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The Maximum Principle : Theoretical and Numerical Applications

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Abstract

This dissertation aims to study the maximum principle in partial differential equations and to highlight its theoretical and practical importance. First, the fundamental concepts related to partial differential equations, Sobolev spaces, and weak solutions are introduced. Then, the weak and strong maximum principles are presented along with some fundamental results such as Hopf's lemma and the uniqueness of solutions. The study also discusses applications of this principle to the heat equation and certain boundary value problems. On the numerical side, approximation methods were adopted to study the behavior of solutions and verify the theoretical results through simulations and graphical representations. The obtained results showed that the numerical solutions are consistent with the theoretical properties of the maximum principle, confirming its effectiveness in the analysis of partial differential equations and the study of the stability of their solutions.

Key words: Maximum principle, Partial differential equations, Elliptic equations, Parabolic equations, Finite difference methods, Discrete maximum principle, Stability, Convergence.

ملخص

تهدف هذه المذكرة إلى دراسة مبدأ القيم العظمى في المعادلات التفاضلية الجزئية مع إبراز أهميته النظرية والتطبيقية. تم في البداية تقديم المفاهيم الأساسية المتعلقة بالمعادلات التفاضلية الجزئية، فضاءات سوبولاف، والحلول الضعيفة. بعد ذلك تم التطرق إلى مبدأ القيم العظمى الضعيف والقوي مع عرض بعض النتائج الأساسية مثل مبرهنة Hopf ووحدانية الحلول. كما تناولت الدراسة تطبيقات هذا المبدأ على معادلة الحرارة وبعض مسائل القيم الحدية. وفي الجانب العددي، تم اعتماد طرق تقريبية لدراسة سلوك الحلول والتحقق من النتائج النظرية بواسطة المحاكاة والرسم البياني. أظهرت النتائج توافق الحلول العددية مع الخصائص النظرية لمبدأ الأعظمية، مما يؤكد فعاليته في تحليل المعادلات التفاضلية الجزئية ودراسة استقرار حلولها.

الكلمات المفتاحية: مبدأ القيم العظمى، المعادلات التفاضلية الجزئية، المعادلات الإهليلجية، المعادلات المكافئة، طريقة الفروق المنتهية، مبدأ القيم العظمى المتقطع، الاستقرار، التقارب.

Résumé

Ce mémoire a pour objectif d'étudier le principe du maximum dans les équations aux dérivées partielles tout en mettant en évidence son importance théorique et pratique. Dans un premier temps, les notions fondamentales relatives aux équations aux dérivées partielles, aux espaces de Sobolev et aux solutions faibles ont été présentées. Ensuite, le principe du maximum faible et fort a été étudié, avec l'exposition de quelques résultats fondamentaux tels que le lemme de Hopf et l'unicité des solutions. Cette étude aborde également les applications de ce principe à l'équation de la chaleur ainsi qu'à certains problèmes aux limites. Sur le plan numérique, des méthodes d'approximation ont été adoptées afin d'étudier le comportement des solutions et de vérifier les résultats théoriques à l'aide de simulations et de représentations graphiques. Les résultats obtenus ont montré la concordance des solutions numériques avec les propriétés théoriques du principe du maximum, ce qui confirme son efficacité dans l'analyse des équations aux dérivées partielles et dans l'étude de la stabilité de leurs solutions.

Mots-clés: Principe du maximum, Équations aux dérivées partielles, Équations elliptiques, Équations paraboliques, Méthodes des différences finies, Principe du maximum discret, Stabilité, Convergence.

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Dedication

I dedicate this work to:

My father Djfafla Mohammed, My mather Boukhelfa Sadia whose prayers, love, sacrifices, and endless support have always been the source of my strength and success.

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Sabrinal Djefafla

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Notations

Ω	:	An open interval of $\mathbb{R}$.
$\partial\Omega$	:	Boundary of Ω.
$\bar{\Omega} = \Omega \cup \partial\Omega$	:	The adhesion of Ω.
$\dfrac{\partial u}{\partial t}, u_t$	:	Partial derivative of u with respect to t.
$\dfrac{\partial u}{\partial x}, u_x$	:	Partial derivative of u with respect to x.
$\dfrac{\partial^2 u}{\partial x^2}, u_{xx}$	:	Second partial derivative of u with respect to x.
∇u	:	$\left(\frac{\partial u}{\partial x_1}(x), \frac{\partial u}{\partial x_2}(x), ..., \frac{\partial u}{\partial x_n}(x)\right) = Grad\, u,\ x \in \Omega \subset \mathbb{R}^N$.
$\nabla.u$	:	$\sum_{i=1}^{i=n} \frac{\partial u}{\partial x_i}(x) = div\, u,\ x \in \Omega \subset \mathbb{R}^N$.
Δu	:	$\sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2}(x) =$ Laplacien operator of u, $x \in \Omega \subset \mathbb{R}^N$.
$L^2\left(]0,1[\times]0,t_0[\right)$	:	$\left\{u :]0,1[\times]0,t_0[\to \mathbb{R}; u \text{ is mesurable and } \int_0^{t_0}\int_0^1 \lvert u(x,t)\rvert^2 < \infty\right\}$.
$C(\Omega)$	:	Set of continuous functions on Ω.
$C^k(\Omega)$	:	Space of k times continuousely differentiable functions on Ω.
$a.e.$	:	almost everywhere.
$u_i = u(x_i, t)$	:	Semi discrete approximation of u at point x_i.
$u_i^n = u(x_i, t_n)$	:	Discrete approximation of u at point (x_i, t_n).
$L^1(\Omega)$	:	Integrable functions.
$L^1_{\text{loc}}(\Omega)$	:	Locally integrable functions.
$supp(f)$	:	Support of function f.
$C_c^\infty(\Omega)$	:	The set of smooth functions with compact support.

Introduction

The Maximum Principle is considered one of the most fundamental principles in the theory of partial differential equations. It has played a central role in studying the qualitative properties of solutions, such as existence, uniqueness, stability, and a priori estimates. The early foundations of this principle date back to the work of mathematicians in the nineteenth century, particularly in the study of elliptic equations related to potential theory. The first essential contributions to this concept are attributed to Pierre-Simon Laplace and Carl Neumann, and it was later developed further through the work of Henri Poincaré, Eberhard Hopf, and other researchers in mathematical analysis and partial differential equations.[1, 3].

The Maximum Principle has become an essential tool in modern analysis because it allows the study of solution behavior without requiring explicit knowledge of the solutions themselves. The core idea of this principle is that the maximum or minimum value of a solution to a partial differential equation is typically attained on the boundary of the domain, rather than in its interior, under certain conditions[4]. . This property is of great importance in understanding the physical nature of phenomena modeled by partial differential equations, such as heat transfer, wave propagation, fluid motion, and various diffusion processes.

Moreover, the importance of the Maximum Principle is not limited to the theoretical aspect; it also extends to numerical and computational applications. In recent decades, increasing attention has been devoted to studying numerical schemes that preserve this principle when approximating solutions, leading to the concept of the Discrete Maximum Principle. This concept is essential for ensuring the stability of numerical methods and for preserving the physical properties of approximate solutions, particularly when using finite difference and finite element methods to solve partial differential equations[5]. .

The aim of this thesis is to study the Maximum Principle from both theoretical and numerical perspectives, highlighting its main applications in elliptic and parabolic partial differential equations. We also aim to investigate certain numerical methods that preserve this principle and analyze their properties in terms of consistency, stability, and convergence.

This thesis is organized into three main chapters:

The first chapter presents some fundamental concepts and results in functional analysis that are necessary for understanding the subsequent theoretical study. We review normed spaces and Banach spaces, followed by Sobolev spaces and key inequalities such as Poincaré's inequality and Sobolev inequalities. We also discuss differential operators of elliptic, parabolic, and hyperbolic types, along with the definitions of classical and weak solutions. Finally, we introduce the main boundary and initial conditions, such as Dirichlet, Neumann, and Robin conditions.

The second chapter is devoted to the theoretical study of the Maximum Principle. We begin by presenting the classical Maximum Principle in its weak and strong forms, along with Hopf's lemma. We then study applications of this principle to elliptic equations, particularly Laplace and Poisson equations, as well as second-order linear elliptic operators. Next, we address the Maximum Principle for parabolic equations such as the heat equation, in addition to comparison principles. At the end of the chapter, we present the main consequences of this principle, including proofs of uniqueness of solutions, derivation of a priori estimates, and stability results.

The third chapter focuses on the numerical aspect of the Maximum Principle. We study the concept of the Discrete Maximum Principle within the framework of finite difference methods. We begin by presenting the numerical discretization of partial differential equations, followed by a discussion of consistency, stability, and convergence. We then examine sufficient conditions for ensuring the validity of the Discrete Maximum Principle, along with the study of numerical schemes that preserve it. Finally, we provide several applications and numerical experiments illustrating the effectiveness of these methods and their importance in obtaining stable and physically meaningful numerical solutions.

Through this work, we aim to establish a connection between the theoretical and numerical aspects, while emphasizing the fundamental role played by the Maximum Principle in the study of partial differential equations and their various scientific applications.

Chapter 1

Mathematical preliminaries

Functional analysis provides the fundamental mathematical framework for the study of partial differential equations and the analysis of the properties of their solutions. In order to understand the maximum principle and its applications, it is necessary to introduce some essential mathematical concepts and tools used in this field. Therefore, in this chapter, we present the main functional spaces and fundamental inequalities, together with differential operators, types of solutions, and boundary and initial conditions, which constitute the basis for the theoretical and numerical study developed in the following chapters.

1.1 Functional analysis tools

1.1.1 Banach and Hilbert spaces

Definition 1.1 (Metric space)**.** *Let X be a set. A **metric** (or distance) on X is a function*

$$d : X \times X \longrightarrow \mathbb{R}_+$$

such that for all $x, y, z \in X$, the following properties hold:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$,

4. $d(x,y) \leq d(x,z) + d(z,y)$ *(triangle inequality).*

*The pair (X,d) is called a **metric space**.*

Definition 1.2 (Convergent sequence)**.** *Let (X,d) be a metric space. A sequence (x_n) in X is said to **converge** to $x \in X$ if*

$$\lim_{n\to+\infty} d(x_n, x) = 0.$$

If the limit exists, it is unique.

Definition 1.3 (Cauchy sequence)**.** *Let (X,d) be a metric space. A sequence (x_n) in X is called a **Cauchy sequence** if*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N,\ d(x_n, x_m) < \varepsilon.$$

Definition 1.4 (Complete metric space)**.** *A metric space (X,d) is said to be **complete** if every Cauchy sequence in X converges to an element of X.*

Definition 1.5. *Let E be a vector space over a field $\mathbb{K}$. A mapping $N : E \to \mathbb{R}_+$ is called a **semi-norm** on E if and only if the following two conditions are satisfied:*

1. *$N(x+y) \leq N(x) + N(y)$, for all $x, y \in E$,*

2. *For every $\lambda \in \mathbb{K}$ and every $x \in E$,*

$$N(\lambda x) = |\lambda| N(x).$$

Definition 1.6. *A mapping $N : E \to \mathbb{R}_+$ is called a **norm** on E if it is a semi-norm and satisfies the additional condition*

$$N(x) = 0 \Rightarrow x = 0.$$

Definition 1.7. *Let E be a vector space and let N be a norm on E. The pair (E,N) is called a **normed space**.*

Proposition 1.1. *Let $(E, \|\cdot\|)$ be a normed space. Then the mapping*

$$d : E \times E \to \mathbb{R}_+$$

$$(x,y) \mapsto \|x-y\|$$

*is a metric on E, called the **metric associated with the norm** $\|\cdot\|$.*

Definition 1.8 (Banach space)**.** *Let $(E, \|\cdot\|)$ be a normed space. We say that $(E, \|\cdot\|)$ is a* ***Banach space*** *if the metric space (E, d), where the distance d is defined by*

$$d(x, y) = \|x - y\|,$$

is complete.

Exemple 1.1.

1. *The space of real numbers $\mathbb{R}$, $\|x\| = |x|, \quad \forall x \in \mathbb{R}$.*
2. *The space of complex numbers $\mathbb{C}$, $\|z\| = |z| = \sqrt{z\overline{z}}, \quad \forall z \in \mathbb{C}$.*
3. *The space $\mathbb{R}^n$ for every positive integer $n \in \mathbb{N}^*$, $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad$ where $x = (x_1, x_2, \ldots, x_n)$.*

Definition 1.9 (Inner product)**.** *Let E be a real (or complex) vector space. We say that the mapping*

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}, \quad (or\ \mathbb{C}),$$

is an inner product if it satisfies:

1. ***Linearity in one argument:***
$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle.$$
2. ***Symmetry:***
$$\langle x, y \rangle = \langle y, x \rangle.$$
3. ***Positive definiteness:***
$$\langle x, x \rangle \geq 0 \quad and \quad \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

Definition 1.10 (Hilbert space)**.** *A vector space H equipped with an inner product $\langle \cdot, \cdot \rangle$ is called a* ***Hilbert space*** *if:*

H is a complete inner product space. That is, every Cauchy sequence in H converges to an element in H.

For more details, the reader is referred to [1, 3, 4, 5].

1.1.2 Sobolev spaces L^p

Let $\Omega \subset \mathbb{R}^n$ be a measurable open domain.

Definition 1.11. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. We denote by $L^1(\Omega)$ the space of (equivalence classes of) measurable real-valued functions f on Ω such that*

$$\int_\Omega |f(x)|\, d\mu(x) < \infty.$$

That is,

$$L^1(\Omega) = \left\{ f \text{ measurable} \;\middle|\; \int_\Omega |f|\, d\mu < \infty \right\}.$$

The norm on $L^1(\Omega)$ is defined by

$$\|f\|_{L^1} = \int_\Omega |f(x)|\, d\mu(x).$$

Definition 1.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $1 < p < \infty$. We define*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ is measurable and } \int_\Omega |f(x)|^p\, d\mu(x) < \infty \right\}.$$

The norm on $L^p(\Omega)$ is defined by

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p\, d\mu(x) \right)^{1/p}.$$

It can be shown that $\|\cdot\|_p$ defines a norm on $L^p(\Omega)$.

Definition 1.13. *[12] We define*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and essentially bounded} \}.$$

The norm on $L^\infty(\Omega)$ is defined by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| = \inf\{ C > 0 \;;\; |f(x)| \le C \text{ a.e. on } \Omega \}.$$

The functional $\|\cdot\|_\infty$ defines a norm on $L^\infty(\Omega)$.

Notation 1. *Let $1 \le p \le \infty$. The conjugate exponent p' is defined by*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

1.1.3 L^2 space

The fundamental example is the space $L^2(\Omega)$ of square-integrable functions

$$L^2(\Omega) = \left\{ f \text{ measurable on } \Omega \;\middle|\; \int_\Omega |f|^2 < \infty \right\},$$

with norm

$$\|f\|_{L^2} = \left(\int_\Omega |f(x)|^2 \, dx \right)^{1/2}.$$

The associated inner product in $L^2(\Omega)$ is defined by

$$(u, v)_{L^2} = \int_\Omega u(x)v(x) \, dx.$$

We identify two functions in $L^2(\Omega)$ if they are equal almost everywhere

$$u = v \text{ in } L^2(\Omega) \quad \Longleftrightarrow \quad u(x) = v(x) \text{ for almost every } x \in \Omega.$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open domain, equipped with its natural inner product. Then $L^2(\Omega)$ is a Hilbert space.*

Definition 1.14. *$C_c^\infty(\Omega)$ is the set of smooth functions with compact support, where*

$$supp(f) = \{x / f(x) \neq 0\}.$$

Theorem 1.2. *The space $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, i.e*

$$\forall u \in L^2(\Omega), \quad \exists (u_n) \subset C_c^\infty(\Omega) \text{ such that } u_n \to u \text{ in } L^2(\Omega).$$

1.1.4 Weak derivatives

Definition 1.15 (Locally integrable functions)**.** *[6] A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **locally integrable** if its restriction to every compact set (or every bounded interval in $\mathbb{R}$) is integrable, i.e.*

$$\int_a^b |f(t)| \, dt < \infty \quad \text{for all } a < b.$$

The space of locally integrable functions is denoted by $L^1_{\text{loc}}(\Omega)$.

Exemple 1.2.

1. *Every continuous function is locally integrable.*

2. *The function*

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is not locally integrable near $x = 0$.

Let $u \in C^1(\Omega)$ and $\varphi \in C_0^1(\Omega)$. By integration by parts,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi \frac{\partial u}{\partial x_i}\, dx, \qquad i = 1, \dots, n.$$

The left-hand side is meaningful if $u \in L^1_{\text{loc}}(\Omega)$, whereas the right-hand side requires $u \in C^1(\Omega)$.

Definition 1.16. *Let $u \in L^1_{\text{loc}}(\Omega)$. If there exists a function $v_i \in L^1_{\text{loc}}(\Omega)$ such that*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega v_i \varphi\, dx, \qquad \forall \varphi \in C_0^\infty(\Omega),$$

then v_i is called the weak partial derivative of u with respect to x_i.

More generally, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, a function u admits a weak derivative $D^\alpha u$ if there exists v_α such that

$$\int_\Omega u\, D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi\, dx, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

1.1.5 Sobolev spaces $H^m(\Omega)$

Definition 1.17 ($H^1(\Omega)$space)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. We define*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1, \dots, n,\ \textit{in the weak sense} \right\}.$$

The inner product on $H^1(\Omega)$ is defined by

$$(u, v)_{H^1} = \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega uv\, dx,$$

and the associated norm is

$$\|u\|_{H^1} = \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx\right)^{1/2}.$$

Equipped with this inner product, $H^1(\Omega)$ is a Hilbert space.

Definition 1.18 ($H_0^1(\Omega)$ space)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We define*

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}.$$

If Ω has a Lipschitz boundary, then every function in $H_0^1(\Omega)$ has zero trace on $\partial\Omega$.

Moreover, the seminorm

$$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$$

defines a norm on $H_0^1(\Omega)$ which is equivalent to the usual H^1 norm, thanks to the Poincaré inequality.

Definition 1.19 (Higher order Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $m \in \mathbb{N}$. We define*

$$H^m(\Omega) = \left\{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega),\ \forall \alpha \in \mathbb{N}^n,\ |\alpha| \leq m,\ \textit{in the weak sense}\right\}.$$

The norm and semi-norm on $H^m(\Omega)$ are defined by

$$\|v\|_{m,\Omega} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|^2_{L^2(\Omega)}\right)^{1/2},$$

$$|v|_{m,\Omega} = \left(\sum_{|\alpha| = m} \|D^\alpha v\|^2_{L^2(\Omega)}\right)^{1/2}.$$

$H^m(\Omega)$ is a Hilbert space with this norm.

Definition 1.20. *$H_0^m(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^m(\Omega)$.*

1.1.6 Sobolev space $W^{m,p}(\Omega)$

Definition 1.21. *More generally, for every $1 \leq p \leq +\infty$ and for every $m \in \mathbb{N}$, the Sobolev spaces are defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega)\ ;\ D^\alpha u \in L^p(\Omega),\ \forall \alpha \in \mathbb{N}^n,\ |\alpha| \leq m\}.$$

These spaces are endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p \, dx \right)^{1/p}.$$

The space $\left(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)}\right)$ is a Banach space.

In particular case $p = 1$, we have

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ i = 1, \ldots, n \right\}.$$

Theorem 1.3. *For $1 \leq p \leq \infty$:*

1. *The norm on $W^{1,p}(\Omega)$ is given by*

$$\|u\|_{W^{1,p}} = \begin{cases} \left(\|u\|_{L^p}^p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^p \right)^{1/p}, & p < \infty, \\ \max\left(\|u\|_{L^\infty}, \left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty} \right), & p = \infty. \end{cases}$$

2. *If $p = 2$, then $H^1(\Omega)$ is a Hilbert space with the inner product defined above.*

3. *If $1 < p < \infty$, then $W^{1,p}(\Omega)$ is reflexive.*

4. *If $1 \leq p < \infty$, then $W^{1,p}(\Omega)$ is separable.*

Definition 1.22 ($W_0^{1,p}(\Omega)$ space)**.** *Let $1 \leq p < \infty$. We define*

$$W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p}(\Omega)},$$

that is, the closure of $C_c^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$.

We set

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}(\Omega)$ equipped with the $W^{1,p}$ norm is a separable Banach space and is reflexive if $1 < p < \infty$. In the case $p = 2$, the space $H_0^1(\Omega)$ is a Hilbert space endowed with the H^1 inner product.

Remark 1.1. *If $\Omega = \mathbb{R}^N$, since $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we obtain*

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N).$$

If $\Omega \neq \mathbb{R}^N$, then in general

$$W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega).$$

However, if $\mathbb{R}^N \setminus \Omega$ is sufficiently thin and $p < N$, then equality may hold in special cases.

Lemma 1.1. *If $u \in W^{1,p}(\Omega)$ and $\operatorname{supp}(u)$ is a compact subset of Ω, then*

$$u \in W_0^{1,p}(\Omega).$$

1.1.7 Sobolev–Poincaré inequalities

Theorem 1.4 (Poincaré)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is bounded and $1 \leq p < \infty$. Then there exists a constant $c = c(p)$ such that*

$$\int_\Omega |u|^p \, dx \leq c \, (\operatorname{diam}(\Omega))^p \int_\Omega |Du|^p \, dx,$$

for every $u \in W_0^{1,p}(\Omega)$.

Remark 1.2. *The above inequality implies that $W_0^{1,p}(\Omega) \subset L^p(\Omega)$ for every $1 \leq p < \infty$, provided that $\Omega \subset \mathbb{R}^n$ is bounded.*

Another important consequence is that if $Du = 0$ almost everywhere and $u \in W_0^{1,p}(\Omega)$, then $u = 0$ almost everywhere.

Theorem 1.5 (Sobolev inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $1 \leq p < n$.*

Then there exists a constant $C = C(n, p, \Omega)$ such that

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|Du\|_{L^p(\Omega)},$$

for every $u \in W_0^{1,p}(\Omega)$*, where*

$$p^* = \frac{np}{n-p}$$

is called the Sobolev critical exponent.

Remark 1.3. *This inequality implies the continuous embedding*

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

for $1 \leq p < n$*.*

Remark 1.4. *The Poincaré inequality implies that the seminorm*

$$|u|_{W^{1,p}} = \|\nabla u\|_{L^p(\Omega)}$$

is equivalent to the full $W^{1,p}$ *norm on* $W_0^{1,p}(\Omega)$*. Therefore, on* $W_0^{1,p}(\Omega)$ *we may define the norm by*

$$\|u\| = \|\nabla u\|_{L^p(\Omega)},$$

which is equivalent to the standard Sobolev norm.

Remark 1.5. *The Poincaré inequality plays a fundamental role in obtaining a priori estimates for solutions of elliptic and parabolic problems.*

In particular, it allows us to control the L^p *norm of a solution by its gradient, which is essential in energy methods and in the proof of existence and uniqueness results.*

For more details on functional analysis tools, the reader is referred to [1, 3, 5, 6, 8, 20].

1.2 Differential operators

Definition 1.23 (Differential operator)**.** *A* ***differential operator*** *is an operator that acts on a function by taking its derivatives. In general, a differential operator is expressed as a linear combination of derivatives of different orders with coefficient functions.*

More precisely, let $\Omega \subset \mathbb{R}^n$ *be an open set and let* u *be a sufficiently smooth function defined on* Ω*. A* ***linear differential operator of order*** m *is defined by*

$$Lu(x) = \sum_{|\alpha| \leq m} a_\alpha(x)\, D^\alpha u(x),$$

where

- $a_\alpha(x)$ *are given coefficient functions defined on* Ω,

- $\alpha = (\alpha_1, \dots, \alpha_n)$ *is a multi-index,*

- $|\alpha| = \alpha_1 + \cdots + \alpha_n$,

- $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

The ***order of the differential operator*** *is the highest order of derivatives appearing in the operator.*

1.2.1 Classification of second-order PDEs

Definition 1.24. *[10] Let* $\Omega \subset \mathbb{R}^n$ *be an open set. A linear second-order differential operator is given by*

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

where the coefficients a_{ij}, b_i, c *are given functions on* Ω.

Remark 1.6. *We often assume that*

$$a_{ij}(x) = a_{ji}(x),$$

so that the matrix $A(x) = (a_{ij}(x))$ *is symmetric.*

Definition 1.25 (Elliptic operator)**.** *The operator* L *is called uniformly elliptic if there exists* $\theta > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ x \in \Omega. \tag{1.1}$$

Exemple 1.3. *An important example is the Laplacian operator*

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

This operator appears in many partial differential equations such as the Laplace equation and the Poisson equation [8].

Definition 1.26 (Hyperbolic operator)**.** *[10] A partial differential equation is called hyperbolic if the eigenvalues of the matrix $A(x)$ associated with the second-order terms have different signs.*

These equations typically describe wave propagation phenomena.

Exemple 1.4. *A classical example of a hyperbolic equation is the wave equation*

$$u_{tt} - \Delta u = 0.$$

Definition 1.27 (Parabolic operator)**.** *[10] A second-order operator is called parabolic if the matrix $A(x)$ is positive semi-definite, i.e.,*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^n,\ x \in \Omega.$$

Exemple 1.5. *The classical example of a parabolic equation is the heat equation*

$$u_t - \Delta u = 0.$$

1.2.2 Weak and classical solutions

Motivation of the weak formulation: Let $\Omega \subset \mathbb{R}^n$ and consider the boundary-value problem

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where L is a second-order elliptic operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u.$$

Assume $a_{ij}, b_i, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

Idea: Suppose that u is sufficiently smooth. Multiply the equation by a test function $v \in C_c^\infty(\Omega)$ and integrate over Ω

$$\int_\Omega Lu\, v\, dx = \int_\Omega fv\, dx.$$

Using integration by parts on the highest-order term and using $v = 0$ on $\partial\Omega$, we obtain

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} v + cuv \right) dx = \int_\Omega fv\, dx.$$

No boundary terms appear because the test function vanishes on the boundary.

By density arguments, the same identity remains valid for all $v \in H_0^1(\Omega)$ provided that $u \in H_0^1(\Omega)$.

Bilinear form: We define the bilinear form associated with L by

$$B(u,v) := \int_\Omega \left(\sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b_i u_{x_i} v + cuv \right) dx, \quad u, v \in H_0^1(\Omega).$$

This form is called the bilinear form associated with the operator L.

Definition 1.28 (Weak solution)**.** *[8, 13] A function $u \in H_0^1(\Omega)$ is called a weak solution of the boundary-value problem if*

$$B(u,v) = (f,v), \quad \textit{for all } v \in H_0^1(\Omega),$$

where

$$(f,v) = \int_\Omega f v \, dx,$$

denotes the inner product in $L^2(\Omega)$.

This identity is called the variational formulation of the problem.

Definition 1.29 (Classical solution)**.** *[8, 13] A function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is called a classical solution if it satisfies*

$$Lu = f \quad \textit{pointwise in } \Omega, \quad u = 0 \ \textit{on } \partial\Omega.$$

Nonhomogeneous boundary conditions:

If $u = g \quad$ on $\partial\Omega$, and $\partial\Omega$ is of class C^1, assume that g is the trace of some function $w \in H^1(\Omega)$. Define $\tilde{u} = u - w$. Then $\tilde{u} \in H_0^1(\Omega)$ and satisfies

$$L\tilde{u} = f - Lw \quad \text{in } \Omega, \qquad \tilde{u} = 0 \quad \text{on } \partial\Omega.$$

Thus nonhomogeneous boundary conditions can be reduced to the homogeneous case.

1.2.3 Lax–Milgram theorem

First, we introduce the following notations: Let $u, v \in H_0^1(]0,1[)$, and define

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx, \qquad L(v) = \int_0^1 f(x)v(x)\,dx.$$

The mappings a and L are respectively a continuous bilinear form and a continuous linear functional on $H_0^1(]0,1[)$, and the problem

$$\text{Find } u \in H_0^1(]0,1[) \text{ such that } a(u,v) = L(v) \quad \forall v \in H_0^1(]0,1[),$$

is equivalent to the abstract formulation. The Lax–Milgram theorem guarantees that this problem admits a unique solution under very general conditions.

Let V be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Assume the following:

(H1) **Continuity:** There exists $M \in \mathbb{R}$ such that

$$|a(u,v)| \leq M\|u\|\,\|v\|, \quad \forall u, v \in V.$$

(H2) **Coercivity:** There exists $\alpha > 0$ such that

$$a(u,u) \geq \alpha\|u\|^2, \quad \forall u \in V.$$

Theorem 1.6 (Lax–Milgram theorem)**.** *[13] If the bilinear form a satisfies (H1)–(H2) and L is a continuous linear functional on V, then there exists a unique $u \in V$ such that*

$$a(u,v) = L(v), \quad \forall v \in V.$$

Definition 1.30 (Classical solution)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain, and consider a second-order PDE*

$$\mathbb{L}[u](x) = f(x), \quad x \in \Omega,$$

with boundary conditions on $\partial\Omega$. A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is called a classical solution if:

1. *u satisfies the PDE pointwise in Ω.*

2. *u satisfies the boundary conditions pointwise on $\partial\Omega$.*

Remark 1.7.

1. *The classical solution differs from a weak solution, which allows derivatives to exist in a weak sense (i.e., in Sobolev spaces) and not necessarily pointwise.*

2. *If a weak solution belongs to an appropriate Sobolev space and satisfies certain regularity conditions, it can be shown that the weak solution is also a classical solution.*

1.3 Boundary and initial conditions

In the study of differential equations, boundary conditions play a fundamental role in ensuring the well-posedness and uniqueness of solutions. In this section, we present the most common types of boundary conditions for second-order differential equations, namely Dirichlet, Neumann, and Robin conditions, together with their corresponding variational formulations.

1.3.1 Dirichlet conditions

We consider the one-dimensional Poisson problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} u''(x) = f(x), \quad x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

To derive the variational formulation, let $v \in C^1([0,1])$ be a test function such that $v(0) = v(1) = 0$. Multiplying the equation by v and integrating over $(0,1)$, we obtain

$$-\int_0^1 u''(x)v(x), dx = \int_0^1 f(x)v(x), dx.$$

Using integration by parts, we get

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \quad \forall v \in H_0^1(0,1).$$

We introduce the Sobolev space

$$H_0^1(0,1) = \left\{u \in L^2(0,1); u' \in L^2(0,1), u = 0 \text{ on } \partial(0,1)\right\}.$$

Definition 1.31. *The integral or variational formulation of the problem (1.2) consists in finding a function $u \in H_0^1(0,1)$ such that*

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx, \quad \forall v \in H_0^1(0,1). \tag{1.3}$$

Definition 1.32. *A function u satisfying the variational formulation (1.3) is called a* ***weak solution*** *of the original differential problem.*

Dirichlet boundary conditions prescribe the values of the solution on the boundary [16].

1.3.2 Neumann conditions

We now consider the problem where the boundary conditions involve the derivative of the solution

$$\begin{cases} u''(x) + u(x) = f(x), \quad x \in (0,1), \\ u'(0) = \alpha, \quad u'(1) = \beta. \end{cases} \tag{1.4}$$

Multiplying the equation by a test function $v \in C^1([0,1])$ and integrating, we obtain

$$\int_0^1 -u''(x)v(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

After integration by parts, we get

$$\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx + \beta v(1) - \alpha v(0).$$

We introduce the space

$$H^1(0,1) = \left\{u \in L^2(0,1); u' \in L^2(0,1)\right\}.$$

The variational formulation is:

Find $u \in H^1(0,1)$ such that

$$\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx + \beta v(1) - \alpha v(0), \quad \forall v \in H^1(0,1). \tag{1.5}$$

Neumann boundary conditions prescribe the flux (derivative) of the solution at the boundary [16].

1.3.3 Robin conditions

Finally, we consider boundary conditions that combine both the function and its derivative

$$\begin{cases} u''(x) + u(x) = f(x), \quad x \in (0,1), \\ u'(0) + \alpha u(0) = 0, \\ u'(1) + \beta u(1) = 0. \end{cases} \tag{1.6}$$

Multiplying by a test function v and integrating, we obtain

$$\int_0^1 -u''(x)v(x)\,dx + \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

After integration by parts

$$\int_0^1 u'(x)v'(x)\,dx - [u'(x)v(x)]_0^1 + \int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx.$$

Using the Robin conditions, we obtain

$$\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx - \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 f(x)v(x)\,dx.$$

Thus, the variational formulation reads:

Find $u \in H^1(0,1)$ such that

$$\int_0^1 u'(x)v'(x)\,dx + \int_0^1 u(x)v(x)\,dx - \alpha u(0)v(0) + \beta u(1)v(1) = \int_0^1 f(x)v(x)\,dx, \quad \forall v \in H^1(0,1). \tag{1.7}$$

Robin boundary conditions represent a weighted combination of Dirichlet and Neumann conditions and arise naturally in many physical applications such as heat transfer and convection phenomena [16].

Chapter 2

Maximum principles for elliptic and parabolic problem

The maximum principle is considered one of the most fundamental results in the theory of partial differential equations because of its major role in the study of the qualitative behavior of solutions. It allows the derivation of important results such as uniqueness of solutions, a priori estimates, and comparison principles, without requiring the explicit knowledge of the solution itself. In this chapter, we present the theoretical study of the maximum principle for elliptic and parabolic equations, together with its main applications and fundamental consequences.

2.1 Preliminary definitions

Let $f : A \to \mathbb{R}$, where A is a subset of a metric space.

Definition 2.1. *[9] We say that f admits a (global) maximum if there exists a point $p \in A$ such that*

$$f(x) \leq f(p) \quad \forall x \in A,$$

then

$$\max_{x \in A} f(x) = f(p).$$

Definition 2.2. *We say that f admits a local maximum at $p \in A$ if there exists $\delta > 0$ such that*

$$f(x) \leq f(p) \quad \forall x \in B(p, \delta) \cap A.$$

Definition 2.3 (Supremum principle)**.** *If f is continuous on a compact set A, then the supremum is attained and there exists $p \in A$ such that*

$$\sup_{x \in A} f(x) = f(p).$$

Definition 2.4 (Sub-solution and Super-solution)**.** *[9] Let us consider the boundary value problem*

$$\begin{cases} \mathbb{L}u = f(x, u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

*A function $\alpha \in W^{2,p}(\Omega)$ is called a **sub-solution** of (2.1) if*

$$\mathbb{L}\alpha \leq f(x, \alpha) \quad \text{in } \Omega, \quad \text{and} \quad \alpha \leq 0 \quad \text{on } \partial\Omega.$$

*A function $\beta \in W^{2,p}(\Omega)$ is called a **super-solution** of (2.1) if*

$$\mathbb{L}\beta \geq f(x, \beta) \quad \text{in } \Omega, \quad \text{and} \quad \beta \geq 0 \quad \text{on } \partial\Omega.$$

2.2 Simple application

Let we have the model problem

$$\begin{cases} -u''(x) = f(x), \ x \in (a, b), \\ u(a), u(b) \text{ are given.} \end{cases}$$

The following theorem give the maximum principle of this problem.

Theorem 2.1. *[9] If $u \in C^2[a, b]$ and $f(x) \leq 0$ on $[a, b]$, then we have the following:*

1. *Either $u(x) < \max\{u(a), u(b)\}$ for all $x \in (a, b)$, or u is constant on $[a, b]$;*

2. *If $u(b) > u(a)$, then either the outward derivative $u'(b) > 0$, or u is constant on $[a, b]$;*

3. *If $u(a) > u(b)$, then the outward derivative $-u'(a) > 0$.*

Proof. Firstly, we have

$$u(x) = u(a) + \int_a^x u'(t)\,dt. \tag{2.2}$$

Integration by parts for $\int_a^x u''(t)(x-t)\,dt$, gives

$$u(x) = u(a) + u'(a)(x-a) + \int_a^x u''(t)(x-t)\,dt, \quad a \le x \le b.$$

To eliminate the unknown $u'(a)$ in favor of the "data" $u(a)$, $u(b)$, and u'', at $x = b$, then

$$u'(a) = \frac{1}{b-a}\Big\{u(b) - u(a) - \int_a^b u''(t)(b-t)\,dt\Big\}, \quad a \le x \le b, \tag{2.3}$$

substituting (2.2) into (2.3), we conclude

$$u(x) = v(x) - \int_a^b G(x,t)\,u''(t)\,dt, \qquad (a \le x \le b), \tag{2.4}$$

where v is the solution of $v'' = 0$ given by

$$v(x) := u(a) + \frac{u(b)-u(a)}{b-a}(x-a), \quad \text{on } [a,b],$$

and

$$G(x,t) := \begin{cases} \frac{(b-x)(t-a)}{b-a} & \text{if } x \le t, \\ \frac{(x-a)(b-t)}{b-a} & \text{if } x \ge t. \end{cases}$$

So $G > 0$ in $(a,b)\times(a,b)$ and $G = 0$ on the boundary of the square. Since $u'' \ge 0$ on $[a,b]$, it follows from (2.4) that

$$v(x) - u(x) \ge 0, \quad a \le x \le b, \tag{2.5}$$

with strict inequality for $x \in (a,b)$ if $u''(x_0) > 0$ at some point $x_0 \in [a,b]$, because $u'' > 0$ on an interval of positive length, by continuity of u''.

i. To prove assertion (1) of the preceding example, we consider three cases:

- If $u(a) \ne u(b)$, then $u(x) < \max\{u(a), u(b)\}$ for $x \in (a,b)$.

- If $u(a) = u(b)$ and $u''(x_0) > 0$ at some $x_0 \in [a,b]$, then by (2.5), it follows that $u(x) < v(x) = u(a)$ for $x \in (a,b)$.

- If $u(a) = u(b)$ and $u'' = 0$ on $[a, b]$, then $u(x) = v(x) = u(a)$ for $x \in [a, b]$.

ii. Differentiating (2.2) replacing x by b, or forming the analogue of (2.3) for $u'(b)$, we get

$$u'(b) = \frac{1}{b-a}\Big\{u(b) - u(a) + \int_a^b (t-a)u''(t)\,dt\Big\} \geq 0 \quad \text{when } u(b) \geq u(a),$$

with equality only in third case.

iii. If $u(a) > u(b)$, then $u'(a) < 0$ by (2.3). □

The following remark is almost trivial, as it is an implication of a strict differential inequality. Although the modification is small, this remark plays a role in the subsequent proofs.

Remark 2.1. *If $u \in C^2(\Omega)$ and $\Delta u > 0$ in Ω, then u cannot have a local maximum (let alone a global one) at a point of Ω.*

Proof. Suppose (by contradiction) that u admits a local maximum at $q \in \Omega$. Then, for each $j \in \{1, \ldots, N\}$, we have

$$\partial_j u(q) = 0 \quad \text{and} \quad \partial_j^2 u(q) \leq 0.$$

Consequently, $(\Delta u)(q) \leq 0$. □

2.3 Maximum principle for elliptic equations

This section develops the *maximum principle* for second-order elliptic partial differential equations.

Maximum principle methods are based upon the observation that if a C^2 function u attains its maximum over an open set U at a point $x_0 \in U$, then

$$Du(x_0) = 0, \qquad D^2u(x_0) \leq 0, \tag{2.6}$$

the latter inequality meaning that the symmetric matrix $D^2u = (u_{x_i x_j})$ is nonpositive definite at x_0. Deductions based upon (2.6) are consequently pointwise in character, and are thus utterly different from the integral-based energy methods set (see [8]).

In particular we will need to require that our solutions u are at least C^2, so that it makes sense to consider the pointwise values of Du, D^2u. (In view of the regularity theory, we know however that a weak solution is this smooth, at least provided the coefficients, etc. are sufficiently regular, see §6.3 [8]). As we will shortly learn, it is also most appropriate now to consider elliptic operators L having the *nondivergence form*

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu, \tag{2.7}$$

where the coefficients a_{ij}, b_i, c are continuous and the uniform ellipticity condition (1.1) holds. We continue also to assume, without loss of generality, the symmetry condition

$$a_{ij} = a_{ji} \qquad (i, j = 1, \ldots, n).$$

2.3.1 Weak maximum principle

First, we identify circumstances under which a function must attain its maximum (or minimum) on the boundary. We always assume $\Omega \subset \mathbb{R}^n$ is open, bounded.

Theorem 2.2 (Weak maximum principle)**.** *[8] Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \equiv 0 \quad in\ \Omega$*

(i) If

$$Lu \leq 0 \quad in\ \Omega, \tag{2.8}$$

then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) If

$$Lu \geq 0 \quad in\ \Omega, \tag{2.9}$$

then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Proof.

1. Let us first suppose we have the strict inequality

$$Lu < 0 \quad \text{in } \Omega, \tag{2.10}$$

and yet there exists a point $x_0 \in \Omega$ with

$$u(x_0) = \max_{\overline{\Omega}} u. \tag{2.11}$$

Now at this maximum point x_0, we have

$$Du(x_0) = 0, \tag{2.12}$$

and

$$D^2 u(x_0) \leq 0. \tag{2.13}$$

2. Since the matrix $A = (a^{ij}(x_0))$ is symmetric and positive definite, there exists an orthogonal matrix $O = (O_{ij})$ so that

$$OAO^T = \operatorname{diag}(d_1, \ldots, d_n), \quad OO^T = I, \tag{2.14}$$

with $d_k > 0$ $(k = 1, \ldots, n)$. Write $y = x_0 + O(x - x_0)$. Then $x - x_0 = O^T(y - x_0)$, and so

$$u_{x_i} = \sum_{k=1}^n u_{y_k} O_{ik}, \quad u_{x_i x_j} = \sum_{k,l=1}^n u_{y_k y_l} O_{ik} O_{jl}, \quad (i, j = 1, \ldots, n).$$

Hence at the point x_0,

$$\begin{aligned} \sum_{i,j=1}^n a_{ij} u_{x_i x_j} &= \sum_{k,l=1}^n \sum_{i,j=1}^n a_{ij} u_{y_k y_l} O_{ik} O_{jl} \\ &= \sum_{k=1}^n d_k u_{y_k y_k} \leq 0, \quad (\text{from } (2.14)), \end{aligned} \tag{2.15}$$

since $d_k > 0$ and $u_{y_k y_k}(x_0) \leq 0$ $(k = 1, \ldots, n)$, according to (2.13).

3. Thus at x_0

$$Lu = -\sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} \geq 0,$$

in light of (2.13) and (2.15). So (2.10) and (2.11) are incompatible, and we have a contradiction.

4. In the general case that (2.9) holds, write

$$u^\varepsilon(x) := u(x) + \varepsilon e^{\lambda x_1} \quad (x \in \Omega),$$

where $\lambda > 0$ will be selected below and $\varepsilon > 0$. Recall the uniform ellipticity condition implies $a_{ii}(x) \geq \theta$ $(i = 1, \ldots, n,\ x \in \Omega)$. Therefore

$$\begin{aligned} Lu^\varepsilon &= Lu + \varepsilon L(e^{\lambda x_1}) \\ &\leq \varepsilon e^{\lambda x_1}\left[-\lambda^2 a^{11} + \lambda b^1\right] \\ &\leq \varepsilon e^{\lambda x_1}\left[-\lambda^2 \theta + \|b\|_{L^\infty}\lambda\right] \\ &< 0 \quad \text{in } \Omega. \end{aligned} \tag{2.16}$$

Provided we choose $\lambda > 0$ sufficiently large. Then according to steps 1 and 2 above

$$\max_{\Omega} u^\varepsilon = \max_{\partial\Omega} u^\varepsilon.$$

Let $\varepsilon \to 0$ to find

$$\max_{\Omega} u = \max_{\partial\Omega} u.$$

This proves (i).

5. Since $-u$ is a subsolution whenever u is a supersolution, assertion (ii) follows. □

Remark 2.2. *A function satisfying (2.8) is called a sub-solution. We are thus assuming a sub-solution attains its maximum on $\partial\Omega$. Similarly, if (2.9) holds, u is a super-solution and attains its minimum on $\partial\Omega$.*

Theorem 2.3 (Weak maximum principle for $c \geq 0$)**.** *[8] Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and*

$$c \geq 0 \quad \text{in } \Omega.$$

(i) If

$$Lu \leq 0 \quad \text{in } \Omega,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+. \tag{2.17}$$

(ii) Likewise, if

$$Lu \geq 0 \quad \text{in } \Omega,$$

then

$$\min_{\overline{\Omega}} u \geq -\max_{\partial\Omega} u^-. \tag{2.18}$$

Proof.

1. Let u be a sub-solution and set $V := \{x \in \Omega \mid u(x) > 0\}$. Then

   $$Ku := Lu - cu \leq -cu \leq 0 \quad \text{in } V.$$

   The operator K has no zeroth-order term and consequently Theorem 2.2 implies

   $$\max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial\Omega} u^+.$$

   This gives (2.17) in the case that $V \neq \emptyset$. Otherwise $u \leq 0$ everywhere in Ω, and (2.17) likewise follows.

2. Assertion (ii) follows from (i) applied to $-u$, once we observe that

   $$(-u)^+ = u^-.$$

□

Remark 2.3. *In particular, if $Lu = 0$ in Ω, then*

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{2.19}$$

2.3.2 Strong maximum principle

We next substantially strengthen the foregoing assertions, by demonstrating that a subsolution u cannot attain its maximum at an interior point of a connected region at all, unless u is constant. This statement is the strong maximum principle, which depends on the following subtle analysis of the outer normal derivative $\frac{\partial u}{\partial \nu}$ at a boundary maximum point.

Theorem 2.4 (Strong maximum principle)**.** *Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and*

$$c \equiv 0 \quad \textit{in } \Omega.$$

Suppose also Ω is connected, open and bounded.

1. *If*

$$Lu \leq 0 \quad in\ \Omega$$

and u attains its maximum over $\bar{\Omega}$ at an interior point, then u is constant within Ω.

2. *Similarly, if*

$$Lu \geq 0 \quad in\ \Omega$$

and u attains its minimum over $\bar{\Omega}$ at an interior point, then u is constant within Ω.

Proof. Write $M := \max_{\Omega} u$ and $C := \{x \in \Omega \mid u(x) = M\}$. Then if $u \not\equiv M$, set

$$V := \{x \in \Omega \mid u(x) < M\}.$$

Choose a point $y \in V$ satisfying $\text{dist}(y, C) < \text{dist}(y, \partial\Omega)$, and let B denote the largest ball with center y whose interior lies in V. Then there exists some point $x^0 \in C$, with $x^0 \in \partial B$. Clearly V satisfies the interior ball condition at x^0; whence Hopf's Lemma, (i), implies

$$\frac{\partial u}{\partial \nu}(x^0) > 0.$$

But this is a contradiction: since u attains its maximum at $x^0 \in \Omega$, we have $Du(x^0) = 0$. $\square$

If the zeroth-order term c is nonnegative, we have this version of the strong maximum principle:

Theorem 2.5 (Strong maximum principle with $c \geq 0$). *[8] Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and*

$$c \geq 0 \quad in\ \Omega.$$

Suppose also Ω is connected.

1. *If*

$$Lu \leq 0 \quad in\ \Omega$$

and u attains a nonnegative maximum over $\bar{\Omega}$ at an interior point, then

$$u \text{ is constant within } \Omega.$$

2. Similarly, if

$$Lu \geq 0 \quad \text{in } \Omega$$

and u attains a nonpositive minimum over $\bar{\Omega}$ at an interior point, then

$$u \text{ is constant within } \Omega.$$

The proof is like that above, except that we use statement (ii) in Hopf's Lemma.

2.3.3 Hopf's Lemma

Lemma 2.1 (Hopf's Lemma). *[8] Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and*

$$c \equiv 0 \quad \text{in } \Omega.$$

Suppose further

$$Lu \leq 0 \quad \text{in } \Omega,$$

and there exists a point $x^0 \in \partial\Omega$ such that

$$u(x^0) > u(x) \quad \text{for all } x \in \Omega. \tag{2.20}$$

Assume finally that Ω satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset \Omega$ with $x^0 \in \partial B$.

(i) Then

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where ν is the outer unit normal to B at x^0.

(ii) If

$$c \geq 0 \quad \text{in } \Omega,$$

the same conclusion holds provided

$$u(x^0) \geq 0.$$

Remark 2.4. *The importance of (i) is the strict inequality: that $\frac{\partial u}{\partial \nu}(x^0) \geq 0$ is obvious. Note that the interior ball condition automatically holds if $\partial\Omega$ is C^2.*

Exemple 2.1. *Laplace and Poisson equations are fundamental examples of elliptic partial differential equations. They arise in many physical applications such as electrostatics and heat conduction.*

Laplace equation: *The Laplace equation is defined as*

$$\Delta u = 0, \tag{2.21}$$

where the Laplace operator is given by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{2.22}$$

A function satisfying (2.21) is called a harmonic function. These functions satisfy the maximum principle, which states that the maximum and minimum values are attained on the boundary of the domain.

Poisson equation: *The Poisson equation is a generalization of the Laplace equation:*

$$\Delta u = f, \tag{2.23}$$

where f is a given function representing a source term.

This equation models various physical phenomena, such as gravitational and electrostatic potentials, where f represents the density of sources.

Remark 2.5. *Both equations are linear elliptic equations of second order and play a central role in the study of Maximum principles and uniqueness of solutions.*

2.4 Maximum principle for parabolic equations

Parabolic partial differential equations play a central role in the study of time-dependent processes. A fundamental example is the heat equation.

2.4.1 The heat equation

Suppose a thin rod of length l is located in the interval $(0, l)$ along the x-axis. We assume that the material of the rod is homogeneous. Heat can be added to or removed from the rod, and we assume that u represents the temperature at every point of the rod, which depends only on x and time t.

Under certain assumptions on the physical properties of the rod, the differential equation governing the heat flux (in suitable units) in the rod is given by

$$\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial u}{\partial t}(x,t) = f(x,t). \tag{2.24}$$

The function f represents the rate of heat removal from the rod. The temperature $u(x,t)$ satisfies a maximum principle that is somewhat different from the one established for elliptic equations and inequalities.

Assume that $u(x,t)$ satisfies the strict inequality

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} > 0, \tag{2.25}$$

in a domain Ω of the (x,t)-plane.

It is clear that u cannot have a local maximum at any interior point, because at such a point we would have

$$\frac{\partial^2 u}{\partial x^2} \leq 0 \quad \text{and} \quad \frac{\partial u}{\partial t} = 0. \tag{2.26}$$

This leads to a contradiction with $\mathcal{L}[u] > 0$. We will not only extend this statement to solutions of $\mathcal{L}[u] \leq 0$, but we will also show that for operators of this type, the maximum principle takes a stronger form.

To illustrate a typical problem, we assume that the rod described above has its temperature prescribed initially (i.e., at time $t = 0$), and that the temperatures at the ends of the rod are known functions of time.

The principle of causality indicates that the temperature distribution at any fixed time T is not affected by changes in the rod that occur at a time $t > T$. Thus, it is natural to consider the rectangular domain

$$\Omega = \{0 < x < l,\ 0 < t < T\}, \tag{2.27}$$

in the (x,t)-plane.

We assume that the temperature $u(x,t)$ is known on three sides of Ω

$$S_1 : \{x = 0,\ 0 \leq t \leq T\}, \quad S_2 : \{0 \leq x \leq l,\ t = 0\}, \quad S_3 : \{x = l,\ 0 \leq t \leq T\}.$$

For physical reasons, we expect that this information, together with the fact that the temperature u satisfies the equation

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \tag{2.28}$$

in Ω, is sufficient to determine the temperature uniquely throughout Ω.

The uniqueness of the solution can be easily established as a corollary of the following maximum principle [14].

Theorem 2.6. *Suppose that $u(x,t)$ satisfies the inequality*

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \le 0,$$

in the rectangular domain Ω given by (2.27). Then the maximum of u on the closure $\overline{\Omega}$ must occur on one of the three sides S_1, S_2, or S_3.

Proof. Let M be the maximum value of u attained on S_1, S_2, and S_3. Assume that there exists a point $P(x_0,t_0)$ in Ω such that $u(P) = M_1 > M$, and we derive a contradiction.

We define the auxiliary function

$$w(x,t) = \frac{M_1 - M}{2l^2}(x - x_0)^2. \tag{2.29}$$

Since $u \le M$ on S_1, S_2, and S_3, we define

$$v(x,t) = u(x,t) + w(x,t) \le M + \frac{M_1 - M}{2} < M_1. \tag{2.30}$$

Moreover,

$$v(x_0,t_0) = u(x_0,t_0) = M_1. \tag{2.31}$$

We also have

$$\mathcal{L}[v] = \mathcal{L}[u] + \mathcal{L}[w] \ge \mathcal{L}[u] + \frac{M_1 - M}{l^2} > 0 \tag{2.32}$$

throughout Ω.

Conditions (2.30) and (2.31) show that v must attain its maximum either at an interior point of Ω or on the open boundary segment

$$S_4 = \{0 < x < l,\ t = T\}. \tag{2.33}$$

However, inequality (2.32) shows that v cannot have an interior maximum. At a maximum on S_4, we have

$$\frac{\partial^2 v}{\partial x^2} \leq 0, \quad \text{which implies that} \quad \frac{\partial v}{\partial t} < 0.$$

Thus v would have been larger at earlier times, and therefore the maximum in Ω cannot occur on S_4.

Hence, the assumption $u(x_0, t_0) > M$ leads to a contradiction [14]. □

2.5 Consequences and applications

2.5.1 Uniqueness of Solutions

One of the most important consequences of the maximum principle is the uniqueness of solutions to boundary value problems.

Consider the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{2.34}$$

in a rectangular domain $\Omega = \{0 < x < l, 0 < t < T\}$, together with initial and boundary conditions.

Theorem 2.7. *[19] If two functions $u_1(x,t)$ and $u_2(x,t)$ satisfy the same heat equation and the same initial and boundary conditions, then $u_1 = u_2$ in Ω.*

Proof. Let $w = u_1 - u_2$. Then w satisfies

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0, \tag{2.35}$$

in Ω, with homogeneous boundary and initial conditions. By the maximum principle, the maximum and minimum of w are attained on the boundary. Since $w = 0$ on the boundary, it follows that $w = 0$ in Ω. Hence $u_1 = u_2$. □

2.5.2 A priori estimates

A priori estimates are bounds on the solution that depend only on the given data, not on the explicit form of the solution.

For a solution u of the heat equation, the maximum principle implies

$$\max_{\Omega} |u(x,t)| \leq \max_{\partial\Omega} |u(x,t)|. \tag{2.36}$$

More generally, if u satisfies

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{2.37}$$

then under suitable conditions on f, one can derive estimates of the form

$$\|u\|_{\infty} \leq C \left(\|f\|_{\infty} + \|u\|_{\partial\Omega} \right), \tag{2.38}$$

where C is a constant depending on the domain.

2.5.3 Comparison principle

Theorem 2.8 (Comparison principle)**.** *[19] Let u and v be two functions such that*

$$\mathcal{L}[u] \leq \mathcal{L}[v] \tag{2.39}$$

in Ω, and suppose that

$$u \leq v \quad \text{on } \partial\Omega. \tag{2.40}$$

Then

$$u \leq v \quad \text{in } \Omega. \tag{2.41}$$

Proof. Consider $w = u - v$. Then w satisfies

$$\mathcal{L}[w] \leq 0 \tag{2.42}$$

and $w \leq 0$ on the boundary. By the maximum principle, $w \leq 0$ in Ω, which implies $u \leq v$. □

2.5.4 Holomorphic functions and the maximum principle

Definition 2.5. *[2] A function f is said to be holomorphic on Ω if, for every $a \in \Omega$, the limit*

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists in $\mathbb{C}$. In this case, this limit is denoted by $f'(a)$, and the resulting mapping is called the derivative of f on Ω.

Remark 2.6. *We denote by $H(\Omega)$ the set of holomorphic functions on Ω. It is a $\mathbb{C}$-algebra containing polynomial functions on Ω, but not the function $z \mapsto \bar{z}$.*

Remark 2.7. *Since $\mathbb{C} \simeq \mathbb{R}^2$, we can view $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ as a function of two real variables. We write $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.*

Theorem 2.9 (Cauchy–Riemann equations)**.** *[2] Let $a \in \Omega$. Then f is holomorphic at a if and only if f is differentiable at a and*

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a), \qquad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

In this case, the differential $Df(a) : h \mapsto f'(a)h$ is a similarity transformation of the plane.

Definition 2.6. *[2] A curve is a mapping $\gamma : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{C}$ which is continuous and piecewise C^1. The curve is said to be closed if $\gamma(\alpha) = \gamma(\beta)$. We denote by $\gamma^* \subset \mathbb{C}$ its image (which is compact). If f is continuous on γ^*, we define*

$$\int_\gamma f(z)\, dz = \int_\alpha^\beta f(\gamma(t))\, \gamma'(t)\, dt.$$

This integral does not depend on any C^1-bijective reparametrization of γ.

Definition 2.7 (Winding number)**.** *[17] Let γ be a curve and $a \notin \gamma^*$. The index (winding number) of γ around a is defined by*

$$\mathrm{Ind}_\gamma(a) = \frac{1}{2i\pi} \int_\gamma \frac{dz}{z-a}.$$

Theorem 2.10. *[17] Let $\Omega = \mathbb{C} \setminus \gamma^*$, which is open. The map*

$$a \in \Omega \mapsto \mathrm{Ind}_\gamma(a)$$

takes integer values, is constant on each connected component of Ω, and is zero on the unbounded connected component of Ω.

Theorem 2.11 (Cauchy theorem for convex domains)**.** *[17] Let $f \in H(\Omega)$, where Ω is convex, and let γ be a closed curve in Ω. Then*

$$\int_\gamma f(z)\, dz = 0.$$

Moreover, for any $a \in \Omega \setminus \gamma^*$,

$$f(a)\ \mathrm{Ind}_\gamma(a) = \frac{1}{2i\pi} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Theorem 2.12 (Holomorphic implies analytic)**.** *[15] Every holomorphic function is representable by a power series in Ω. More precisely, for every $a \in \Omega$, and every $r > 0$ such that $D(a,r) \subset \Omega$, for all $z \in D(a,r)$,*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2i\pi} \int_{C(a,r)} \frac{f(w)}{(w-a)^{n+1}}\, dw.$$

Theorem 2.13 (Isolated zeros theorem)**.** *[15] Let f be holomorphic on a connected domain Ω. If f is not identically zero, then the set of zeros*

$$Z(f) = \{a \in \Omega : f(a) = 0\}$$

has no accumulation point in Ω. Moreover, for any zero a of f, there exists an integer $m \geq 1$ (order of f at a) and $g \in H(\Omega)$ non-zero at a such that

$$f(z) = (z-a)^m g(z), \quad \forall z \in \Omega.$$

Remark 2.8. *In this case, $Z(f)$ is at most countable.*

Corollary 1 (Analytic continuation principle)**.** *Let f and g be holomorphic on a connected domain Ω. If $f = g$ on a set possessing an accumulation point in Ω, then $f = g$ on the whole domain Ω.*

Remark 2.9. *It is important that the accumulation point lies in Ω. Consider $z \mapsto \sin(\pi/z)$ in the half-plane $\{\mathrm{Re}(z) > 0\}$.*

Application 2.1. *The Fourier transform of the Gaussian $t \in \mathbb{R} \mapsto e^{-t^2/2}$ is the Gaussian $\xi \in \mathbb{R} \mapsto \sqrt{2\pi} e^{-\xi^2/2}$.*

Theorem 2.14 (Maximum principle)**.** *Let f be a non-constant holomorphic function on a connected domain Ω. Then $|f|$ does not attain a local maximum inside Ω.*

Corollary 2. *Let Ω be a bounded open set and f holomorphic on Ω, continuous on $\overline{\Omega}$. Then $|f|$ attains its supremum on the boundary $\partial\Omega$.*

Application 2.2 (Schwarz lemma). *Let $f : D(0,1) \to D(0,1)$ be holomorphic with $f(0) = 0$. Then for all $|z| < 1$:*

$$|f(z)| \leq |z| \quad \textit{and} \quad |f'(0)| \leq 1.$$

Moreover, if there exists $0 < |z_0| < 1$ such that $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then f is a rotation around the origin.

Theorem 2.15 (Open mapping theorem). *Let f be a non-constant holomorphic function on a connected domain Ω. Let $a \in \Omega$ and $m \geq 1$ be the order of $f - f(a)$ at a. Then there exists an open neighborhood $V \subset \Omega$ of a, $r > 0$, and a biholomorphism $\varphi : V \to D(0,r)$ such that*

$$f(z) = f(a) + \varphi(z)^m, \quad \forall z \in V.$$

Corollary 3. *Under the above assumptions, f is an open map. Hence $f(\Omega)$ is connected and open.*

Chapter 3

Discrete maximum principle (numerical aspect)

The importance of the maximum principle is not limited to the theoretical aspect, but also extends to the numerical analysis of partial differential equations. Modern numerical methods aim to preserve the essential qualitative properties of solutions, among which the maximum principle plays a central role. This led to the development of the discrete maximum principle, which is an important tool for ensuring the stability of numerical solutions and preserving their physical meaning. In this chapter, we study this principle within the framework of finite difference methods and analyze the properties of some numerical schemes that preserve it.

3.1 Study of a 2D elliptic problem

Let $\Omega =]0,1[\times]0,1[$. We aim to study a numerical scheme for the following problem

$$\begin{cases} -\Delta u(x,y) + \alpha \dfrac{\partial u}{\partial x}(x,y) = f(x,y), & (x,y) \in \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\alpha > 0$ is a given real number and $f \in C(\overline{\Omega})$ is given function. We denote by u the exact solution of (3.1) and assume that $u \in C^4(\overline{\Omega})$.

3.1.1 Discretized problem

First, we define the uniform mesh of Ω. Let $N \in \mathbb{N}$, we set $h = \dfrac{1}{N+1}$ and denote $u_{i,j}$ the approximate value of $u(ih, jh)$, $(i,j) \in \{0, 1, \dots, N+1\}$. We define

$$f_{i,j} = f(ih, jh).$$

Using centered finite differences, we obtain

$$-\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \alpha \frac{u_{i+1,j} - u_{i-1,j}}{2h} = f_{i,j}.$$

Thus the discrete problem is

$$\begin{cases} -\left(\dfrac{1}{h^2} + \dfrac{\alpha}{2h}\right) u_{i+1,j} - \left(\dfrac{1}{h^2} - \dfrac{\alpha}{2h}\right) u_{i-1,j} + \dfrac{4}{h^2} u_{i,j} - \dfrac{1}{h^2} u_{i,j+1} - \dfrac{1}{h^2} u_{i,j-1} = f_{i,j}, \\ u_{0,j} = u_{N+1,j} = u_{i,0} = u_{i,N+1} = 0, \end{cases} \tag{3.2}$$

$(i,j) \in \{1, \dots, N\}$.

3.1.2 Consistency study

Since $u \in C^4(\overline{\Omega})$, and from the Taylor expansions, we have

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{h^2}{4!}\left(\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{\partial^4 u}{\partial x^4}(\xi_{i-1}, y_j)\right), \tag{3.3}$$

$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + \frac{h^2}{4!}\left(\frac{\partial^4 u}{\partial y^4}(\theta_i, y_j) + \frac{\partial^4 u}{\partial y^4}(\theta_{i-1}, y_j),\right), \tag{3.4}$$

such that $x_i < \xi_i < x_{i+1}$, $x_{i-1} < \xi_{i-1} < x_i$, $y_i < \theta_i < y_{i+1}$ and $y_{i-1} < \theta_{i-1} < y_i$,

For $\dfrac{\partial u}{\partial x}(x_i, y_j)$, we have

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{h^2}{2 \times 3!}\left(\frac{\partial^3 u}{\partial x^3}(\mu_i, y_j) - \frac{\partial^3 u}{\partial x^3}(\mu_{i-1}, y_j)\right), \tag{3.5}$$

where $x_i < \mu_i < x_{i+1}$ and $x_{i-1} < \mu_{i-1} < x_i$.

Thus, from (3.3), (3.4) and (3.5), the consistency error is given by

$$\begin{aligned}
R_{i,j} &= -\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{\partial^2 u}{\partial y^2}(x_i, y_j) + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \\
&\quad + \alpha \frac{\partial u}{\partial x}(x_i, y_j) - \alpha \frac{u_{i+1,j} - u_{i-1,j}}{2h} \\
&= \frac{h^2}{24}\left(\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{\partial^4 u}{\partial x^4}(\xi_{i-1}, y_j) + \frac{\partial^4 u}{\partial y^4}(\theta_i, y_j) + \frac{\partial^4 u}{\partial y^4}(\theta_{i-1}, y_j)\right) \\
&\quad + \alpha \frac{h^2}{12}\left(\frac{\partial^3 u}{\partial x^3}(\mu_i, y_j) - \frac{\partial^3 u}{\partial x^3}(\mu_{i-1}, y_j)\right).
\end{aligned}$$

Hence,

$$|R_{i,j}| \leq \frac{h^2}{12}\left(\max_{[0,1]\times[0,1]}\left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| + \max_{[0,1]\times[0,1]}\left|\frac{\partial^4 u}{\partial y^4}(x,y)\right| + 2\alpha \max_{[0,1]\times[0,1]}\left|\frac{\partial^3 u}{\partial x^3}(x,y)\right|\right)$$

$$\Rightarrow \|R\|_\infty \leq Ch^2 \underset{h\to 0}{\longrightarrow} 0,$$

Thus, the scheme is consistent of order 2.

3.1.3 Stability

Proposition 3.1. *The scheme (3.2) is stable in the sense*

$$\|u\|_\infty \leq C\|f\|_\infty,$$

under the condition $\frac{1}{h^2} - \frac{\alpha}{2h} \geq 0.$

3.1.4 Convergence

For $i, j \in \{0, \ldots, N+1\}^2$, we define

$$e_{i,j} = u(ih, jh) - u_{i,j},$$

thus, from from (3.3), (3.4) and (3.5), we obtain

$$\frac{4}{h^2}e_{i,j} - \left(\frac{1}{h^2} + \frac{\alpha}{2h}\right)e_{i+1,j} - \left(\frac{1}{h^2} - \frac{\alpha}{2h}\right)e_{i-1,j} - \frac{1}{h^2}e_{i,j+1} - \frac{1}{h^2}e_{i,j-1} = R_{i,j},$$

$i, j = 1, \ldots, N.$

Under the condition of stability $\left(\frac{1}{h^2} - \frac{\alpha}{2h} \geq 0\right)$, we have

$$\max_{(i,j)\in\{1,\ldots,N\}^2} |e_{i,j}| \leq \frac{1}{2}Ch^2 \longrightarrow 0 \quad \text{as } h \to 0.$$

Therefore, the scheme is convergent.

Remark 3.1. *This chapter does not aim to present a comprehensive study of the finite difference method. Instead, it is primarily concerned with illustrating the validation of the maximum principle for this class of problems. For a more detailed treatment of the theoretical foundations, including proofs of consistency, stability, and convergence, the reader is referred to the relevant literature ([7, 11, 13, 18]).*

3.1.5 Discrete maximum principle

Proposition 3.2. *In scheme (3.2), if* $\frac{1}{h^2} - \frac{\alpha}{2h} \geq 0$ *and* $\omega_{i,j}$ *satisfies*

$$-\left(\frac{1}{h^2} + \frac{\alpha}{2h}\right)\omega_{i+1,j} - \left(\frac{1}{h^2} - \frac{\alpha}{2h}\right)\omega_{i-1,j} + \frac{4}{h^2}\omega_{i,j} - \frac{1}{h^2}\omega_{i,j+1} - \frac{1}{h^2}\omega_{i,j-1} \leq 0,$$

$\forall i, j = 1, \ldots, N$. *Then,*

$$\omega_{i,j} \leq \max_{(n,m)\in\gamma}(\omega_{n,m}), \quad \forall i, j = 1, \ldots, N,$$

where

$$\gamma = \{i \in \{0, N+1\} \text{ or } j \in \{0, N+1\}\}$$

is the set of boundary points of Ω_h.

Proof. The family $\omega_{i,j}$, $(i,j) \in \{1,\ldots,N\}^2$, satisfies

$$\frac{1}{h^2}(\omega_{i,j} - \omega_{i,j+1}) + \frac{1}{h^2}(\omega_{i,j} - \omega_{i,j-1}) + \left(\frac{1}{h^2} - \frac{\alpha}{2h}\right)(\omega_{i,j} - \omega_{i-1,j}) + \left(\frac{1}{h^2} + \frac{\alpha}{2h}\right)(\omega_{i,j} - \omega_{i+1,j}) \leq 0.$$

Let $M = \max\{\omega_{i,j} \mid (i,j) \in \{0,\ldots,N+1\}^2\}$ and $m = \max\{\omega_{i,j} \mid (i,j) \in \gamma\}$.

Clearly, $m \leq M$; it remains to show that $M \leq m$. Let

$$A = \left\{(i,j) \in \{0,\ldots,N+1\}^2 \text{ such that } \omega_{i,j} = M\right\},$$

and let $(\bar{i}, \bar{j}) \in A$ such that

$$\bar{i} = \max\{i \mid (i,j) \in A\}, \quad \bar{j} = \max\{j \mid (\bar{i}, j) \in A\}.$$

We distinguish two cases:

1. If $\bar{i} = 0, N+1$ or $\bar{j} = 0, N+1$, then $(\bar{i}, \bar{j}) \in \gamma$, and hence

$$M = \omega_{\bar{i},\bar{j}} \leq m.$$

2. Otherwise, $\bar{i} \notin \{0, N+1\}$ and $\bar{j} \notin \{0, N+1\}$, thus $(\bar{i}, \bar{j}) \in \{1, \ldots, N\}^2$. Then,

$$\frac{1}{h^2}(\omega_{\bar{i},\bar{j}} - \omega_{\bar{i},\bar{j}+1}) + \frac{1}{h^2}(\omega_{\bar{i},\bar{j}} - \omega_{\bar{i},\bar{j}-1}) + \left(\frac{1}{h^2} - \frac{\alpha}{2h}\right)(\omega_{\bar{i},\bar{j}} - \omega_{\bar{i}-1,\bar{j}})$$

$$+ \left(\frac{1}{h^2} + \frac{\alpha}{2h}\right)(\omega_{\bar{i},\bar{j}} - \omega_{\bar{i}+1,\bar{j}}) \leq 0,$$

which is impossible since $M = \omega_{\bar{i},\bar{j}}$, hence

$$\omega_{\bar{i},\bar{j}} - \omega_{\bar{i},\bar{j}+1} \geq 0, \quad \omega_{\bar{i},\bar{j}} - \omega_{\bar{i},\bar{j}-1} \geq 0,$$

$$\omega_{\bar{i},\bar{j}} - \omega_{\bar{i}-1,\bar{j}} \geq 0, \quad \omega_{\bar{i},\bar{j}} - \omega_{\bar{i}+1,\bar{j}} \geq 0.$$

Thus, we have shown that $M \leq m$, that is

$$\omega_{i,j} \leq \max_{(n,m)\in\gamma} (\omega_{n,m}), \quad \forall i, j = 1, \ldots, N.$$

□

3.1.6 Numerical example

Consider the 2D heat problem (3.1), with $\alpha = 2$ and $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y) + 2\pi \cos(\pi x) \sin(\pi y)$. In this case, we can easily verify that the exact solution to the problem is the following

$$U(x, y) = \sin(\pi x). \sin(\pi y).$$

To obtain numerical results validating the theoretical analysis, we implemented a MATLAB code based on the finite difference method described above.

```
% Finite Difference Method (Centered Scheme)
% -Delta u + alpha * du/dx = f
clear; clc;
% Parameters
N = 20;
```

```
alpha = 2;       % choose any value
h = 1/(N+1);
% Grid
x = linspace(0,1,N+2);
y = linspace(0,1,N+2);
% RHS function
f = @(x,y) 2*pi^2*sin(pi*x).*sin(pi*y) + alpha*pi*cos(pi*x).*sin(pi*y);
% Exact solution (for test)
u_exact = @(x,y) sin(pi*x).*sin(pi*y);
% Number of unknowns
n = N*N;
% Sparse matrix
A = sparse(n,n);
b = zeros(n,1);
% Index mapping
index = @(i,j) (j-1)*N + i;
for j = 1:N
for i = 1:N
k = index(i,j);
xi = x(i+1);
yj = y(j+1);
% Center
A(k,k) = 4/h^2;
% Left (i-1,j)
if i > 1
A(k,index(i-1,j)) = -1/h^2 - alpha/(2*h);
end
% Right (i+1,j)
if i < N
A(k,index(i+1,j)) = -1/h^2 + alpha/(2*h);
end
```

```
% Bottom (i,j-1)
if j > 1
A(k,index(i,j-1)) = -1/h^2;
end
% Top (i,j+1)
if j < N
A(k,index(i,j+1)) = -1/h^2;
end
% RHS
b(k) = f(xi,yj);
end
end
% Solve system
U = A\b;
% Reshape
U_mat = reshape(U, N, N);
% Add boundary (zero Dirichlet)
U_full = zeros(N+2, N+2);
U_full(2:N+1,2:N+1) = U_mat;
% Exact solution
[X,Y] = meshgrid(x,y);
U_exact = u_exact(X,Y);
% Error
err = max(max(abs(U_full - U_exact)));
fprintf('Max error = %e\n', err);
% Plot numerical solution
figure (1)
surf(X,Y,U_full);
colorbar
title('Numerical Solution (Centered Scheme)');
xlabel('x'); ylabel('y');
```

```
print -dpdf Figure-1.pdf
open Figure-1.pdf
% Plot exact solution
figure (2)
surf(X,Y,U_exact);
title('Exact Solution');
colorbar
xlabel('x'); ylabel('y');
print -dpdf Figure-2.pdf
open Figure-2.pdf
```

The following figures illustrate the results obtained by running the code for two different values of N. In each case, both the exact and approximate solutions are plotted in separate figures.

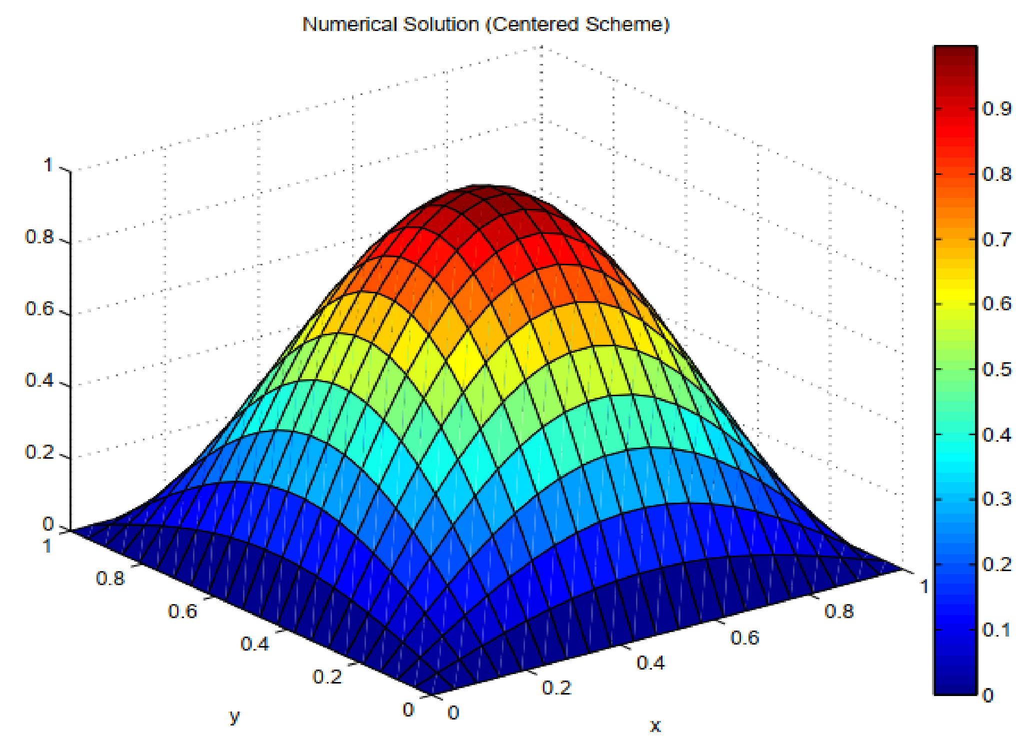

Figure 3.1: Numerical solution of problem (3.1) for $N = 20$ and $\alpha = 2$.

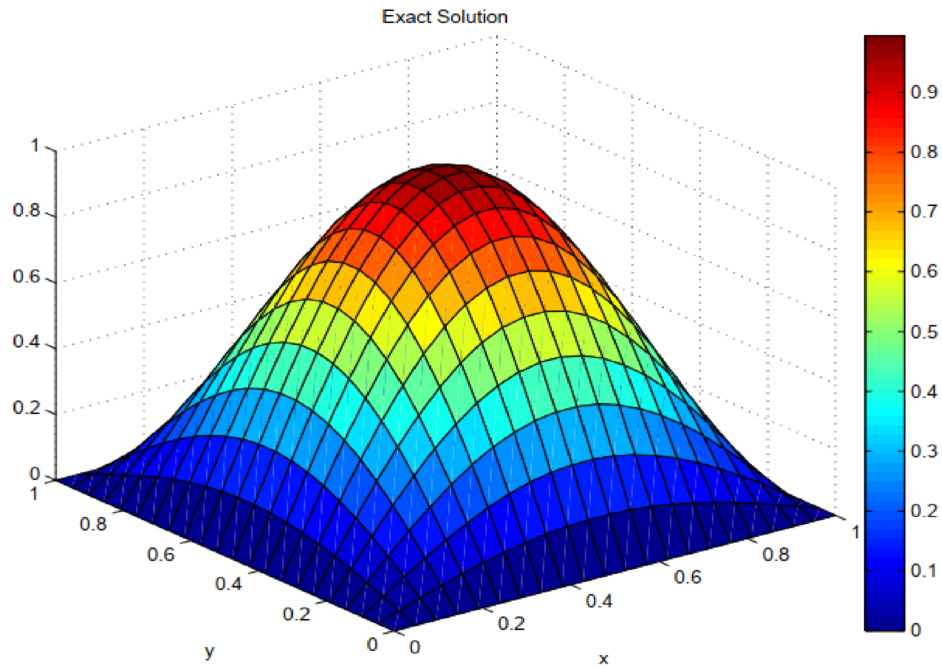

Figure 3.2: Exact solution of problem (3.1) for $N = 20$ and $\alpha = 2$.

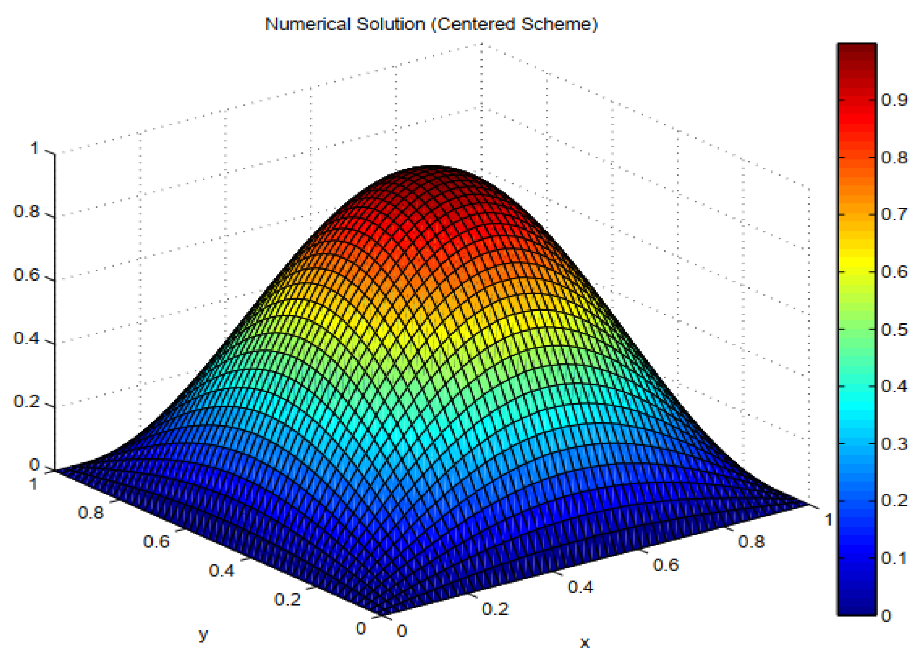

Figure 3.3: Numerical solution of problem (3.1) for $N = 50$ and $\alpha = 2$.

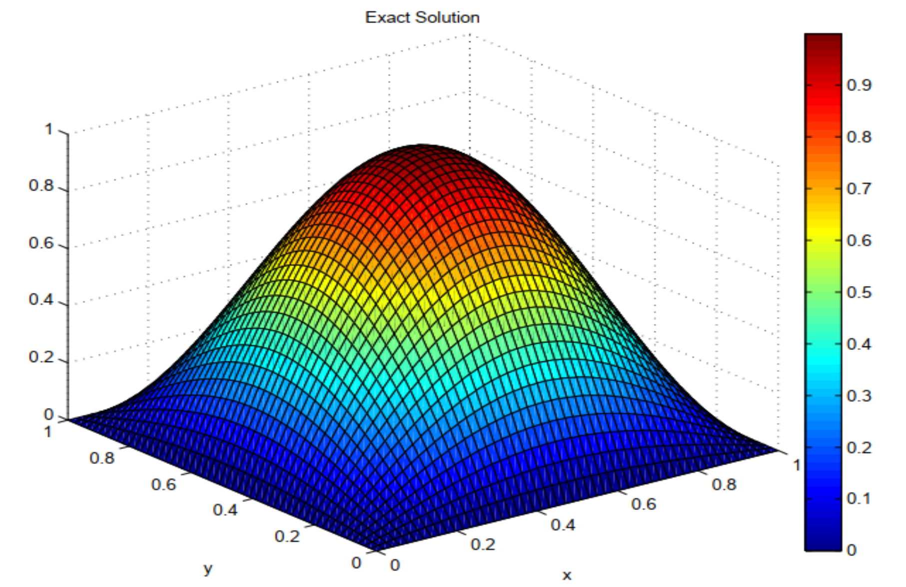

Figure 3.4: Exact solution of problem (3.1) for $N = 50$ and $\alpha = 2$.

The results illustrated in Figures 3.1-3.4, representing the exact and numerical solutions obtained from the MATLAB implementation, confirm the validity of both the continuous and discrete maximum principles for problem (3.1). In particular, we observe that the minimum values of the solution occur at the boundaries of the interval ($[0, 1] \times [0, 1]$). This behavior is consistent with the fact that the right-hand side of the equation is positive.

3.2 Study of the heat equation in one dimension

Exemple 3.1. *Consider the nonhomogeneous heat equation*

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin(\pi x), & 0 < x < 1,\ t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 \leq x \leq 1. \end{cases} \tag{3.6}$$

The exact solution is given by

$$u(x,t) = \frac{1 - e^{-(\pi^2+1)t}}{\pi^2 + 1}\sin(\pi x). \tag{3.7}$$

Theorem 3.1 (Existence and uniqueness). *Let $f \in C([0,T]; L^2(0,1))$ and $u_0 \in L^2(0,1)$. Then the problem (3.6) admits a unique solution $u \in C([0,T]; L^2(0,1))$.*

Moreover, if $u_0 \in C^\infty(0,1)$ and f is smooth, then the solution becomes smooth for $t > 0$ (regularizing effect of the heat equation).

3.2.1 Continuous maximum principle

Under the assumptions of the previous theorem and assuming that the source term is non-negative, let u be the solution of problem (3.6). If

$$u_0(x) \geq 0 \quad \text{for all } x \in (0,1), \quad \text{and} \quad f(x,t) \geq 0 \text{ in } (0,1) \times (0,T),$$

then

$$u(x,t) \geq 0 \quad \forall (x,t) \in (0,1) \times (0,T).$$

This result is obtained directly after using the Theorem 2.6 chapter 2.

3.2.2 Discretization of the problem

The discretization consists in defining a set of points t_n, $n = 0, \ldots, M$ in the interval $[0,T]$ and a set of points x_i, $i = 0, \ldots, N+1$. For simplicity, we consider constant steps in time and space. Let

$$h = \frac{1}{N+1}, \qquad k = \frac{T}{M}$$

be the spatial and temporal discretization steps, respectively. We set

$$t_n = nk, \quad n = 0, \ldots, M, \qquad x_i = ih, \quad i = 0, \ldots, N+1.$$

The discrete unknowns are denoted by $u_i^n \approx u(x_i, t_n)$ for $i = 1, \ldots, N$ and $n = 0, \ldots, M$.

The time derivative is approximated using the explicit Euler method

$$u_t(x_i, t_n) \approx \frac{u_i^{n+1} - u_i^n}{k},$$

and the second spatial derivative is approximated by

$$u_{xx}(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}.$$

The source term is approximated by

$$f(x_i, t_n) = e^{-t_n} \sin(\pi x_i).$$

We obtain the following scheme

$$\begin{cases} \dfrac{u_i^{n+1} - u_i^n}{k} = \dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + e^{-t_n} \sin(\pi x_i), \quad i = 1, \dots, N, \ n = 0, \dots, M-1, \\ u_i^0 = 0, \quad i = 1, \dots, N, \\ u_0^n = u_{N+1}^n = 0, \quad n = 0, \dots, M. \end{cases} \tag{3.8}$$

The scheme can also be written in explicit form

$$u_i^{n+1} = u_i^n + \frac{k}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + k\, e^{-t_n} \sin(\pi x_i).$$

3.2.3 Consistency of the scheme

Let $\bar{u}_i^n = u(x_i, t_n)$ be the value of the exact solution at x_i and t_n.

The consistency error R_i^n at (x_i, t_n) can be written as the sum of time and space consistency errors

$$R_i^n = \hat{R}_i^n + \tilde{R}_i^n,$$

with

$$\tilde{R}_i^n = \frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{k} - u_t(x_i, t_n),$$

and

$$\hat{R}_i^n = \frac{2\bar{u}_i^n - \bar{u}_{i+1}^n - \bar{u}_{i-1}^n}{h^2} - u_{xx}(x_i, t_n).$$

Proposition 3.3. *[7] The scheme (3.8) is consistent of order 1 in time and order 2 in space, i.e., there exists $c > 0$ depending only on u such that*

$$|R_i^n| \leq c(k + h^2).$$

3.2.4 Stability of the scheme

From the previous proposition, the exact solution satisfies

$$\|u^n\|_{L^\infty((0,1)\times(0,T))} \leq \|u_0\|_{L^\infty(0,1)}.$$

If the spatial step is chosen correctly, we obtain a discrete analogue of this property.

Definition 3.1. *A scheme is said to be L^∞-stable if the numerical solution is bounded in L^∞, independently of the mesh size.*

Proposition 3.4. *[7] If the stability condition $\lambda = \dfrac{k}{h^2} \leq \dfrac{1}{2}$ is satisfied, then the scheme (3.8) is L^∞-stable in the sense that*

$$\sup_{i=1,\dots,N;\, n=1,\dots,M} |u_i^n| \leq \|u_0\|_{L^\infty(0,1)}.$$

3.2.5 Convergence

Definition 3.2. *Let u be the exact solution of the problem and (u_i^n) the solution of the numerical scheme. The discretization error at the point (x_i, t_n) is defined by*

$$e_i^n = u(x_i, t_n) - u_i^n.$$

Theorem 3.2. *Under the assumptions of the existence and uniqueness theorem and under the stability condition, there exists a constant $C > 0$, depending only on the exact solution, such that*

$$\|e^{n+1}\|_\infty \leq \|e^0\|_\infty + TC(k + h^2),$$

for all $i = 1, \dots, N$, $n = 0, \dots, M-1$.

3.2.6 Discrete maximum principle

We consider the explicit finite difference scheme (3.8)

$$u_i^{n+1} = u_i^n + \frac{k}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + k\, e^{-t_n} \sin(\pi x_i).$$

Let us denote $\mu = \frac{k}{h^2}$. Then the scheme can be rewritten as

$$u_i^{n+1} = (1 - 2\mu)u_i^n + \mu u_{i-1}^n + \mu u_{i+1}^n + k\, e^{-t_n} \sin(\pi x_i).$$

Theorem 3.3 (Discrete maximum principle). *Assume that* $0 < \mu \le \frac{1}{2}$, *and that*

$$e^{-t_n} \sin(\pi x_i) \ge 0, \quad \textit{for all } i = 0, ..., N+1;\ n = 0, ..., M.$$

Then the numerical solution satisfies

$$u_i^n \ge 0 \quad \textit{for all } i = 0, \dots, N+1,\ n = 0, \dots, M.$$

Moreover, the maximum satisfies

$$\max_i u_i^{n+1} \le \max_i u_i^n + k \max_i \big(e^{-t_n} \sin(\pi x_i)\big).$$

Proof. We proceed by induction on n.

From the initial condition,

$$u_i^0 = 0 \quad \text{for all } i = 0, ..., N,$$

hence

$$u_i^0 \ge 0.$$

Assume that for some $n \ge 0$,

$$u_i^n \ge 0 \quad \text{for all } i = 0, ..., N.$$

Using the scheme,

$$u_i^{n+1} = (1 - 2\mu) u_i^n + \mu u_{i-1}^n + \mu u_{i+1}^n + k\, e^{-t_n} \sin(\pi x_i).$$

Since $0 < \mu \le \frac{1}{2}$, we have

$$1 - 2\mu \ge 0, \quad \mu \ge 0.$$

Thus all coefficients are nonnegative and satisfy

$$(1 - 2\mu) + \mu + \mu = 1.$$

Hence the first three terms form a convex combination:

$$(1 - 2\mu) u_i^n + \mu u_{i-1}^n + \mu u_{i+1}^n \ge 0.$$

Moreover, since

$$e^{-t_n}\sin(\pi x_i) \geq 0,$$

we obtain

$$k\,e^{-t_n}\sin(\pi x_i) \geq 0.$$

Therefore,

$$u_i^{n+1} \geq 0.$$

Maximum estimate: Let $M^n = \max_i u_i^n$. Then

$$u_{i-1}^n \leq M^n, \quad u_i^n \leq M^n, \quad u_{i+1}^n \leq M^n.$$

Thus,

$$(1-2\mu)u_i^n + \mu u_{i-1}^n + \mu u_{i+1}^n \leq M^n.$$

Hence,

$$u_i^{n+1} \leq M^n + k \max_i \left(e^{-t_n}\sin(\pi x_i)\right).$$

Taking the maximum over i, we get

$$M^{n+1} \leq M^n + k \max_i \left(e^{-t_n}\sin(\pi x_i)\right).$$

This concludes the proof. □

Remark 3.2. *The above discrete maximum principle is a classical result for explicit finite difference schemes for parabolic equations. It follows from the monotonicity of the scheme under the CFL condition $\mu \leq \frac{1}{2}$, see for instance LeVeque [13] or Smith [18].*

3.2.7 Numerical results

To ensure that the discrete maximum principle holds for the numerical solution of the nonhomogeneous heat equation (3.6), we implemented a MATLAB code based on the finite difference scheme described above, under appropriate assumptions guaranteeing consistency, stability, and convergence.

```
clc; clear all;
N = input('N=');
M = input('M=');
a = 0;
b = 1;
T = 0.2;
h = (b-a)/(N+1);
k = T/M;
t = 0:k:T;
alpha = 0;
betta = 0;
x = a+h:h:b-h;
F = k*sin(pi*x);
F(1) = F(1) + alpha*k/h^2;
F(end) = F(end) + betta*k/h^2;
A = zeros(N,N);
U = zeros(N,M+1);
U(:,1) = zeros(1,N);
A = diag((1-2*k/h^2)*ones(1,N)) ...
+ diag((k/h^2)*ones(1,N-1),-1) ...
+ diag((k/h^2)*ones(1,N-1),+1);
for n = 1:M
U(:,n+1) = A*U(:,n) + exp(-t(n))*F';
end
U = [alpha*ones(1,M+1); U; betta*ones(1,M+1)];
x = [a, x, b];
%%%%%%%%%%%% Exact solution %%%%%%%%%%
Uexacte = zeros(N+2,M+1);
for n = 1:M+1
for i = 1:N+2
```

```
Uexacte(i,n) = ((1-exp(-(pi^2+1)*t(n)))/(pi^2+1)) ...
* sin(pi*x(i));
end
end
figure(444)
[tg, xg] = meshgrid(t, x);
surf(tg, xg, U)
grid on
xlabel('t')
ylabel('x')
zlabel('Approximate-solution')
figure(445)
surf(tg, xg, Uexacte)
grid on
xlabel('t')
ylabel('x')
zlabel('Exact solution')
```

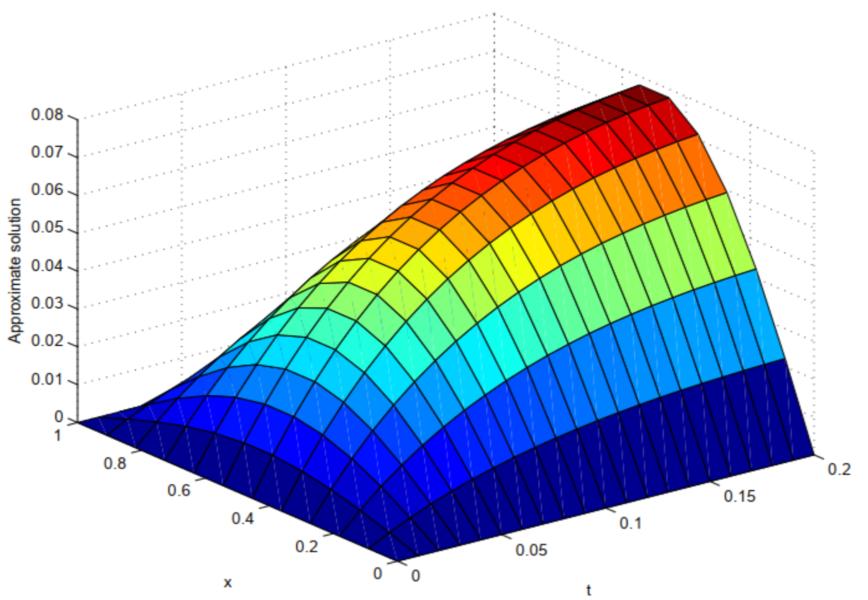

Figure 3.5: Numerical solution of problem (3.6) for $N = 10$ and $M = 20$.

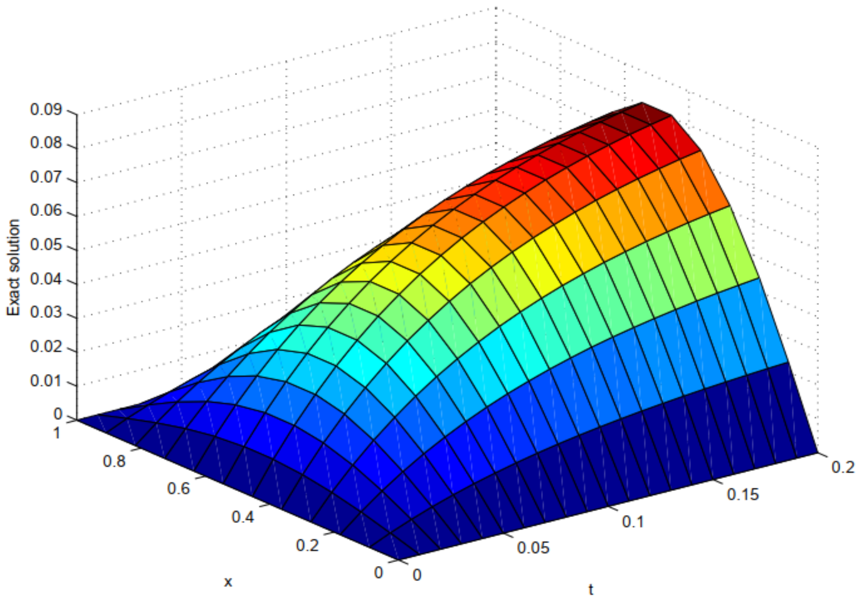

Figure 3.6: Exact solution of problem (3.6) for $N = 10$ and $M = 20$.

From the Figures 3.5 and 3.6, we observe that at each time instant, the solution reaches its extrema on the boundaries, either at 0 or at 1. This confirms the theoretical results established in chapters two and three.

Conclusion

In this thesis, we have studied the Maximum Principle from both theoretical and numerical perspectives. We began by presenting the main fundamental concepts in functional analysis and partial differential equations, such as Sobolev spaces, differential operators, and boundary and initial conditions.

We then devoted the theoretical part to the study of the Maximum Principle for elliptic and parabolic equations. In this context, we introduced both the weak and strong forms of the principle, as well as Hopf's lemma, together with their applications to Laplace and Poisson equations and the heat equation. These results allowed us to establish several fundamental properties of solutions, including uniqueness, stability, positivity, and the derivation of a priori estimates.

On the numerical side, we studied the Discrete Maximum Principle within the framework of finite difference methods. We discussed the notions of consistency, stability, and convergence, and examined the conditions that ensure numerical schemes preserve the qualitative properties of the original problem. This property is essential for obtaining stable and physically meaningful numerical solutions.

Through this study, it has been shown that the Maximum Principle is a fundamental tool in both the mathematical and numerical analysis of partial differential equations, due to the important results it provides in understanding the behavior of solutions and ensuring their stability.

Finally, this work can be extended in the future to include nonlinear equations, finite element methods, and multi-dimensional problems. Preserving the Maximum Principle remains an important and active topic in scientific research.

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