, {}_{a}^{ABC}D_{t}^{\alpha}(t-a)^{k} = 0, k = 0, 1, ..., n$. Moreover ,

$${}_{a}^{ABC}D_{t}^{\alpha} f(t) = 0$$

if $f(t)$ is a constant function.
lemma 4 [1] Let $\alpha \in (0,1]$ and $\varpi(0) = 0$, Then the solution of

$${}^{ABC}D_{0}^{\alpha} f(t) = \varpi(t), t \in [0,1],$$

$f(0) = c$
is given by

$$f(t) = c + {}^{AB}I_{0^{+}}^{\alpha} \varpi(t).$$

5.1 Results of applying the Banach's Principle of Contraction theorem to nonlocal problems

This section is devoted to obtaining formula of the solution to problem (5.1)–(5.2). Then we prove the existence and uniqueness of solution for problem (5.1)– (5.2) by means of Schauder's fixed point theorem and Banach's fixed point theorem . Moreover, we also discuss the continuous dependence of solutions to such equations on arbitrary data.
lemma 5 :
Let $0 < \alpha \leq 1$ and $\sum_{k=1}^{m} \beta_k \neq 0$.Then the solution of problem (5.1)–(5.2) can be indicated by the fractional integral equation

$$u(t) = A(u_0 - \sum_{k=1}^{m} \beta_k {}^{AB}I_{0,\tau_k}^{\alpha} F_u(\tau_k)) + {}^{AB}I_{0,t}^{\alpha} F_u(t), \tag{5.11}$$

where F_u is the solution of the functional integral equation

$$F_u(t) = f(t, Au_0 - A\sum_{k=1}^{m} \beta_k {}^{AB}I_{0,\tau_k} F_u(\tau_k) + {}^{AB}I_{0,t}^{\alpha} F_u(t), F_u(t)) \tag{5.12}$$

and

$$A := (\sum_{k=1}^{m} \beta_k)^{-1}.$$

proof :
Set ${}^{ABC}D_{0,t}^{\alpha} u(t) = F_u(t)$ in(5.1). Then we get

$$F_u(t) = f(t, u(t), F_u(t)).$$

Applying ${}^{AB}I_{0,t}^{\alpha}$ on both sides of (5.1) and using Lemma 1, we have

$$u(t) = u(0) + {}^{AB}I^{\alpha}_{0,t}F_u(t). \tag{5.13}$$

putting $t = u_0 = \sum_{k=1}^{m} \beta_k u(0) + {}^{AB}I^{\alpha}_{0,\tau_k}F_u(\tau_k)$,
which implies

$$u(0) = (\sum_{k=1}^{m} \beta_k)^{-1}[u_0 - \sum_{k=1}^{m} \beta_k {}^{AB}I^{\alpha}_{0,\tau_k}F_u(\tau_k)]$$

Since

$$A = (\sum_{k=1}^{m} \beta_k)^{-1}$$

, we get

$$u(t) = A(u_0 - \sum_{k=1}^{m} \beta_k {}^{AB}I^{\alpha}_{0,\tau_k}F_u(\tau_k)) + {}^{AB}I^{\alpha}_{0,t}F_u(t).$$

Here, F_u is the solution of equation $F_u(t) = f(t, u(t), F_u(t)), i.e,$

$$F_u(t) = f(t, Au_0 - A\sum_{k=1}^{m} \beta_k {}^{AB}I^{\alpha}_{0,\tau_k}F_u(\tau_k) + {}^{AB}I^{\alpha}_{0,t}F_u(t), F_u(t)).$$

Now, we consider the following hypotheses:

(H$_1$: There exists a constant L$_1$ > 0 such that)
$|f(t, x, y) - f(t, x', y')| \leq L_1(|x - x'| + |y - y'|)$
for all $t \in [0, \chi]$ and $x, x', y, y' \in R$.

(H$_2$:There exists a constant k>0 such that)
$|f(t, x, y)| \leq k(1 + |x| + |y|)\forall(t, x, y) \in [0, \chi] * R * R.$

5.2 Existence and Uniqueness of the Solution

Theorem 4.2.1
Let $f : [0, \chi]R^2 \to R$ is continuous . If $(H2)$ holds with $k \neq 1$, and

$$\eta_1 := \frac{k}{1-k}[\frac{(|A|\sum_{k=1}^{m}|\beta_K| + 1)(1-\alpha)}{M(\alpha)} + \frac{|A|\sum_{k=1}^{m}|\beta_K|\tau_K^{\alpha} + \chi^{\alpha}}{M(\alpha)\Gamma(\alpha)}] < 1, \tag{5.15}$$

then problem (5.1)–(5.2) has at least one solution $u \in C[0, \chi]$.
Proof :
Let the operator $T : C[0, \chi] \to C[0, \chi]$ by

$$(Tu)(t) = Au_0 - A\sum_{k=1}^{m} \beta_k {}^{AB}I^{\alpha}_{0,\tau_k}F_u(\tau_k) + {}^{AB}I^{\alpha}_{0,t}F_u(t), \tag{5.16}$$

where

$$F_u(t) = f(t, Au_0 - A\sum_{k=1}^{m} \beta_K {}^{AB}I^{\alpha}_{0,\tau_k} F_u(\tau_k) + {}^{AB}I^{\alpha}_{0,t}F_u(t), F_u(t)). \tag{5.17}$$

we have

$$({}_0^{ABC}D_t^{\alpha}T_u)(t) = Au_0({}_0^{ABC}D_t^{\alpha}1)(t)$$

$+{}_0^{ABC}D_t^{\alpha}{}_0^{AB}I_t^{\alpha}F_u(t)$
=F$_u(t)$.
Since $Tu \in C[0,\chi]$ and ${}_0^{ABC}D_t^{\alpha}u(t) = F_u(t)$ in equation (5.1), it follows that

$$({}_0^{ABC}D_t^{\alpha}T_u)(t) = f(t, u(t), {}_0^{ABC}D_t^{\alpha}u(t))).$$

As $f(t, u(t), {}_0^{ABC}D_t^{\alpha}u(t)$ is continous on $[0,\chi]$,then ${}_0^{ABC}D_t^{\alpha}T_u(t) \in C[0,\chi]$.
Let $r \geq \frac{\eta_2}{1-\eta_1}$ and $B_r = u \in C[0, chi] : \|u\| \leq r$,where B_r is a nonempty, closed, convex, and bounded subset of $C[0,\chi]$ and

$$\eta_2 = |Au_0| + \eta_1. \tag{5.18}$$

The proof is presented in numerous steps as follows.
Step 1. $TB_r \subseteq B_r$.
For $t \in [0,\chi]$ we get

$$|Tu(t)| \leq |Au_0| + |A|\sum_{k=1}^{m}|\beta_k|_0^{AB}I_{\tau_k^{\alpha}|F_u(\tau_k|+{}_0^{AB}I_t^{\alpha}|F_u(t)|}$$

$\leq |Au_0| + |A|\sum_{k=1}^{m}|\beta_k|[\frac{1-\alpha}{M(\alpha)}|F_u(\tau_k)| + \frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{(}\tau_k)(\tau_k -$
$s)^{\alpha-1}|F_u(s)|ds]\frac{1-\alpha}{M(\alpha)}|F_u(t)| + \frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha} - 1|F_u(s)|dsand$

$$|F_u(t)| = |f(t, u(t), F_u(t))| \leq k(1 + |u(t)| + |F_u(t)|).$$

This

$$|F_u(t)| \leq \frac{k(1+|u(t)|)}{1-k} \tag{5.19}$$

It follows from (5.15) and (5.18) that, for each $u \in B_r$,

$$|Tu(t)| \leq |Au_0| + |A|\frac{k(1+r)}{1-k}\sum_{k=1}^{m}|\beta_k|(\frac{1-\alpha}{M(\alpha)} + \frac{\tau_k^{\alpha}}{M(\alpha)\Gamma(\alpha)}) + \frac{k(1+r)}{1-k}(\frac{1-\alpha}{M(\alpha)} + \frac{t^{\alpha}}{M(\alpha)\Gamma(\alpha)})$$

$$= |Au_0| + \frac{k}{1-k}[(|A|\sum_{k=1}^{m}|\beta_k| + 1)\frac{(1-\alpha)}{M(\alpha)} + \frac{|A|\sum_{k=1}^{m}|\beta_k|\tau_k^{\alpha} + t^{\alpha}}{M(\alpha\alpha)\Gamma(\alpha)}] +$$

$$\frac{k}{1-k}[(|A|\sum_{k=1}^{m}|\beta_k| + 1)\frac{(1-\alpha)}{M(\alpha)} + \frac{|A|\sum_{k=1}^{m}|\beta_k|\tau_k^{\alpha} + t^{\alpha}}{M(\alpha)\Gamma(\alpha)}]r$$

$\leq \eta_2 + \eta_1 r \leq r.$
*Step*2. T is continuous .
Let u_n be sequence such that $u_n \to u$as $n \to \infty$.Then

$$|Tu_n(t) - Tu(t)| \leq |A| \sum_{k=1}^{m} |\beta_k| {}_0^{AB}I_{\tau_k}^{\alpha} |F_{u_n}(\tau_k) F_u(\tau_k)|$$

$+{}_0^{AB}I_t^{\alpha} |F_{u_n}(t) - F_u(t)|$

$$\leq |A| \sum_{k=1}^{m} |\beta_k| [\frac{1-\alpha}{M(\alpha)} |F_{u_n}(\tau_k) - F_u(\tau_k)| + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^{\tau} {}_k(\tau_k - s)^{\alpha-1} |F_{u_n}(s) - F_u(s)| ds]$$

$$+\frac{1-\alpha}{M(\alpha)} |F_{u_n}(t) - F_u(t)| + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_{u_n}(s) - F_u(s)| ds$$

$$\leq \frac{(|A| \sum_{k=1}^{m} |\beta_k| + 1)(1-\alpha)}{M(\alpha)} \|F_{u_n}(.) - F_u(.)\|$$

$$+\frac{|A| \sum_{k=1}^{m} |\beta_k| \tau_k^{\alpha} + \chi^{\alpha}}{M(\alpha)\Gamma(\alpha)} \|F_{u_n}(.) - F_u(.)\|.$$

Since $F_u(.) = f(., u(.), F_u(.)) \in C[0, \chi]$, it follows that $\|T_{u_n}(.) - T_u(.) \to 0$ as $n \to \infty$, which proves the required result.
Step 3.T is compact
We show that TB_r is relatively compact. Clearly,TB_r is uniformly bounded due to Step 1. It remains to show that $TB_r$is equicontinuous. Let $t_1, t_2 \in [0, \chi]$ such that $0 \leq t_1 \leq t_2 \leq \chi$.Then

$$|Tu(t_2 - Tu(t_1)|$$

$$= |{}_0^{AB}I_{t_2}^{\alpha} F_u(t_2) - {}_0^{AB}I_{t_1}^{\alpha} F_u(t_1)|$$

$$\leq |\frac{1-\alpha}{M(\alpha)} F_u(t_2)| + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t {}_2(t_2 - s)^{\alpha-1} |F_u(s)| ds$$

$$-\frac{1-\alpha}{M(\alpha)} |F_u(t_1)| - \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t {}_1(t_1 - s)^{\alpha-1} |F_u(s)|| ds|$$

$$\leq \frac{1-\alpha}{M(\alpha)} |F_u(t_2) - F_u(t_1)|$$

$$+\frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t {}_1|(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}||F_u(s)| ds$$

$$+\frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |F_u(s)| ds.$$

It is clear from (5.11) that each of them $u \in B_r$

$$|Tu(t_2) - Tu(t_1)| \leq \frac{1-\alpha}{M(\alpha)}|F_u(t_2) - F_u(t_1)|$$

$$+\frac{k(1+r)}{1-k}\frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}|ds$$

$$+\frac{k(1+r)}{1-k}\frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds$$

$$= \frac{1-\alpha}{M(\alpha)}|F_u(t_2) - F_u(t_1)|$$

$$+\frac{k(1+r)}{1-k}\frac{1}{M(\alpha)\Gamma(\alpha)}[t_1^{\alpha} + (t_2-t_1)^{\alpha} - t_2^{\alpha}]$$

$$\frac{k(1+r)}{1-k}\frac{1}{M(\alpha)\Gamma(\alpha)}(t_2-t_1)^{\alpha}$$

$$\leq \frac{1-\alpha}{M(\alpha)}|F_u(t_2) - F_u(t_1)| + \frac{k(1+r)}{1-k}\frac{2(t_2-t_1)^{\alpha}}{M(\alpha)\Gamma(\alpha)}$$

Since $F_u(.) = f(., u(.), F_u(.)) \in C[t_1, t_2]$, it follows that $|T_u(t_2) - T_u(t_1)| \to 0ast_2 \to t_1$. 3 and the Arzela-Ascoli theorem, we arrive at T being Continuous and compact. ABC (5.1) non-local problem - (5.2) has at least one solution in B_r.

Theorem 4.2.2 :
Assume that $f : [0, \chi]R_2 \to R$ is continuous. If (H_1) holds with $L1 = 1$, then problem (5.1)–(5.2) has a unique solution $\in C[0, \chi]$ provided that

$$\Upsilon := \frac{L_1}{1-L_1}[\frac{(|A||\sum_{k=1}^{m}|\beta_k| + 1)(1-\alpha)}{M(\alpha)} + \frac{|A|\sum_{k=1}^{m}|\beta_k|\tau_k^{\alpha} + \chi^{\alpha}}{M(\alpha)\Gamma(\alpha)}] < 1. \tag{5.20}$$

Proof :
Let's prove that T defined by (5.8) has a fixed point
Let u,$u^{'} \in C[0, \chi]$and $t \in [0, \chi]$.Then

$$|(Tu)(t) - (Tu^{'})(t)| \leq |A|\sum_{k=1}^{m}|\beta_k|{}_0^{AB}I_{\tau_k}^{\alpha}|F_u(\tau_k) - F_{u'}(\tau_k)| + {}_0^{AB}I_t^{\alpha}|F_u(t) - F_{u'}(t)| \tag{5.21}$$

We have for each $t \in [0, \chi]$,

$$|F_u(t) - F_{u'}(t)| = |f(t, u(t), F_u(t)) - f(t, u^{'}(t), F_{u'}(t))|$$

$$\leq L_1(|u(t) - u^{'}(t)| + |F_u(t) - F_{u'}(t)|.$$

This

$$|F_u(t) - F_{u'}(t)| \leq \frac{L_1}{1-L_1}|u(t) - u^{'}(t)|. \tag{5.22}$$

Substituting (5.14) into (5.13), we find

$$|(Tu)(t) - (Tu^{'})(t)| \leq \frac{|A|L_1}{1-L_1}\sum_{k=1}^{m}|\beta_k|{}_0^{AB}I_{\tau_k}^{\alpha}|u(\tau_k) - u^{'}(\tau_k)|$$

$$+\frac{L_1}{1-L_1}{}_0^{AB}I_t^{\alpha}|u(t) - u^{'}(t)|$$

$$= \frac{|A|L_1}{1-L_1}\sum_{k=1}^{m}|\beta_k|[\frac{1-\alpha}{M(\alpha)}|u(\tau_k) - u^{'}(\tau_k)| + \frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{\tau_K}(\tau_K - s)^{\alpha-1}|u(s) - u^{'}(s)|ds]$$

$$+\frac{L_1}{1-L_1}[\frac{1-\alpha}{M(\alpha)}|u(t) - u^{'}(t)| + \frac{\alpha}{M(\alpha)}\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}|u(s) - u^{'}(s)|ds]$$

$$\leq \frac{L_1}{1-L_1}[\frac{(|A|\sum_{k=1}^{m}|\beta_k|+1)(1-\alpha)}{M(\alpha)}\frac{|A|\sum_{k=1}^{m}|\beta_k|\tau_k^{\alpha} + \chi^{\alpha}}{M(\alpha)\Gamma(\alpha)}]\|u - u^{'}\|.$$

Hence, by (5.12), T is the contraction. and we deal with- Suppose T has a fixed point which is a solution to the problem (5.1) - (5.2)

5.3 Example :

Let the following non-local problem be of type ABC:

$${}_0^{ABC}D^{\frac{1}{3}}u(t) = \frac{t^2}{8}(1 + \frac{|u(t)| + |{}_0^{ABC}D^{\frac{1}{3}}|u(t)|}{1 + |u(t)| + |{}_0^{ABC}D^{\frac{1}{3}}u(t)|}), t \in [0, \frac{1}{2}] \tag{5.23}$$

with nonlocal conditions

$$\frac{1}{8}u(\frac{1}{6}) + \frac{3}{8}u(\frac{1}{4}) + \frac{1}{2}u(\frac{1}{3}) = 1, \tag{5.24}$$

where $0 < (\tau_1 = \frac{1}{6}, \tau_2 = \frac{1}{4}, \tau_3 = \frac{1}{3}) < \frac{1}{2}, (\beta_1 = \frac{1}{8}, \beta_2 = \frac{3}{8}, \beta_3 = \frac{1}{2}) > 0 (k = 1, 2, 3)(m = 3), \alpha = \frac{1}{3}$.
set $f(t, u, v) = \frac{t^2}{8}(1 + \frac{u}{(1+u+v)} + \frac{v}{(1+u+v)})$for $(t, u, v) \in [0, \frac{1}{2}]R^2$Clearly, the function $f(0, u(0), v(0)) = 0$.Let $u, u^{'}, v, v^{'} \in R$and $t \in [0, \frac{1}{2}]$.Then we have

$$|f(t, u, v)| \leq \frac{t^2}{8}(1 + \frac{|u|}{(1+|u|+|v|)} + \frac{|v|}{(1+|u|+|v|)}) \leq \frac{1}{8}(1 + |u| + |v|).$$

. By choosing $M(\frac{1}{3}) = 1$ we canfind that $\eta_1 \approx 0.23 < 1$,It follows from Theorem 4.2.1 that problem (5.23)-(5.24) has a solution on $[0, \frac{1}{2}]$.

Conclusion

We have presented The definition of fractional derivation with the concept of Atangana-Baleanu We presented some results existences and uniqueness of solutions to certain Cauchy problems of derivative equations of fractional orders. We have treated linear cases by the Laplace transform of fractional derivatives The problem of the existence of solutions fractional order derivative equations remain among the important problems that require much more research.

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Abstract

The definition of Atangana-Baleanu of fractional derivative is one of the latest definitions to improve the fractional derivative. This definition has been applied to some Cauchy fractional problems and some nonlocal implicit problems. It deals with linear cases with Laplace transform , Banach fixed point theorem.

Keywords and phrases main: nonlocal non singular kernel, Atangana-Baleanu fractional derivative ,Atangana-Baleanu fractional integral ,nonlocal implicit problem ,fixed point .

Résumé

La définition Atangana-Baleanu de la dérivation fractionnaire est l'une des dernières définitions pour améliorer la dérivation fractale. Cette définition a été appliquée à certains problèmes fractionnaires de Cauchy et à certains problèmes implicites non locaux. Elle traite des cas linéaires avec transformée de Laplace et théorème du point fixe de Banach.

Mots et phrases clés : noyau non singulier non local ,dérivée fractionnaire d'Atangana-Baleanu, intégrale fractionnaire d'Atangana-Baleanu ,non local problème implicite ,point fixe .

الملخص

يعد تعريف أطنجانا بلاينو للاشتقاق الكسري أحد أحدث التعريفات لتحسين الاشتقاق الكسري ، هذا التعريف تم تطبيقه على بعض مسائل كوشي الخطية ، درسنا وجود ووحدانية حلول المعادلات الخطية بواسطة تحويل لابلاس ، ودرسنا مسألة ضمنية غير محلية بواسطة أنماط النقطة الثابتة لباناخ .

الكلمات والعبارات المفتاحية : النواة غير الشادة الغير محلية ، مشتق أطنجانا بلاينو الكسري ، تكامل أطنجانا بلاينو الكسري ، مسألة ضمنية غير محلية ، النقطة الثابتة .