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by:

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Study of Spectrum lines Exposure in Magnetized Plasma using Path Integrals Formalism

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Dedication

I would be remiss my family, especially

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my siblings "FATMA", "AMARA", "MED SEGUIR" and "CHAMA",

and my daughter "ELIA".

Their belief in me has kept my spirits and motivation high during this process.

Contents

Abbreviations and Nomenclature

D-T	Deuterium and tritium
D-D	Deuterium and Deuterium
ICF	Inertial Confinement Fusion
MIF	Magneto-Inertial Fusion
JET	Join European Torus
ITER	International Thermonuclear Experimental Reactor
TOKAMAK	Toroidal Chamber with Magnetic Coils
FWHM	Full Width at Half Maximum
ICRH	Ion Cyclotron Resonance Heating
ECRH	Electron Cyclotron Resonance Heating
CCFE	Culham Center for Fusion Energy
EFDA	European Fusion Development Agreement
NU	Nikiforov-Uvarov
N_e	Electronic density
N_i	Ionic density
Z	Ionizations number
e	Electron charged
E_c	Kinetic energy
T	Temperature
K_B	Boltzmann's constant
h	Planck's constant
r_0	Landau lenth
ϵ_0	Vacuum permittivity
λ_D	Debye's length
ω_e	Electronic Plasma Frequency
m_e	Electron mass
v_{th}	Thermal velocity
v_{T_e}	Electron thermal velosity
L	Dimension system
N_D	Particles number in Debye sphere
g_p	coupling parameter

E_p	Potential energy
τ_{col}	The average lifetime between two collisions
$\overrightarrow{M_L}$	The orbital magnetic moment
$\overrightarrow{M_S}$	The spin magnetic moment
$\overrightarrow{M_I}$	The nuclear magnetic moment
$\overrightarrow{L}$	The total orbital moment
$\overrightarrow{S}$	The spin angular moment
$\overrightarrow{I}$	The nucleus spin angular moment
τ_p	The energy confinement time
W	The plasma total energy
P_{losses}	The energy losses in The fusion plasma
$\Omega_{c,\alpha}$	Cyclotron frequency
ρ_c	Larmor raduis
B_θ	The poloidal field
B_ϕ	The toroidal field
I_ϕ	The total current flowing through the coil
$\overrightarrow{A}$	The vector potential
$\overrightarrow{B}$	The external magnetic field
ε_0	The magnetic field strength along z-axis
μ_B	Bohr's magneton
μ	The reduced mass
B_0	The toroidal magnetic field strength

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General Introduction

The science of plasma physics dating from the 20th century was born from the study of gas discharges [1–3]. Since 1920 (it was in 1923 that the physicists Langmuir and Tonks first introduced the term "plasma" to designate the ionized gas contained in a type of discharge) [4], this discipline has developed considerably due to its industrial and academic interests (natural environments, industrial applications, etc.). Integrating most of the knowledge of modern physics, in nature, plasma is the fourth state of matter and follows, in increasing order of temperature, the solid, liquid, and gaseous states. Plasma is currently thought to make up 99% of matter [1–3]. Therefore, plasma physics holds an important place in the study of natural environments, especially astrophysical ones, as well as ionized gases produced in the laboratory. The complexities of this problem necessitate investigation into all branches of physics, including statistical mechanics, collision theory, transport equations, and so on.

Spectroscopy is often used as a mean of diagnosis in plasmas because it is a "probe" that does not disturb the medium [5]. The radiation emitted by the plasma is decomposed either by refraction or by diffraction, and depending on whether is in emission or absorption, we can see bright or dark lines that stand out on a continuous spectrum [1, 3, 5]. We want to deduce information about the physical conditions (temperature, densities, velocity fields, electric field, magnetic field, and so on) that exist in this continuum from the intensity of the continuous spectrum, specifically the intensity or profile of the lines. Theoretical modeling of spectral lines provides very satisfactory predictions of their shapes. this has allowed, in many cases, the precise realization of the diagnosis in terms of the density and the plasma temperature. However, there remain some disagreements between the different models, on one hand, and between the observations and the predictions of the models, on the other hand. These relate, for example, to the dynamic effects of particles, in particular when several of them act simultaneously on the emitter.

The study of electromagnetic wave propagation in a magnetised plasma is the subject of a fundamental research axis in the context of several scientific and technological applications. Among these applications, we can cite thermonuclear fusion. Controlled thermonuclear fusion is a process for exploiting nuclear energy without nuclear waste, without risk, and inexhaustible (deuterium is found in sufficient quantity in the water of the oceans, and tritium can be produced in the fusion reaction itself). One of the great technological challenges of today is the realisation of this fusion to produce electrical energy. Indeed, the energy of thermonuclear fusion has been known for several decades in the sun and in the stars. The principle of fusion is to collide light atoms with each other to produce heavier ones while releasing energy under special conditions of temperature (100 million degrees) and pressure. Many projects to produce fusion reactions have been born. The first magnetic confinement experiments on plasma were carried out in 1940. In Russia, it was in 1968 that a particular form of reactor was discovered that was more stable and more promising for carrying out fusion by magnetic confinement. It is the Tokamak configuration.

Our problem is about the extent of the results that would be obtained from the magnetic field geometry chosen to approximate a simulation of reality in the tokamak reactor.

The most advanced and successful method to achieve controlled fusion is the confinement of plasmas by sufficient magnetic fields.Finding a magnetic field geometry that efficiently confines the plasma has been a significant component of the confinement problem. A group of toroidal magnets is used to do this by creating a toroidal field. It makes sure that the charged particles are contained inside the torus. It is demonstrated that this confinement is insufficient and that the field lines must be helical along the torus in order to further reduce the leakage of particles towards the walls. To do this, a magnetic field called a poloidal field that is perpendicular to the toroidal field is added. Before the fusion starts, a number of devices and experiments must be ready to ensure that these conditions are met. Feldman et al. talked about how to detect the poloidal magnetic field in a tokamak plasma using Zeeman splitting and the polarisation of magnetic dipole radiation from heavy ions [6, 7]. In the same year (1985), The Toroidal Cusp Experiment (TCX) was equipped with a dye laser-generated resonant fluorescence scattering device by P. Gao et al. [8] Measurements were made of the line shapes of the Balmer series of hydrogen/deuterium in the edge and divertor regions of the Alcator C-Mod tokamak by Welch et al. [9] in 1996. J. D. Hey et al. [10, 11] talked about the Zeeman spectroscopic technique for tokamak diagnosis in 2002. J. D. Hey et al. [10, 11] talked about the Zeeman spectroscopic technique for tokamak diagnosis in 2002. The Dalpha and Halpha spectra were analysed by Koubiti et al. [12] to produce a number of neutral populations with temperatures ranging between $1 - 3$ eV and $10 - 30$ eV . In order to enhance the study, Y. Marandet et al. [13] fitted the model to the experimental spectra using an effective fitting technique based on a genetic algorithm (GA). According to the analysis's findings, there may be a population of neutrals with several hundred electron volts. On the other hand, the positions of emission of hydrogen atoms and oxygen ions in the limiter shadow's boundary region as well as beryllium-like oxygen ions in the core region were measured using the TRIAM-M1 fusion device's poloidal section's difference of the Zeeman patterns [14]. Shikama et al. used a linear polarizer to resolve the s components of the Zeeman spectra in order to calculate the magnetic field strength. They also measured the bulk ion temperature in the core region [15]. For the motional Stark effect (MSE) diagnostic on magnetic fusion devices, M. F. Gu et al. [16] presented comprehensive atomic physics models in 2008. In the same year, J. Rosato et al. [17] explored when the conditions for impact ion broadening were almost satisfied and published calculations of hydrogen Zeeman-Stark line profiles. In order to make measurements, carbon impurities in high-temperature plasma produced spectra that were parallel to the magnetic field. In 2011, A. Iwamae et al.

[18] published the strength of emission lines in the ITER divertor: the intensity of hydrogen isotopes and impurities was recorded with the divertor's central optical system. M. Koubiti et al.'s 2012 study [19] focused on the C IV n = 6-7 (l = 772.6 nm) line and examined the widening mechanisms affecting carbon lines emitted from tokamak divertor plasmas. They simplified the computations by simply taking into account Doppler and Stark broadenings, ignoring the Zeeman effect. These computations, which neglected the impact of the magnetic field, could thus be contrasted with spectra obtained by using a linear polarizer that only transmitted the p component of light with polarisation parallel to the toroidal magnetic field. It was possible to determine the various plasma properties by comparing theoretical profiles to high-resolution observed C IV n = 6-7 line spectra [19]. J. Rosato et al. estimated line shapes and S-matrix components for the first hydrogen Lyman lines in the same year, and they offered two models for concurrently retaining the Stark and Zeeman effects in the impact limit [20]. The emitters' frame of reference's Lorentz electric field $\vec{v} \times \vec{B}$ caused what is known as a "motional" Stark effect, which is a disruption of the atomic energy levels. Wei Gao et al. [21] used a high resolution Optical Spectroscopic Multichannel Analysis (OSMA) equipment in the EAST tokamak to measure the D_α atomic emission spectra in the boundary region of the plasma in 2017 on the basis of passive spectroscopy.

Chapter 1 is essentially an introduction to certain concepts that will be useful later. In this chapter, we present general concepts on controlled thermonuclear fusion, the principle of which is to fuse two light nuclei to have a heavier nucleus and a considerable amount of energy under extreme conditions of pressure and temperature similar to those found in the core of the sun in order to create a plasma. This fourth state of matter is the origin of most of the elements that surround us. We then approach the Lawson criterion, which presents the conditions that must be met to maintain the fusion reaction. Thereafter, we expose the two ways of fusion (inertial and magnetic) as well as the new technique of fusion called magneto-inertial.

The confinement of plasmas by adequate magnetic fields is the most advanced and widely used approach for achieving controlled fusion. Finding a magnetic field geometry that efficiently confines the plasma has been a huge part of the confinement problem. This is accomplished by a group of toroidal magnets producing a toroidal field. It makes sure that the charged particles are confined inside the torus. It is demonstrated that this confinement is insufficient, and the field lines must be helical along the torus in order to further reduce the leakage of particles towards the walls. To do this, a magnetic field called a poloidal field that is perpendicular to the toroidal field is added. Before the fusion starts, a number of devices and experiments must be ready to ensure that these conditions are realized.

Let us limit our focus to hydrogen-like ions because of its simple atomic structure, we are firstly interested in how a nonuniform magnetic field affects the quantum dynamics of ions in fusion plasma. Secondly, The spectral line shapes (Lyman-alpha) of three different types of ions He^+, C^{5+}, and $Ar^{17+}$for various magnetic field intensities have also been determined using the derived eigenenergy. Moreover, as we will see, the Zeeman separation, for large values of B_0(which is a factor in the toroidal magnetic field along $\overrightarrow{u_\phi}$), is much larger than the separation of the fine structure (for example, the latter is about 5 eV , while the Zeeman separation is much larger). As a result, we will proceed without fine structure in the next chapter.

The second chapter is devoted to presenting the quantum mechanical equations that describe the ions in the presence of the nonuniform magnetic field. We discovered a new method for removing degeneracy using a magnetic field. This geometry is a new addition to the study of plasma fusion in the tokamak. In the same chapter, we conduct a discussion and present some spectral line shapes for Ly-alpha of three hydrogen-like ions for different magnetic fields and temperatures.

Since hydrogenoid plasma is a many-body system, we have studied it as a problem in mean field theory. As the spectral properties of the hydrogen atom (energy levels) are well established, this system is the simplest in terms of its atomic structure and thus allows an exact analytical solution. Because of the different plasma ions, each hydrogen atom in the hydrogen plasma is subjected (perturbed) to a time-varying electric field. The response to this disturbance is radiation collected by a spectroscopic measuring instrument. Let us emphasize that the radiation collected is the result of the superposition of individual radiations (individual intensities), each with a probability density that characterizes the disturbing electric field of each hydrogen atom. Hence the collective context that we mentioned at the beginning of the introduction, and following the quantum treatment of the problem, a statistical treatment of the fluctuating electric field is necessary for the accomplishment of this task, it is devoted to the computation of the propagator of the problem. In (chap 3), we propose a form for the electrical field correlation function. It is possible to simulate the collisions of electrons or fast ions with the emitter in plasmas. we find a propagator whose derived lagrangian is similar to that of its hydrogen atom with a charge multiplied by a parameter-dependent function. This function in turn obeys an auxiliary equation type constraint [22]. Next, we present a Coulomb problem with a quadratic term and a time-dependent coefficient. We have solved this problem using canonical, unitary, and spatio-temporal transformations [23]. Finally, we find the wave function of the Hamiltonian of the Coulomb system perturbed by a quadratic term with a time-dependent coefficient. Then, We presented the technique to evaluate the propagator relative to the lagrangian while describing the Coulomb problem, whose charge depends on a parameter function. If the correlation function of the $C_{EE}(t)$ field is chosen as a function inversely proportional to the square of time t, then the function of the parameter depends exponentially on it. We then end up with a propagator (Green's function) similar to that of the Coulomb system; the only difference is that the energy is shifted. We tested our calculation by setting the parameter's function equal to "one", and we found the exact Green's function of the Coulomb system.

Chapter 1

Overview of Controlled Thermonuclear Fusion and Line Profiles

1.1 Introduction

Energy consumption has never been higher than in the two centuries since the industrial revolution. It is a safe bet that this demand will continue to increase during the beginning of this century for two main reasons: on the one hand, because of the increase (the world population, which should pass 10 billion in 2050), and on the other, because the industrialization of countries currently in the process of development will require greater energy needs. Indeed, energy needs will continue to increase while fossil fuel resources (oil, coal, natural gas, etc.) are running out.

In this race for energy, man has tried to control nuclear reactions: fission and fusion are the two ways to extract energy from an atomic nucleus. The first way consists in breaking heavy nuclei like uranium (U^{235}) to produce two lighter ones. It is this fission process that is used in current nuclear power plants. The negative impact of fossil fuels on the environment, in particular the production of long-lived radioactive waste, poses a serious problem for their sustainable extension, to say nothing of the difficulties linked to the safety of fission plants.

The second way consists in fusing two nuclei of light atoms, like hydrogen (H), to produce a heavier one; this is the fusion reaction. Controlled thermonuclear fusion therefore appears to be a promising avenue among alternative energy candidates. However, the scientific but also economic demonstration of fusion as a potential source of energy for the long term remains to be established, making this ambitious project a challenge combining high development costs with great technological complexity.

Controlled thermonuclear fusion has been studied for seven decades. In this chapter, we recall the fundamental notions of fusion plasma and controlled thermonuclear fusion.

1.2 Fusion Plasma

The difference between the states of matter is based on the arrangement of the atoms. In the solid state, the atoms are tight against each other in a rigid network (as in ice, for example). When the temperature rises, we pass to the liquid state (the ice liquefies), where the space between the atoms increases and they can slide relative to each other, which allows the liquid to take the shape of a container. If we heat further, we arrive at the gaseous state: the atoms then move freely, independently of each other (the water has turned into vapour). Finally, when the gas is exposed to intense energy, electrons are torn from the atomic nucleus and form a globally neutral mixture: it is plasma (see figure 1.1).

Plasma, also called a gas of charged particles, is an electrically ionized gas that is macroscopically neutral and characterized by collective effects, so plasma can therefore be considered a mixture of positively charged ions and negatively charged electrons coexisting with neutral atoms and molecules. In the plasmas produced for fusion experiments, the gas is strongly ionized and the atoms are in low proportion. The charged particles in the plasma interact with each other by long-range forces, the Colombian forces; each particle can interact with all the particles, which means that the behavior of the plasma is collective [25].

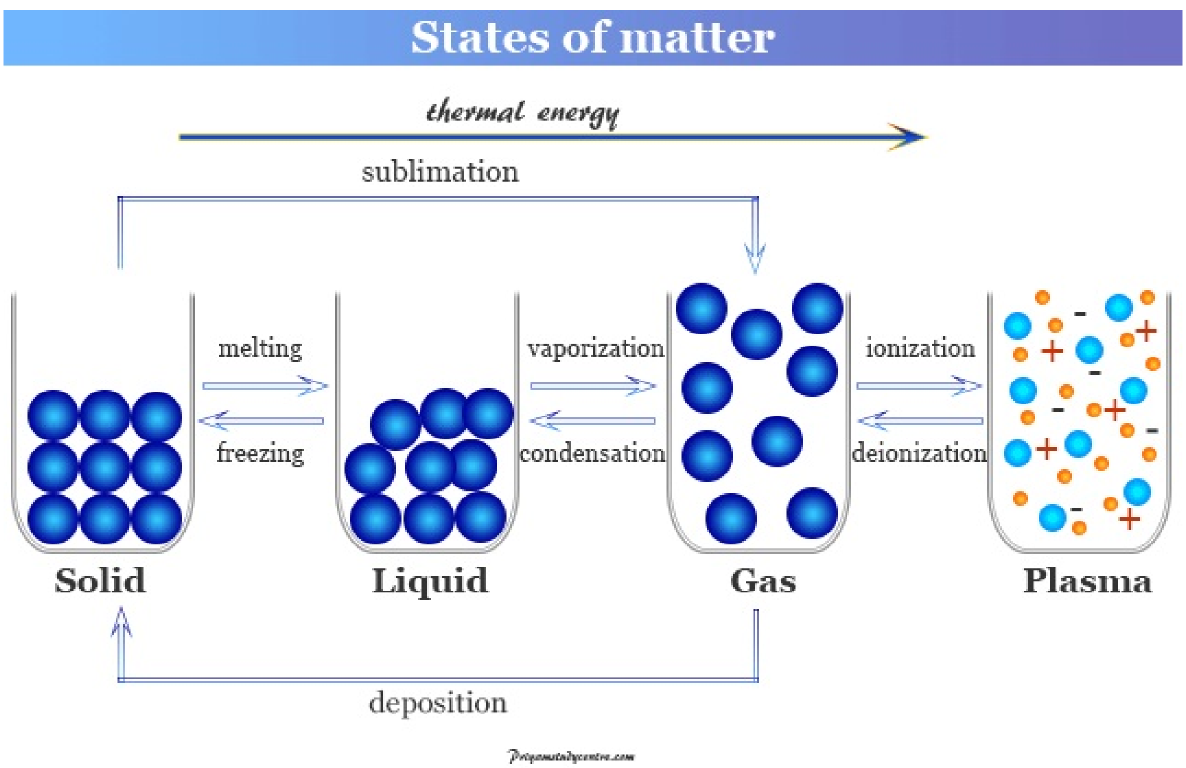

Fig. 1.1: states of matter [24].

1.2.1 Plasma Creation

Plasma creation can be done in several ways:

1. **The Creation of Plasma by Thermal Heating:**

   When the temperature of the solid is increased, the energy of the thermal agitation of the atoms and molecules exceeds the binding energy, which results in a phase change from the solid state to the liquid state. The increased thermal energy of liquid breaks all the bonds between atoms, which results in a phase change to the gaseous state.

   The transition to the plasma state is done by increasing the temperature, which gives very high kinetic energy to the atoms. During collisions between these atoms, electrons are torn off and the gas becomes an ionized gas containing electrons, ions, and neutrals. [25]

2. **The Creation of Plasma by the Application of an Intense Electromagnetic Field:**

   A gas always containing some free charges, the application of an intense electric field can communicate sufficient energy to the most mobile particles, the electrons, and during collision of these electrons with the neutral atoms one creates ions and other electrons. The extra electrons can then be accelerated by the used electromagnetic field, collide with the neutrals, and generate plasma through an electron avalanche ionization mechanism [25, 26].

1.2.2 Plasma Characteristics

An ionized gas must satisfy three conditions to be plasma, which are:

1- The total sum of the charges is zero in a macroscopic volume, so the plasma is globally an electrically neutral medium. It is quasi-neutrality that requires that [26]:

$$-eN_e + eZN_i = 0 \tag{1.1}$$

where:
N_e: The electronic density.
N_iThe ionic density.
Z: The number of ionizations.

2- In a fully ionized plasma, the interactions that occur between particles are electromagnetic and therefore fundamentally different. Unlike interactions between neutral particles, these interactions are long-range, with the field created by a charge decreasing only at $\frac{1}{r^2}$. A given particle can be sensitive to a very close neighbor (near binary interaction), but it is also sensitive to all the others via the electromagnetic fields they create. We speak about "collective" interaction when, in a given region, a particle is mainly subject to the mean field created by all the others. These fields can also be imposed from outside. In the case of a partially ionized plasma, the two types of interactions (short-range and electromagnetic) are involved in the physics of the medium at the same time. Electrons or ions can undergo collisions with neutral particles or with other charged particles [27].

1.2.3 The Fundamental Quantities of Plasma

Plasmas are made up of electrons, ions, and neutrals whose interactions can be described by two main quantities: density and temperature [26].

1. **Density:**

   We define density as the number of particles per unit volume, N_e. If the ions have a charge $+Ze$, the ionic density N_i is obtained by the condition of charge neutrality, $eN_e = ZeN_i$; this is the quasi-neutrality hypothesis where $N_i = \frac{N_e}{Z}$; if Z is 1, then we have $N_e = N_i$ [26].

2. **Temperature:**

   Because a plasma contains ionized species (electrons and ions), it will be necessary to distinguish between electronic temperature $T_e$and ionic temperature T_i. At thermal equilibrium, these two quantities are equal. In plasma physics, the kinetic energy of electrons or ions is measured by their temperature ($E_c = K_B T$), where K_B is Boltzmann's constant.

   - In the case of "cold plasmas", the temperature (energy) of the electrons is much higher than that of the ions. The ions are considered "cold" and will only be able to make possible chemical reactions with their energy.

   - In hot plasmas, the ions are "hot" and therefore more reactive [26].

1.2.4 The Characteristic Lengths of Plasma

1. **Landau Length r_0:**

   Let an electron with kinetic energy $\frac{1}{2}m_e v^2 = T_e$meet another electron. The potential energy of this one due to the presence of the other electron is: $V(r) = \frac{e^2}{4\pi\epsilon_0 r}$.

   where $\epsilon_0$is the vacuum permittivity.

   The Landau length r_0 corresponds to the distance for which the kinetic energy of the electron is equal to the potential energy of the Colombian interaction of the two electrons: $r_0 = \frac{e^2}{4\pi\epsilon_0 T_e}$

   The Landau length is involved in the analysis of collision phenomena and in that of position correlations in a plasma [25].

2. **Debye's Length λ_D:**

   Consider a quasi-neutral plasma. If we try to disturb this plasma by breaking its quasi-neutrality by injecting a test charge, Q, into it, the charged particles of the plasma (electrons and ions) quickly rearrange to form a screen around this charge, called the sphere of Debye, with the aim of limiting the interaction of this charge with the plasma inside a small sphere called the sphere of Debye, of radius $\lambda_D = \sqrt{\frac{\epsilon_0 T_e}{N_e e^2}}$. Therefore, the rest of the plasma does not feel the effect of this charge.

where $N_e$is the electron density of the plasma. T_e and e are respectively the temperature, the charge of the electron.

In the plasma, the potential of the test charge, Q, is screened [26]:

$$V_D(r) = \frac{Ze}{4\pi\epsilon_0 r} e^{\frac{-r}{\lambda_D}} \tag{1.2}$$

1.2.5 The Characteristic Frequencies of Plasma

1. **Plasma Frequency:**

   If an initially quasi-neutral plasma is disturbed by displacing the electrons from their equilibrium position (by an electric field), an electric field due to the space charge (separation of negative and positive charges) is generated. This field generates an electric restoring force, which causes the electrons to oscillate around the equilibrium position at a frequency called natural plasma frequency. The phenomenon is called natural plasma oscillation [25].

   $$\omega_e = \sqrt{\frac{N_e e^2}{\epsilon_0 m_e}} \tag{1.3}$$

   where N_e is the electron density of the plasma, e and m_e are the charge and mass of the electron, respectively, and ϵ_0 is the vacuum permittivity.

   The formula (1.3) gives the frequency of the oscillations associated with the electrons and is, as such, called electronic plasma frequency. An ion plasma frequency can be defined by an analogous formula by replacing the electronic mass m_e by the ion mass m_i. There is a simple relationship between electron thermal velocity v_{T_e}, Debye length λ_D, and electron plasma frequency ω_e [25].

   $$v_{T_e} = \lambda_D \omega_e \tag{1.4}$$

2. **Collision Frequency:**

   The collision frequency is the average frequency between two successive collisions; we distinguish the collisions as electron-ion v_{e-i}, electron-neutral v_{e-n}, and ion-neutral v_{i-n}.

1.2.6 Plasma Criteria

For an ionized medium to be plasma, the Debye screen must be effective, and the plasma oscillation must be able to maintain the oscillating plasma in the equilibrium position where quasi-neutrality is established. For this, conditions on the Debye sphere depend on the charge in the plasma and on the natural plasma frequency. must be verified:

- First criterion: A necessary and obvious condition is that the dimensions of the studied system must be much larger compared to the Debye length. If we assume that the dimension of the system is L, then we must have the relation [25]:

$$L \gg \lambda_D \tag{1.5}$$

- Second criterion: The screening phenomenon being a statistical phenomenon, it is necessary that the number of particles in the Debye sphere be very important. One poses [25, 27]:

$$N_D = \frac{4\pi}{3} N_e \lambda_D^3 \gg 1 \tag{1.6}$$

- Third criterion: To avoid the attenuation of the plasma oscillation by collision, it is necessary that the frequency of collisions be lower than the plasma frequency [25, 27].

$$\omega \gg v \tag{1.7}$$

1.2.7 Classification of Plasmas

There is a very wide variety of plasmas classified into families according to several parameters:

1. **Ideal and Non-ideal Plasma:**

   To classify the plasmas, we use a classification parameter or a coupling parameter given by [27]:

   $$g_p = \frac{E_p}{E_c} = N_e \lambda_D^3 \tag{1.8}$$

   where:

   $E_c = T_e$ is The thermal agitation energy of electrons.

   $E_p = \frac{e^2}{4\pi\epsilon_0} n^{\frac{1}{3}}$ is The potential energy of the particle interaction.

   - If $g_p \gg 1 \Rightarrow E_p \gg E_c$, The plasma is said to be weakly coupled; in this case, the Colombian collisions are dominant.

   - If $g_p \ll 1 \Rightarrow E_p \ll E_c$, the plasma is said to be strongly coupled, in this case the potential energy of the Colombian interaction is negligible compared to the thermal agitation.

2. **Hot Plasmas and Cold Plasmas:**

   The grading parameter is the electron temperature $T_e(\mathrm{eV})$ [27].

   - If $T_e \sim \mathrm{KeV}$; the plasma is said to be hot (thermonuclear fusion plasma, interstellar medium, etc.).

   - If $T_e \sim \mathrm{eV}$; the plasma is said to be cold (electric discharge plasma, etc.).

3. **Dense Plasmas and Non-dense Plasmas:**

   The classification parameter is the electron density $N_e(\mathrm{m}^{-3})$ [27].

   - If $N_e \sim \mathrm{cm}^{-3}$; the plasma is said to be non-dense.

   - If $N_e \sim 10^{19}\mathrm{cm}^{-3}$; the plasma is said to be dense.

4. **Relativistic Plasmas and Non-relativistic Plasmas:**

   The classification parameter is the thermal velocity v_{th} [27].

   - If $v_{th} \ll c$; the plasma is said to be non-relativistic.

   - If $v_{th} \gg c$; the plasma is said to be relativistic.

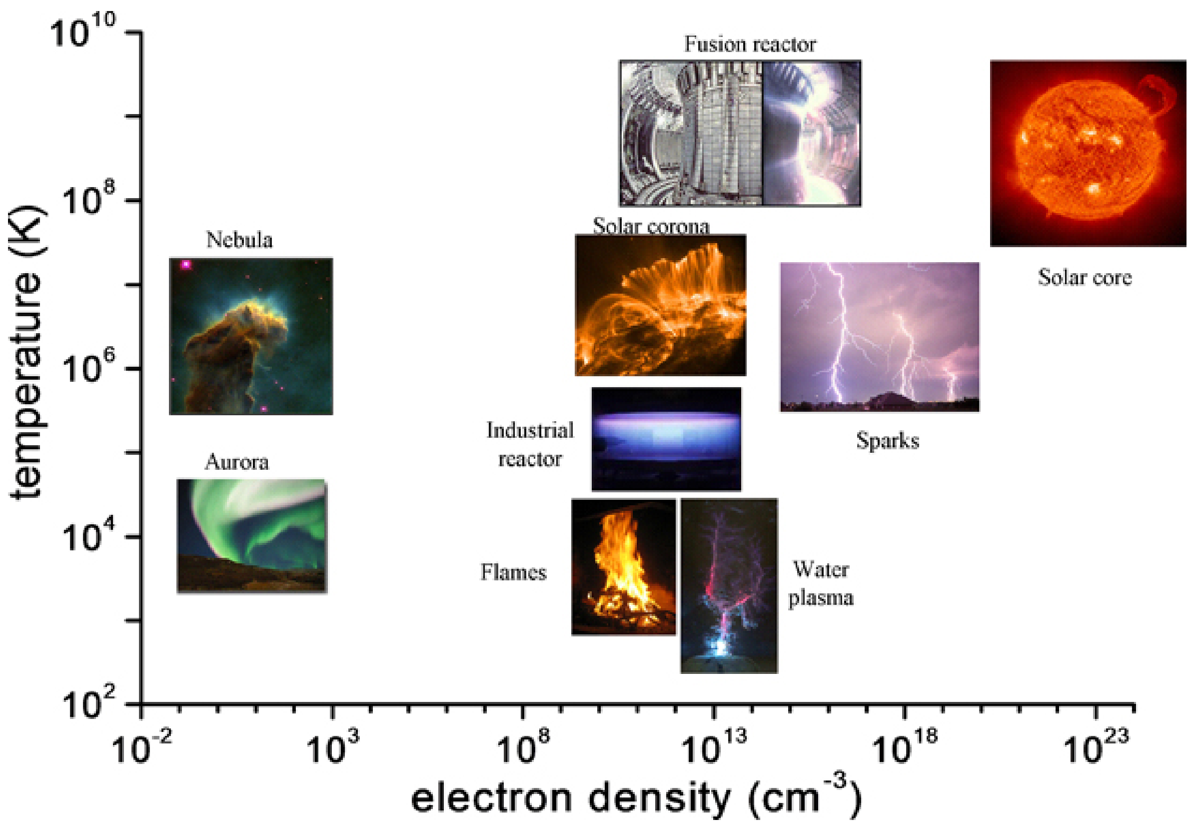

Fig. 1.2: Classification of natural and laboratory plasmas in a logarithmic diagram charges density/temperature [28].

1.3 Broadening and Formality of the Line Profile

Line profiles are an important diagnostic tool in plasma. Generally, the width of a line is measured at mid-height of the line, or the value of the intensity is equal to half of the maximum intensity $I_{\max}$ at the center of the line [29]. These profiles undergo broadenings of several types, of which we can cite:

1.3.1 Natural Broadening

It is known that the energy levels of a quantum system have some natural uncertainty, called "natural broadening" [30], given as:

$$\Delta E = \frac{\hbar}{\tau} \tag{1.9}$$

Where: ΔE is the uncertainty on the level of energy, and τ is the average lifetime of the quantum state, and τ_i is the average lifetime of an excited level of energy E_i; this level has a width of ΔE_i.

In this case, the linewidth at mid-height (FWHM) is given as [30]:

$$\Delta\omega = \delta E_i + \delta E_j \tag{1.10}$$

$\Delta\omega_{ij}$: presents the linewidth at mid-height (FWHM), and the shape of the profile is a Lerentzian profile [30]:

$$g(x) = \frac{1}{\pi}\frac{\delta_1}{x^2 + \delta_1^2} \tag{1.11}$$

where:

$$\delta = \frac{\frac{1}{\tau_i} + \frac{1}{\tau_j}}{4\pi} \tag{1.12}$$

$$x = \omega - \omega_{ij} \tag{1.13}$$

1.3.2 Dopple Broadening

When a particle is moving, its emission frequency differs from when it is stationary; This difference causes a broadening called Doppler broadening.

The widening due to the statistical Doppler effect is related to the distribution of the speeds of the emitter at the temperature T of the medium and at the mass M of the emitter [31].

$$\Delta\omega_D = 7.16 \times 10^{-7}\omega_0\sqrt{\frac{T}{M}} \tag{1.14}$$

ω_D and ω_0 are in eV, T is in K, and M is in Uma. The shape of the profile in this case is a Gaussian [31].

$$g(x) = \frac{1}{\sqrt{\pi}}\frac{1}{B}\exp\left(-\frac{x^2}{B^2}\right) \tag{1.15}$$

where: $B = \Delta\omega_D$ and $x = \omega - \omega_0$.

1.3.3 Collisional Broadening by Neutral Particles

Collisions with neutral particles also cause a broadening of the spectral lines; their width is given as follows [29]:

$$\Delta\omega_{col} = \frac{1}{\tau_{col}} \tag{1.16}$$

with τ_{col} is the average lifetime between two collisions. The profile of a line expanded by the electronic stark effect is a Lorentzian profile.

1.3.4 Collisional Broadening by Charged Particles (Stark Effect)

The interaction of charged particles (ions and electrons) with the emitter results in stark broadening. The electric field at the emitter, also called the plasma microfield, has two components [32].

$$\overrightarrow{E} = \overrightarrow{E_i} + \overrightarrow{E_e} \tag{1.17}$$

where, $\overrightarrow{E_i}$ is a component that the medium's ions combine to form, and $\overrightarrow{E_e}$ is the electrical component made up of all the electrons.

The line that the Stark effect has widened has a profile with a Lorentzian shape.

1.3.5 Zeeman Broadening

The interaction of a magnetic field with the kinetic moments of the atom (movement of the nucleus and the electrons) creates a disturbance; the latter makes it possible to lift the degeneracy of the energy levels, which results in a disturbance of the observed profile [33].

1.3.6 Instrumental Broadening

Measurement and spectroscopy devices cause further broadening of the observed spectral lines. This widening is due, among other things, to the diffraction phenomena of light lines and to the finite dimensions of devices. The constructors apply the enlargements to the corresponding profiles. The profile can take a form:

- A profile of Lorentz
- A profile of Gauss
- A profile of Voigt
- A compound profile [29].

1.4 Interactions Between the Emitter and the Magnetic Field in Plasmas

In the presence of a magnetic field, there is an energy of interaction with the angular moments of the atom (movement of the nucleus and electrons). This disturbance makes it possible to remove the degeneracy of the energy levels and can result in a disturbance of the observed profile.

1.4.1 Effect of the Magnetic Field on the Energy Level System of an Atom

Since 1896, Piter Zeeman has researched the effect of a magnetic field on the energy levels of an atom. He found that each of the lines emitted by the atom subjected to the magnetic field splits into a certain number of equidistant lines, separated by intervals proportional to the magnetic field; this is the Zeeman effect. This field interacts with the magnetic moments present in the atom; Orbital magnetic moment of spin, electron, and nucleus magnetic moments [34]:

$$\overrightarrow{M_L} = \frac{q_e}{2m_e}\overrightarrow{L}, M_S = \frac{q_e}{m_e}\overrightarrow{S}, \overrightarrow{M_I} = \frac{q_e}{2m_p}g_p\overrightarrow{I} \tag{1.18}$$

where $\overrightarrow{M_L}$: is the orbital magnetic moment, $\overrightarrow{M_S}$: is the spin magnetic moment and $\overrightarrow{M_I}$: is the nuclear magnetic moment.

q_e and m_e are respectively the charge and the mass of the electron, q_p and m_p are the charge and mass of the proton; $\overrightarrow{S}$ and $\overrightarrow{L}$ are the total orbital and spin angular moments of the electron, $\overrightarrow{I}$ is the nucleus spin angular momentum and g_p is the g-factor.

The Hamiltonian, which describes the interaction energy of the atom with the magnetic field $\overrightarrow{B}$, is therefore written as [34]:

$$\begin{aligned} H_z &= -\overrightarrow{B}\left(\overrightarrow{M_L} + \overrightarrow{M_S} + \overrightarrow{M_I}\right) = \frac{q_e}{2m_e}\overrightarrow{B}\left(\overrightarrow{L} + 2\overrightarrow{S}\right) - \frac{q_n}{2m_n}g_p\overrightarrow{B}.\overrightarrow{I} \\ &= \overrightarrow{\omega_0}\left(\overrightarrow{L} + 2\overrightarrow{S}\right) + \overrightarrow{\omega_n}.\overrightarrow{I} \end{aligned} \tag{1.19}$$

where $\overrightarrow{\omega_0}$ is the Larmor pulsation defned by:

$$\overrightarrow{\omega_0} = \frac{q_e}{2m_e}\overrightarrow{B} \tag{1.20}$$

and $\overrightarrow{\omega_n}$ is defined by:

$$\overrightarrow{\omega_n} = -\frac{q_n}{2m_n}g_p\overrightarrow{B} \tag{1.21}$$

The total Hamiltonian which describes an atom immersed in a magnetic field is therefore written [34]:

$$H = H_0 + H_f + H_{mag} \tag{1.22}$$

where H_0 is the Hamiltonian of the unpurtured atom.

H_f is the sum of the fine structure terms [34]:

$$H_f = \omega_{mv} + \omega_{so} + \omega_d \tag{1.23}$$

H_{mag} is the Zeeman Hamiltonian [34]:

$$H_{mag} = \overrightarrow{\omega_0}\left(\overrightarrow{L} + 2\overrightarrow{S}\right) \tag{1.24}$$

Depending on the intensity of the field, we are led to distinguish three cases, which correspond to three different calculations:

The magnetic field is relatively weak, so the Hamiltonian H_{mag} can be considered small compared to $H_{f.}$. The fine-structure Hamiltonian $H_0 + H_f$ is then an unperturbed Hamiltonian, and H_{mag} is treated as a perturbation of the states $\mid jnlsjm_j\rangle$: This is the anomalous Zeeman effect.

The magnetic field is said to be strong if H_f is weak compared to H_{mag}. In this case, H_f is treated as a perturbation on $H_0 + H_f$; this is the Paschen-Back effect.

If we completely neglect the term "fine structure," we speak of the normal Zeeman effect (strong field).

When the interactions H_{mag} and H_f are of the same order of magnitude, an intermediate Zeeman effect is obtained. In this case, the problem must be treated without approximations [34].

1.4.2 Anomalous Zeeman Effect

Consider the case where the magnetic field is uniform and parallel to the axis oz. Within the framework of perturbation theory, the Hamiltonian H_{mag} will be considered a perturbation with respect to the unperturbed Hamiltonian $H_0 + H_f$. The calculation of the corrections $E^{(1)}$ in the 1^{st} order approximation leads to using the eigenstates of $H_0 + H_f$ to obtain the diagonal matrix elements of H_{mag}.

Consequently, we use a base indicated $\{\mid nlsjm_j\rangle\}$ (formed from the eigenvectors common to L^2, S^2, J^2 and J_z with $\overrightarrow{J} = \overrightarrow{L} + 2\overrightarrow{S}$) one obtain [35]:

$$E^{(1)}_{mag} = \langle jm_j \mid \omega_0\left(L_z + 2S_z\right) \mid jm_j\rangle \tag{1.25}$$

The notation of states has been lightened since H_f concerns only orbital and spin variables. It is therefore necessary to express the operators L_z and S_z in the base $\{\mid jm_j\rangle\}$:

According to the projection theorem, in a subspacee $\Im(l, s, j)$, we have the following relations [35].

$$\langle L_z\rangle = \frac{\left\langle \overrightarrow{L}.\overrightarrow{J}\right\rangle_{lsj}}{\hbar^2 j\left(j+1\right)}\langle J_z\rangle \tag{1.26}$$

and

$$\langle S_z\rangle = \frac{\left\langle \overrightarrow{S}.\overrightarrow{J}\right\rangle_{lsj}}{\hbar^2 j\left(j+1\right)}\langle J_z\rangle \tag{1.27}$$

where $\left\langle \overrightarrow{L}.\overrightarrow{J} \right\rangle_{lsj}$ and $\left\langle \overrightarrow{S}.\overrightarrow{J} \right\rangle_{lsj}$ denote respectively the average values of the operators $\overrightarrow{L}.\overrightarrow{J}$ and $\overrightarrow{S}.\overrightarrow{J}$ for the states of the system belonging to $\Im(l,s,j)$; we find after calculation [35]:

$$\langle S_z \rangle = \frac{\langle J^2 - L^2 - S^2 \rangle}{2 \langle J^2 \rangle} \langle J_z \rangle \tag{1.28}$$

$$\langle L_z \rangle = \frac{\langle J^2 + L^2 - S^2 \rangle}{2 \langle J^2 \rangle} \langle J_z \rangle \tag{1.29}$$

The eigenvalues of J^2,L^2 and S^2 being:

$$\hbar^2 j\,(j+1)\,,\hbar^2 l(l+1),\hbar^2 s(s+1) \tag{1.30}$$

one obtain [35]:

$$\left\langle \overrightarrow{L_z} + 2\overrightarrow{S_z} \right\rangle = \left(1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right) \langle J_z \rangle \tag{1.31}$$

With this calculation, we show that the operator H_{mag} can lie in the form [35]:

$$H_{mag} = \omega_0 g J_z \tag{1.32}$$

where g is Landau g-factor.

$$g = \left(1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right) \tag{1.33}$$

we calculate [34]:

$$E^{(1)}_{mag} = \langle jm_j \mid \omega_0(L_z + 2S_z) \mid jm_j \rangle \tag{1.34}$$

$$E^{(1)}_{mag} = \omega_0 \left(1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right) \langle jm_j \mid J_z \mid jm_j \rangle \tag{1.35}$$

In the basis [34]:

$$\langle jm_j \mid J_z \mid jm_j \rangle = \hbar m_j \tag{1.36}$$

Then

$$E^{(1)}_{mag} = \left(1 + \frac{j(j+1) - \ell(\ell+1) + s(s+1)}{2j(j+1)}\right) \hbar\omega_0 m_j \tag{1.37}$$

The perturbation energy becomes:

$$E^{(1)}_{mag} = \hbar\omega_0 g m_j \tag{1.38}$$

We note that for a given value of j, l, and s, the disturbance energy depends on m_j, and the degeneracy of the level n is therefore completely lifted. A set of values of the quantum numbers n, I, s, j, and m_j corresponds to an energy level E_{nlsjm_j}, to which is attached a single state value $\mid nlsjm_j\rangle$. The quantum number m_j can take $(2j+1)$ values:

$$m_j = -j, -j+1, ..., j$$

1.4.3 Paschen-Back Effect

In this case, the interaction energy W_{mag} is more important than the terms of the fine structure [34]:

$$H = H_0 + H_{mag} \tag{1.39}$$

W_f One must take it as a Hamiltonian of order zero, H_f is considered a perturbation of order one,The eigenvalues of H_{mag} are easily obtained from the states $\mid nlsm_l m_s\rangle$ which are eigenvectors of L^2,S^2,L_z and S_z.

The total energy without fine structure is [34]:

$$E = E_0 + \omega_0 \hbar(m_l + 2m_s) \tag{1.40}$$

where E_0 is the eigenvalue of H_0.

Let us now consider the Hamiltonian H_f, retaining only the spin-orbit interaction term, which is written [31]:

$$W_{so} = \zeta_{nl} \overrightarrow{L}.\overrightarrow{S} \tag{1.41}$$

where

$$\overrightarrow{L} = \sum \overrightarrow{l_i} \text{ and } \overrightarrow{S} = \sum \overrightarrow{s_i} \tag{1.42}$$

The term W_{so} is sufficient to cause the decay of the E_{mag} levels to appear, and the energy correction is equal, according to the results of perturbation theory, to the mean value of H_f taken in the unperturbed state [33].

$$\langle H_f \rangle = \zeta_{nl} \hbar^2 m_l m_s \tag{1.43}$$

Then a state $\mid nlsm_l m_s\rangle$ corresponds to an energy level E defined by quantum numbers and given by: n, l, s, m_l and m_s [31].

$$E_{tot} = E_0 + \omega_0 \hbar(m_l + 2m_s) + \zeta_{nl} \hbar^2 m_l m_s \tag{1.44}$$

1.4.4 Normal Zeeman Effect

In strong field, the fine structure is completely neglected [34]:

$$H = H_0 + H_{mag} \tag{1.45}$$

In this approximation we study the structure of the level n the values of m_l are equal to $-1, 0, 1$, and $m_s = \pm\frac{1}{2}$ the distinct pairs (m, m_l, m_s) are then the following: $\left(2, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$, $(1, 0)$, $\left(-2, \frac{1}{2}\right)$.

The effect of an intense magnetic field is therefore to reveal five sub-levels that already exist in the expression E given by the relation (1.40).

1.5 Controlled Thermonuclear Fusion

Controlled thermonuclear fusion is a process for exploiting nuclear energy that meets the criteria of sustainable development. In particular, fusion would not participate in the greenhouse effect, use a fuel that is abundant in nature (hydrogen isotopes, one of which is extracted from lithium), and produce radioactive waste with a longer lifespan and a lower level of radiation than that from current fission nuclear power plants. Moreover, a fusion reactor does not create a chain reaction, making it inherently stable and therefore safe. Thermonuclear fusion therefore appears to be a promising avenue alongside renewable energies and new fission power plants [36].

In this part of the chapter, we will try to address the subject of fusion in a global way and summarize the scientific approach that has been necessary for current advances. To do this, we will first present the different fusion reactions. Then, we will briefly dwell on the different ways of obtaining these fusion reactions in laboratories, i.e., the two possible confinements, which are magnetic confinement and inertial confinement. Finally, we will study in more detail magneto-inertial confinement, which is the global theme of this work.

1.5.1 The Different Types of Fusion Reaction

The fusion reaction consists of bringing two light atomic nuclei together sufficiently to form a heavier one, The operation creates energy in kinetic form. At first sight, one might be tempted to obtain fusion energy by using a beam of deuterium D coming from an accelerator and directed towards a target of tritium, or by making use of beams of deuterium and tritium directed towards each other. Although this method, called "beam fusion," allows for D-T fusion reactions, it does not provide a solution for energy production since the atomic collision cross sections are much larger than those relating to the D-T fusion reaction. As a result, most of the particles in the beams lose their energy before they have a chance to trigger a nuclear fusion reaction [37].

The adopted solution consists in forming a high temperature plasma, composed of the nuclei of D and T and the electrons. At thermodynamic equilibrium, the Columbian collisions among plasma particles after a series of collisions produce fusion reactions. This approach to nuclear fusion, which uses the thermal motion of nuclei in a plasma, is called thermonuclear fusion [37].

Several thermonuclear fusion reactions are possible. However, two conditions must be met from the perspective of energy production. Firstly, the reaction chosen must obviously be accompanied by an energy engagement (exoenergetic reaction), which implies the use of light nuclei. Then the reaction cross section should be as high as possible [37].

The main fusion reactions of interest for the production of energy on Earth are [37]:

$$\begin{aligned} D + T &\rightarrow {}^4He(3.5\text{MeV}) + n(14.1\text{MeV}) \\ D + D &\rightarrow {}^3He(0.82\text{MeV}) + n(2.45\text{Mev}) \\ D + D &\rightarrow T(1.01\text{MeV}) + p(3.02\text{MeV}) \\ D +^3 He &\rightarrow {}^4He(3.07\text{MeV}) + p(14.07\text{MeV}) \end{aligned}$$

Where Deuterium ($D \equiv_1^2 H$) and Tritium ($T \equiv_1^3 H$) are isotopes of Hydrogen, respectively of atomic mass 2 and 3.

Note that deuterium (D) is extracted from seawater at a relatively modest cost ($1\$/\text{g}$) compared to the amount of potentially recoverable energy. On the other hand, tritium (T) does not exist in its natural state, but it can be produced by bombarding lithium with a neutron flux.

Figure 1.3 shows the cross sections of the different fusion reactions stated above. Among those, the deuterium-tritium (D-T) reaction appears the most attractive since it has the highest cross section, or more simply, the highest probability. It is on this reaction that the research on controlled fusion is concentrated [39].

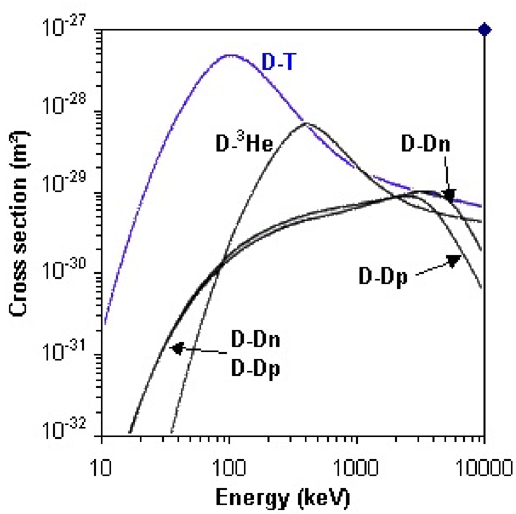

Fig. 1.3: Cross sections of 4 nuclear fusion reactions [38].

Other fusion reactions are more difficult to implement, either due to a lack of adequate fuel or a low cross section.

In a fusion power plant, it is the kinetic energy of the neutron, by slowing it down in a suitable medium, which will be transformed into thermal energy. This energy can then be transformed into electricity with an efficiency of around 30%.

1.5.2 Power Budget

To fuse two positively charged nuclei, it is necessary to bring them to a sufficiently high energy to allow them to cross the Colombian potential barrier by tunnel effect in a sufficient proportion. For the number of nuclei likely to fuse by tunnel effect to be large enough, their energy must be of the order of $10\mathrm{KeV}$ (plasma D-T). In current reactors, the aim is to reach temperatures of around 10 KeV, sufficient to cause fusion reactions [36].

From an energy point of view, the objective of the industrial use of nuclear fusion is to release more energy than the operation of a nuclear fusion reactor consumes. It is therefore necessary to carry out an energy balance of the operation. The mixture of deuterium and tritium is promising in terms of energy since each event produces 17.6 MeV in the form of kinetic energy carried away by the reaction products. The fusion power of a D-T plasma per unit volume is directly a function of the cross section of the reaction as well as the volume densities of deuterium and tritium [40]:

$$P_{Fusion} = N_D N_T \langle \sigma v \rangle E_{Fusion} \tag{1.46}$$

where N_D and N_T are the deuterium density and the tritium density respectively.
$\langle \sigma v \rangle$ is the average fusion reactivity.
σ is the total fusion cross section.
E_{Fusion} is the amount of energy released per reaction (for the D-T reaction).
v is the absolute value of the relative speed of the two nuclei of deuterium and tritium.
$\langle\rangle$ denotes an average over the Maxwell velocity distribution.

Unfortunately, only part of this energy (about 1/5) contributes to the heating of the fusion plasma. This fraction is carried by the alpha particles, which remain confined in the magnetic fields because of their charge, and they will yield their energy to the medium and thus heat the plasma, which therefore receives a power P proportional to the fusion power. Most of the energy from the D-T reaction (about $4/5$ of it) is produced by neutrons of $14.1\mathrm{MeV}$, which are not subjected to the confinement fields and leave the plasma.

This energy must be converted into heat, which is used to obtain steam, which turns the turbines of a power plant.

The energy losses in a fusion plasma are represented by a characteristic time that it would take for the plasma to evacuate all of its energy, called the energy confinement time τ_p, defined by [40]:

$$\tau_p = \frac{W}{P_{losses}} \tag{1.47}$$

where
W :denotes the total energy of the plasma.
P_{losses}: represents energy losses in the fusion plasma.

For the energy balance of the fusion plasma to be positive in the stationary state, the losses must be counterbalanced by the heating of the alpha particles and the external additional heating [40].

$$P_{alpha} + P_{add} \geq P_{losses} \tag{1.48}$$

1.5.3 Ignition Condition and Lawson's Criterion

When the temperature and density of the plasma increase, and therefore the confinement improves, the fusion reactions become more numerous, and the share of the heating power coming from the alpha particles increases, this power can counterbalance all the energy losses. and that no other external heating is necessary for the plasma to continue to burn. This state is called ignition. It is obtained when [42]:

$$P_{alpha} = \frac{P_{Fusion}}{5} = P_{losses} \tag{1.49}$$

We define a factor that quantifies the energy balance of a fusion machine Q as being the ratio between the fusion power and the additional power. This factor is called the amplification factor [42].

$$Q = \frac{P_{Fusion}}{P_{add}} \tag{1.50}$$

- When $Q = 1$. The machine produces as much energy as has been injected. In 1997, researchers at JET (Join European Torus, EU) obtained an amplification factor $Q = 0.69$ for a D-T plasma.

- When $Q \prec 1$, the fusion reactions' power is lower than the power provided by the additional heating.

- When $Q \succ 1$, the fusion reactions' power is greater than the additional heaters' power; this is the ignition limit, and to obtain energy, this limit must be exceeded. Then [42]:

$$P_{Fusion} \succ P_{add} \Rightarrow P_{alpha} \succ P_{losses} \Rightarrow N\tau_p \succ \frac{12}{\langle \sigma v \rangle} \frac{T}{E_{Fusion}} \tag{1.51}$$

where N is the alpha particles density.

The power of the fusion reactions will only compensate for the losses; therefore, the external power is no longer useful, as is the so-called self-sustaining plasma.

In the temperature ranges considered in a tokamak, the cross section can be approximated by the following relationship [42]:

$$\langle \sigma v \rangle = 1.1 \times 10^{-24} \mathrm{T}^2.\mathrm{m}^3.\mathrm{s}^{-1} \tag{1.52}$$

We substitute (1.52) in (1.51), so we get the following criterion to achieve ignition, called Lawson's criterion [42], which indicates that the choices of temperature, density, and confinement time are intrinsically linked:

$$NT\tau_p \succ 3 \times 10^{21} \mathrm{m}^{-1}.\mathrm{KeV.s} \tag{1.53}$$

This triple product $NT\tau_p$ must therefore be greater than a threshold value. The larger it is, the greater the energy gain.

The ignition condition, which is imposed by Lawson's criterion (1.53), considers two possible paths for fusion:

- Obtain a short confinement time with high densities.

- Achieve longer confinement times with a lower density. The first possibility leads to inertial fusion, and the second possibility leads to magnetic fusion (Figure 1.4).

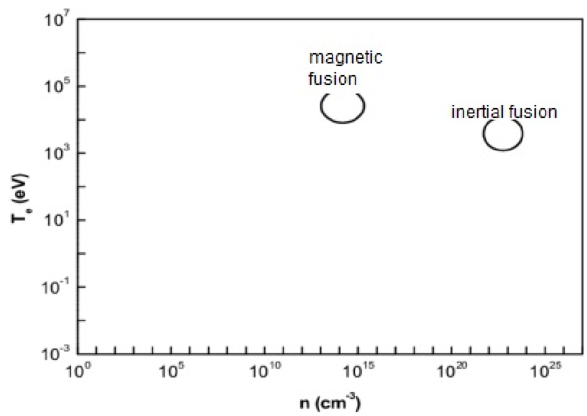

Fig. 1.4: Inertial and magnetic confinement in density [41].

	La fusion magnétique	Inertial Fusion
Particle density N_e (cm^{-3})	10^{14}	10^{26}
Confinement time τ_p (s)	10	10^{-11}
Lawson criterion $N_e\tau_p$ $(\mathrm{s.cm}^{-3})$	10^{15}	10^{15}

Table 1.1: Confinement parameter in inertial and magnetic confinement [41].

1.6 Fusion by Inertial Confinement

In the inertial confinement fusion (ICF) pathway, a small target capsule on the order of a millimeter in diameter containing a mixture of a few milligrams of deuterium and tritium is compressed to high densities and heated to a surrounding temperature $10\mathrm{KeV}$ in such a short time that the nuclei can fuse together and release their fusion energy. More precisely, it is the inertial forces of the mass that maintain the plasma for a very short time, approximately 10^{-11} s. The plasma begins to expand, and the fuel receives a violent centripetal radial movement that produces a shock wave that strongly compresses the fuel and converges towards the center of the target, forming a hot spot. Once the hot spot is ignited, the nuclear reactions must occur in a chain, also providing the necessary temperature for the rest of the fuel. Following the Lawson criterion (1.53), we see that in the ICF approach, the ion density must be at least $N_i = 10^{25}\text{cm}^{-3}$ (Figure 1.5) [44].

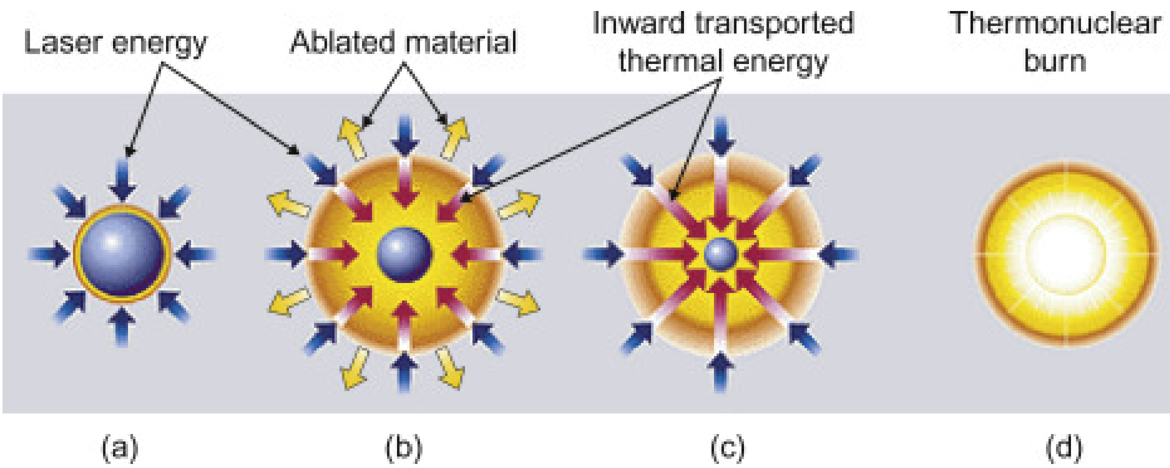

Fig. 1.5: The four stages of a fusion reaction in an inertial-confinement capsule: (a) Laser heating of the outer layer. (b) Ablation of the outer layer compresses the capsule. (c) The core reaches the density and temperature for ignition. (d) The fusion reaction spreads rapidly through the compressed fuel [43].

There are two different types of schemes for achieving fuel heating with power lasers: direct attack and indirect attack [45].

1.6.1 Direct attack

In the direct attack scheme, it simply irradiates the surface of the fuel target with intense laser pulses; the laser uniformly illuminates the ablator (capsule) of the target and transforms it in a short time, much less than the duration of a pulse, creating a crown of plasma around the target. This is because the crown plasma expands in vacuum and the D-T target is compressed rapidly (Figure 1.5(a)).

The laser will illuminate an increasing density profile N_e; the relation linking the wave number K_L, the pulsation ω_L of the incident wave, and that of the local plasma ω_{pe} is given by the dispersion relationship [45]:

$$\frac{K_L^2 c^2}{\omega_L^2} = 1 - \frac{\omega_{pe}^2}{\omega_L^2} \tag{1.54}$$

where $\omega_{pe} = \sqrt{\frac{N_e^2 e^2}{\varepsilon_0 m_e}}$.

The laser cannot propagate in areas where the density is greater than the critical density N_c, defined as the density at which the plasma frequency equalises the laser pulse $\omega_L = \omega_{pe}$ and $\omega_L = \frac{2\pi}{\lambda_L}$.

We therefore obtain [44]:

$$N_c = \frac{N_e \varepsilon_0 \omega_L^2}{e^2} = \frac{1.1 \times 10^{21}}{\lambda^2(\mu\mathrm{m}^2)} \mathrm{cm}^{-3} \tag{1.55}$$

Beyond the critical density, the electrons are heated mainly by collisional absorption in the intensity domain where we are working. These electrons will then transport the energy by thermal conduction to the ablation front and thus heat the central part.

Zone $N_e \prec N_c$ is called an under-dense crown, as opposed to zone $N_e \succ N_c$, which is called over-dense. It is in the latter that the thermal conduction mechanisms that will perpetuate the ablation take place [46].

The energy transfer is satisfactory; however, certain phenomena can disturb the absorption or have other negative effects on the compression and ignition, among which are the parametric instabilities and the symmetry of illumination, which turns out to be very critical. To overcome these problems, the scheme of the indirect attack is considered.

1.6.2 Indirect Attack

In the "indirect attack" scheme, the laser radiation is absorbed in an auxiliary cavity filled with gas containing the target and converted (by a succession of elementary processes of absorption, heating, and re-emission) into X-radiation, which produces the implosion. The cavity material must have a high atomic number (usually gold) to achieve efficient laser-X conversion. The plasma expansion of the cavity walls is prevented by a gas that exists inside the cavities (Figure 1.6 (b)).

This solution ensures better uniformity of X-rays and achieves well-symmetrical compression; on the other hand, it is less interesting than the previous one in terms of energy transfer to the target, and the big drawback is the laser-X-ray conversion, whose efficiency is relatively low [47].

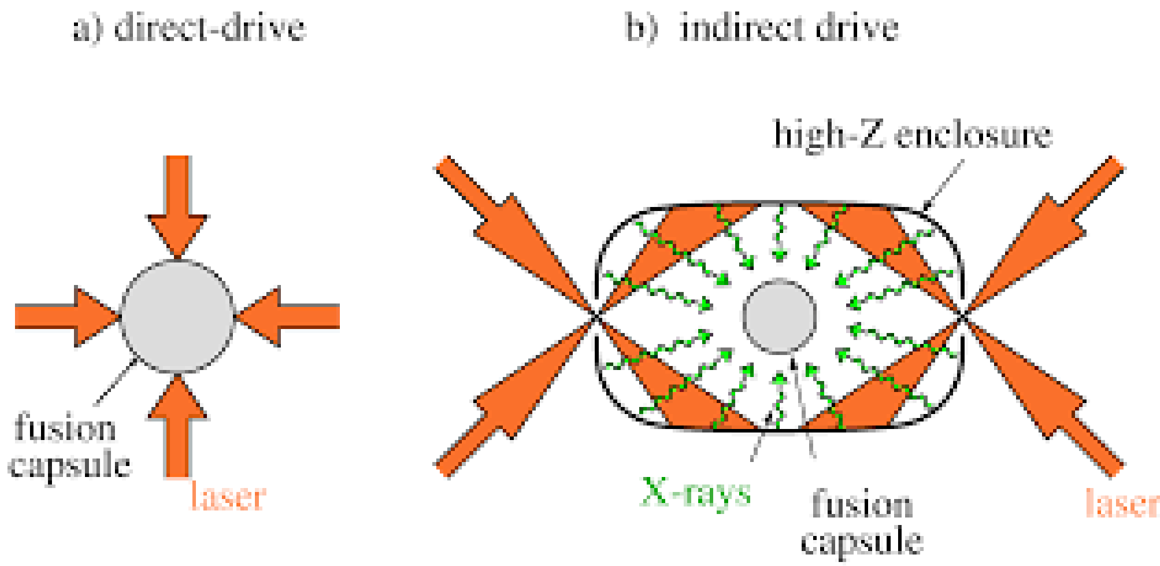

Fig. 1.6: Direct (a) and indirect (b) approach to laser fusion [48].

1.6.3 Fast Ignition

The principle of ICF by rapid igniter consists of separating the compression and heating phases, as proposed in 1994 by M. Tabak et al. [47]. This technique requires super-intense lasers providing pulses of the order of a petawatt (10^{15}W) and whose duration is of the order of a picosecond (10^{-12}s). After focusing, the corresponding intensities can reach $10^{22}\mathrm{Wcm}^{-2}$.

In a first step, an adiabatic compression is carried out with lasers of power $10^{15}\mathrm{Wcm}^{-2}$ in the nanosecond regime (figure 1.7 (a)) [49].

In a second step, after a sufficiently short time, a first laser beam at ultra-high intensity $10^{18}\mathrm{Wcm}^{-2}$ and about $100\mathrm{ps}$ in duration, interacts with the corona of the plasma surrounding the compressed core and digs a channel in the plasma (figure 1.7 (b)).

In the last step, a second very short laser pulse ($10^{20}\mathrm{Wcm}^{-2}, 10\mathrm{ps}$) passes through the channel already created by the first pulse and interacts with the supercritical plasma, which is several tens of $\mu\mathrm{m}$ from the compressed core, depositing its energy on the hot spot (Fig. 1.7(c)).

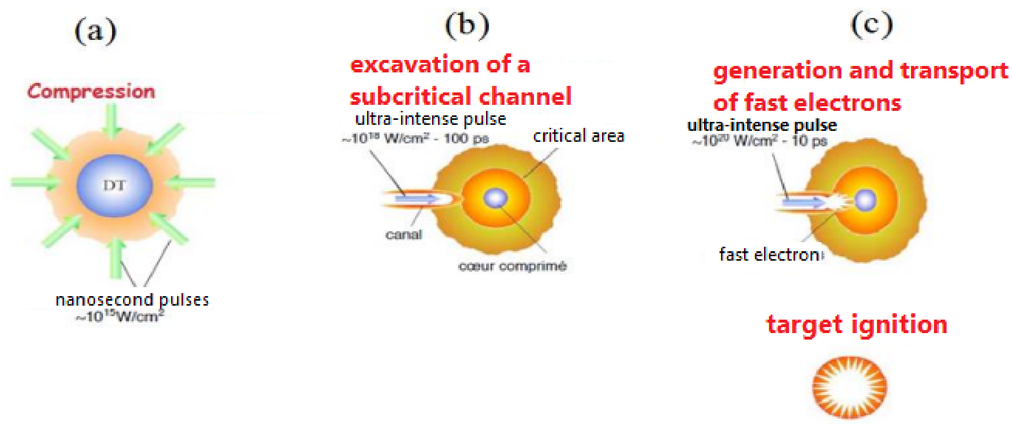

Fig. 1.7: Fast ignition scheme: (a) compression of the target (b) digging of a subcritical channel (c) generation of fast electrons and creation of a hot spot [50].

The central problem of fast ignition is the transport of energy from the interaction region with the super-intense outer laser beam to the compressed D-T. This energy transfer by electrons implies the presence of relativistic electron beams with corresponding currents of about 100MA.

1.7 Fusion by Magnetic Confinement

The conditions necessary to fuse the nuclei in the core of the sun are achieved thanks to the gravitational force, which allows them to be confined and provides sufficient pressure to initiate the fusion reactions. However, on Earth, it is impossible to obtain such pressures, and such a mode of confinement is therefore not possible [53]. The solution adopted therefore consists in confining a large quantity of gas at low pressure at a very high temperature so that a large fraction of the population of particles reaches sufficient kinetic energy to initiate the fusion reactions. Under such conditions, the particles are totally ionised, and the fuel must first be converted into a plasma of ions and electrons. It can be confined by exerting intense magnetic pressure on all of these charged particles. Indeed, when plasma bathes in a magnetic field, the charged particles are subjected to the Lorentz force and wind around the field lines, describing helical trajectories (Figure 1.8).

The natural gyration frequency $\Omega_{c,\alpha}$, called the cyclotron frequency (ionic or electronic, depending on the species considered), is defined by [52]:

$$\Omega_{c,\alpha} = \frac{|q|\,B}{m_\alpha} \tag{1.56}$$

where α:denote the species of the charged particle.
q: denote the particle charge.

The helical radius of motion is called the Larmor radius and is written [53]:

$$\rho_c = \frac{m_\alpha v_\perp}{ZeB} \tag{1.57}$$

with $v_\perp$: is the normal component of its velocity perpendicular to the magnetic field; Z: is the particle charge.

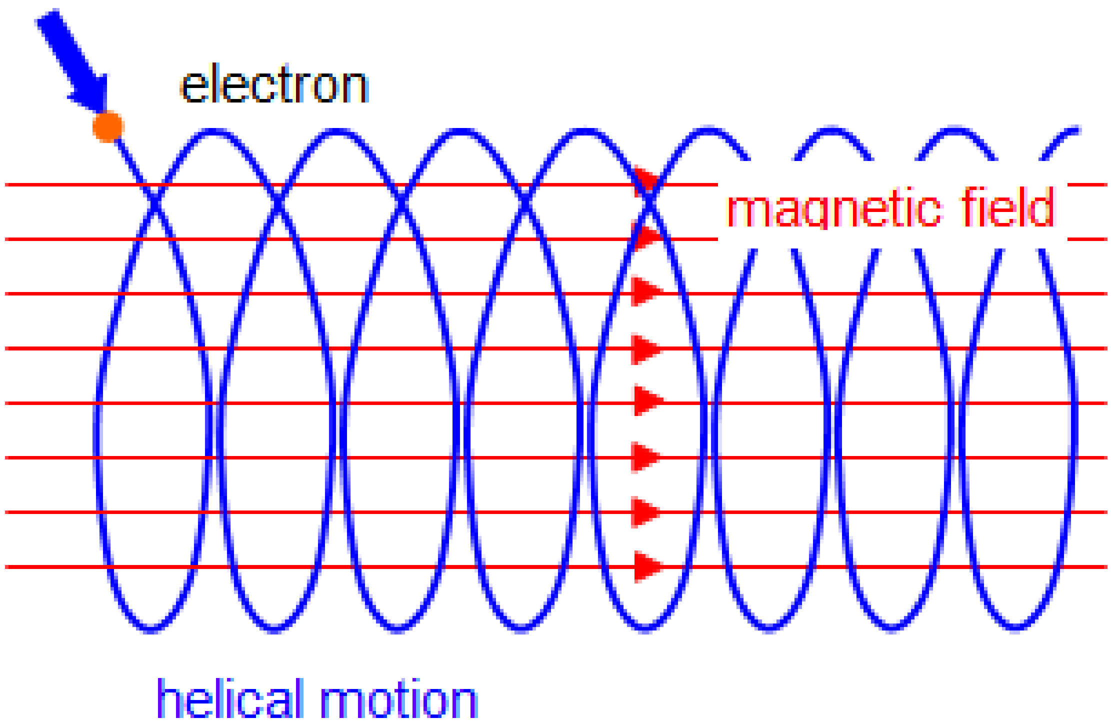

Fig. 1.8: Motion of a charged particle around the field line [51].

In the normal plane of the field line direction, the particle is thus confined and cannot deviate from it by a distance greater than the Larmor radius. On the other hand, in the direction parallel to the field lines, the movement of the particles of the plasma along the lines of the field remains free, and they can escape at the ends of the configuration (Figure 1.8). So it is more optimal to have magnetic field lines close on themselves in a toroidal structure. The magnetic field is thus called a toroidal magnetic field [53]. However, that is not sufficient in such a configuration (Figure 1.9).

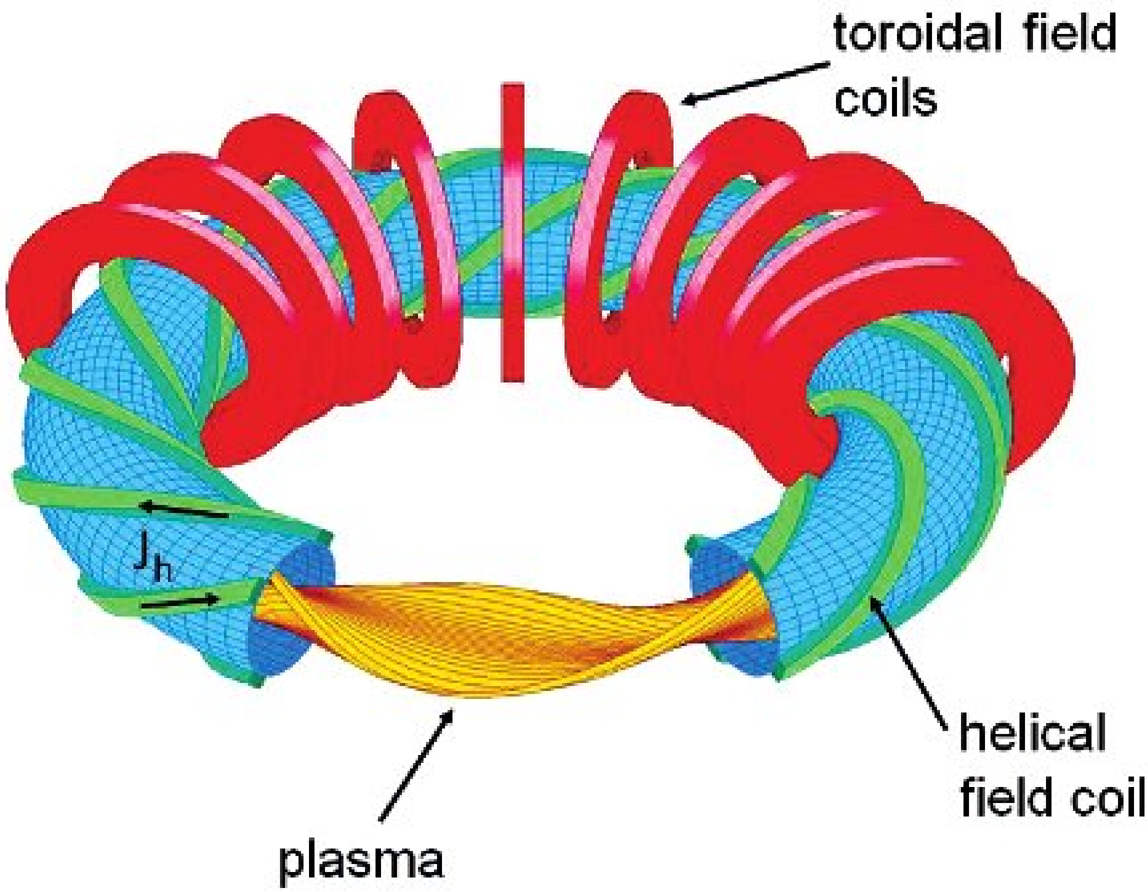

Fig. 1.9: Toroidal field lines alone [52].

Particles are animated in addition to a vertical drift velocity that leaks plasma in a fraction of a second. Field lines should be helical so that drift is compensated during motion. This is accomplished by introducing a poloidal magnetic field that is perpendicular to the toroidal field (Figure 1.10).

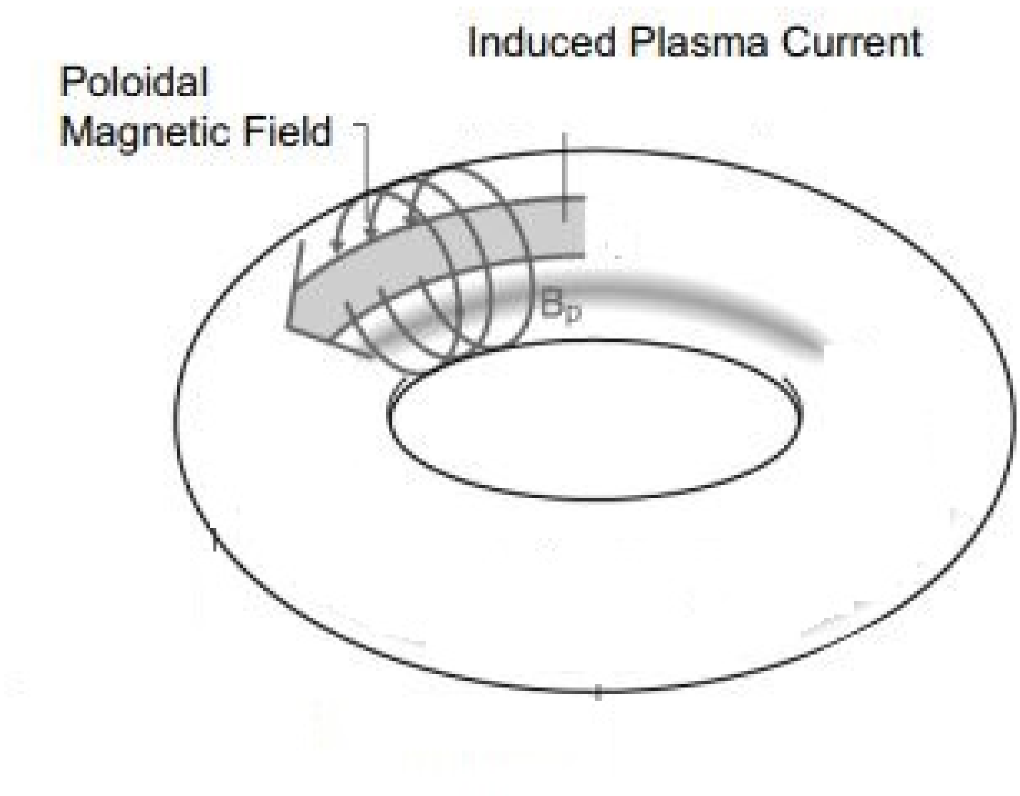

Fig. 1.10: Poloidal field lines alone [54].

Finally, we end up with a complete magnetic configuration, which improves confinement performance (Figure 1.11).

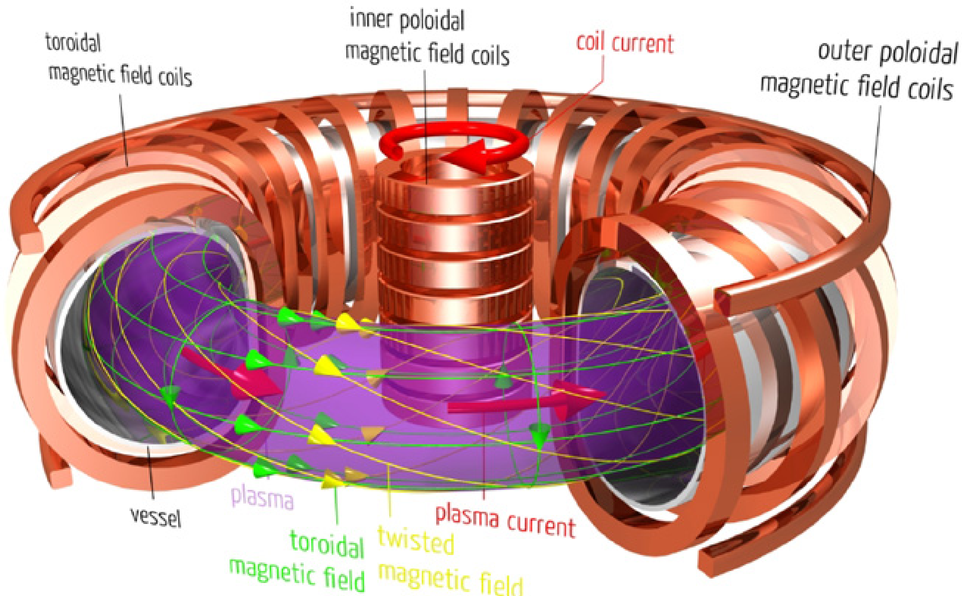

Fig. 1.11: Confinement and trapping of a plasma using a magnetic box in the presence of a toroidal field and a poloidal field [55].

The method employed to confine the plasma magnetically gave rise to a machine; said TOKAMAK.

1.7.1 TOKAMAK

One of the configurations capable of confining plasma using a magnetic field is the use of reactors called Tokamaks. The tokamak is a Russian invention whose term comes from the contraction of toroidalnaya kamera magnitanya katuschka, which means toroidal chamber with magnetic coils [56]. The tokamak configuration is in the form of a toroidally symmetrical vacuum chamber in which a structure of toroidal B_φ and poloidal B_θ magnetic fields is created to confine the plasma.

Toroidal coils create the toroidal magnetic field, and this field's gradient causes vertical drifts perpendicular to these field lines, which are bad for plasma confinement. It is therefore necessary to compensate for these drifts in order to ensure optimal orbital confinement.
A complementary so-called poloidal field B_θ is used for this, generated by a current passing through the plasma, created via a vertical ohmic coil placed in the center of the torus. This poloidal field, generally, is weak $\left(B_\theta = \frac{B_\varphi}{10}\right)$. Additional vertical poloidal field coils are used to compensate for the outward force due to plasma pressure gradients. The result of superimposing the toroidal field on the poloidal field is a helical field that wraps around toric surfaces (figure 1.11) [58].

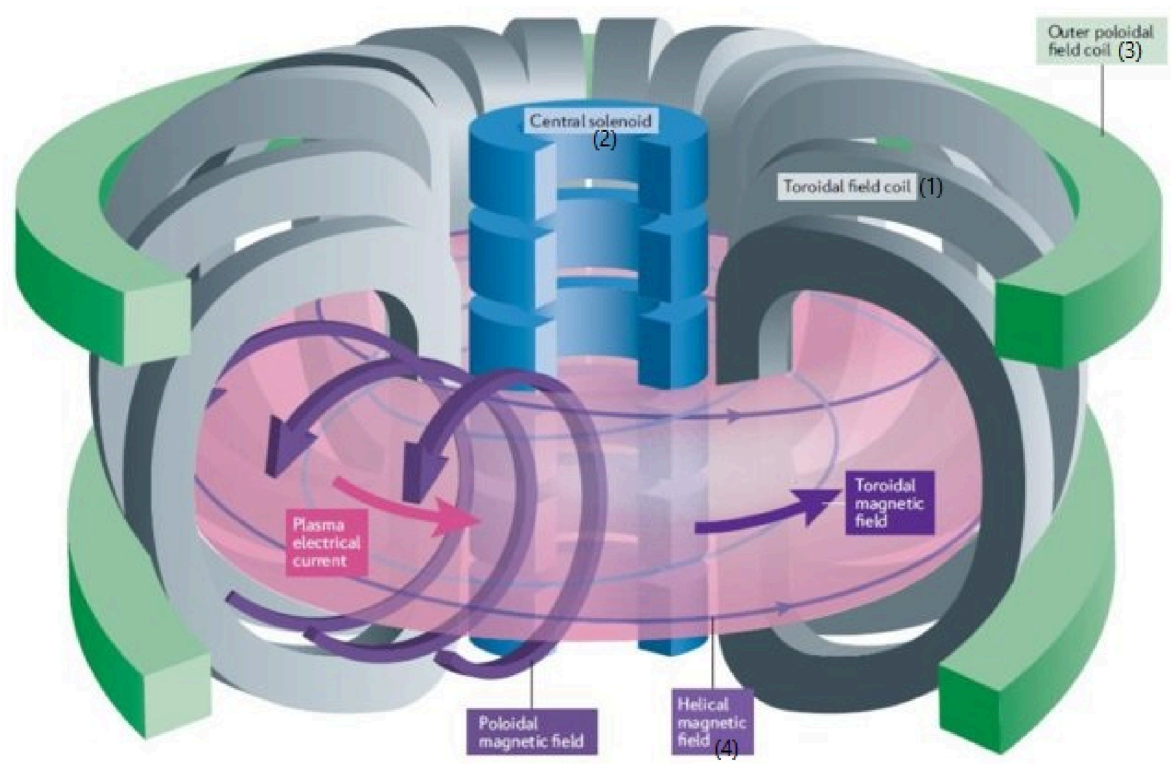

Fig. 1.12: The tokamak configuration is a magnetic structure presenting the arrangement of the various coils around the torus used for the confinement of the plasma. (1) The toroidal magnetic field coils create the toroidal magnetic field. (2) the ohmic coil generating a strong toroidal plasma current at the origin of the poloidal magnetic field (3) the additional poloidal coils used to control the shape and position of the plasma; and (4) the helical-shaped magnetic field line resulting from the addition of the toroidal field and the poloidal field [57].

- Coordinate Systems

  The most common coordinates in the study of tokamak plasmas are the toric coordinates. They are defined by:

  - A large radius R measures the distance from the symmetry of the torus.
  - A toroidal angle φ measures the angular position around the symmetry axis of the torus.
  - The small radius r is the distance from the centre of the toroidal ring to its edge, as noted by a.
  - The toroidal field B_φ is directed around the long circumference of the torus. This field is curved according to the relation [58]:

$$B_\varphi = \frac{\mu_0 I_\varphi}{2\pi R} \tag{1.58}$$

where I_φ: is the total current flowing through the coil.

– The toroidal field B is directed around the short circumference of the torus.

The size of the plasma is characterised by the large radius of the centre (Fig. 1.13).

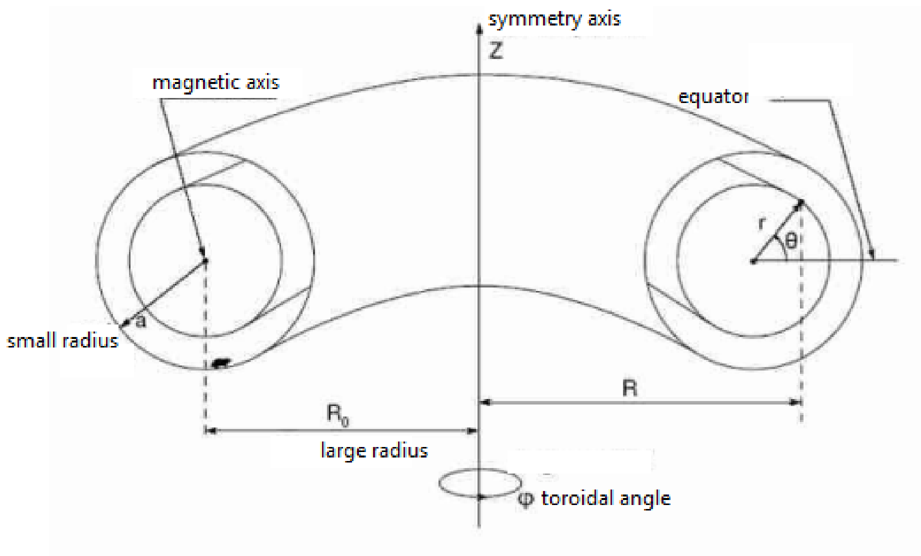

Fig. 1.13: Coordinate system of a TOKAMAK [57].

1.7.2 Plasma Heating and Current Generation

To achieve the ignition range, where the correct heating of the plasma fully compensates the energy losses, it is necessary to use efficient additional heating methods to raise the temperature to about $10\mathrm{KeV}$. Heating is mainly based on several devices implemented on current tokamaks. We will give a brief description of each of them [59].

1.7.2.1 Ohmic Heating (Joule Effect)

In a tokamak, the natural source of plasma heating is the Joule effect. The plasma current, I_φ, directly heats the electrons. As was demonstrated in Section 1.7.1, the poloidal field is created in a tokamak by a current circulating through the torus. A large part of this current is produced via induction using a primary circuit that encircles a toroidal section [59].

The collisional processes between the electrons and the different species of ions make it possible to define an electrical resistivity of the plasma, η, which decreases with the temperature $\left(\eta \sim T_e^{-\frac{3}{2}}\right)$. The hotter the plasma, the less effective the ohmic heating, and therefore the The efficiency of ohmic heating does not allow temperatures of about 1 KeV to be exceeded in large machines. Auxiliary heating means are therefore indispensable [59].

1.7.2.2 Ion and Electronic Cyclotron Heating

To heat the electrons and ions in the plasma, an electromagnetic wave can be used whose frequency is a harmonic, n, of the cyclotron frequency of the electrons and ions ($\Omega_{c,e}$,$\Omega_{c,i}$).

For a fixed pulsation wave ω that propagates perpendicular to the magnetic field $\overrightarrow{B}$, we can write the resonance conditions [59]:

$$\begin{aligned} \Omega_{c,\alpha} &= n \\ \alpha &= e, i \end{aligned} \tag{1.59}$$

1. **Ion cyclotron heating**

   Heating techniques use radio waves of different frequencies to provide additional heat to the plasma. In ion cyclotron resonance heating (ICRH), energy is transferred to plasma ions by a beam of high-intensity electromagnetic radiation with a frequency of 40–55MHz.

   Ion cyclotron heating involves a generator, transmission lines, and an antenna. The generator produces high-power radio frequency waves, which are carried by a transmission line to an antenna located in the vacuum chamber, which in turn sends these waves into the plasma [59].

2. **electronic cyclotron heating**

   The Electron Cyclotron Resonance Heating (ECRH) technique heats plasma electrons through a beam of high-intensity electromagnetic radiation with a frequency of 170 GHz, which is the resonant frequency of the electrons. These electrons then collide with the ions and transfer the absorbed energy to them.

   The electronic cyclotron heating system is also used to bring heat to very specific points of the plasma so as to minimise certain instabilities that could cool it. Compared to the ICRH technique, ECRH heating has the advantage of using a beam capable of propagating in the air, which makes it possible to simplify the design and to use a source far from the plasma and thus facilitate maintenance. The energy will be provided by powerful, high-frequency gyrotrons [59].

1.7.2.3 Heating by Neutral Injection

Probably the most effective way to heat the plasma is to inject beams of neutral particles, deuterium in general, at high energies, from 50 KeV to 2 MeV, depending on the machine. Before being injected, deuterium atoms are accelerated outside the tokamak until they reach a kinetic energy of 1 MeV. Since only positively or negatively charged ions can be accelerated by an electric field, neutral atoms have their electrons stripped to become positively charged ions. In neutral injection devices, ions pass through a cell containing gas, where they pick up their missing electron, and then are injected into the plasma as fast neutral particles [59].

1.7.3 Current and Future Tokamaks

Since the 1950s, increasingly larger tokamaks have been built. Technical innovations such as superconducting coils have also made it possible to improve the performance of tokamaks in terms of both discharge duration and fusion performance. Here we will detail the characteristics of the two machines: the European JET tokamak and the ITER project [53].

1.7.3.1 JET

The "Joint European Torus" is today the largest tokamak in the world. It was built by Europe between 1979 and 1983, is based at the Culham Centre for Fusion Energy (CCFE), near Oxford (UK), and is operated by the EFDA (European Fusion Development Agreement). Its main characteristics are summarized in table 1.2.

large radius	2.96 m
small horizontal radius	1.25 m
small vertical radius	2.1 m
plasma form	X point
toroidal field	3.85 Tesla
plasma current	max 4.8 A
ICF and LH heating	15 MW
Heating by neutral injection	23 MW
plasma volume	100 m^3
discharge duration	up to 60 s
plasma heat energy	10 MJ
plasma magnetic energy	10 MJ

Table 1.2: Main features of JET [60]

JET has 32 water-cooled toroidal copper coils and a system of poloidal coils that allow the formation of plasma. The heating of the plasma is carried out mainly using injectors of neutral particles in a normal or tangential way and HF waves. JET currently holds the world record for fusion power. In the deuterium-tritium campaign that was carried out in 1997, fusion reactions supplied 16 MW of the 22 MJ of energy that these same processes produced. A Q factor of 0.65 was obtained during these experiments [53].

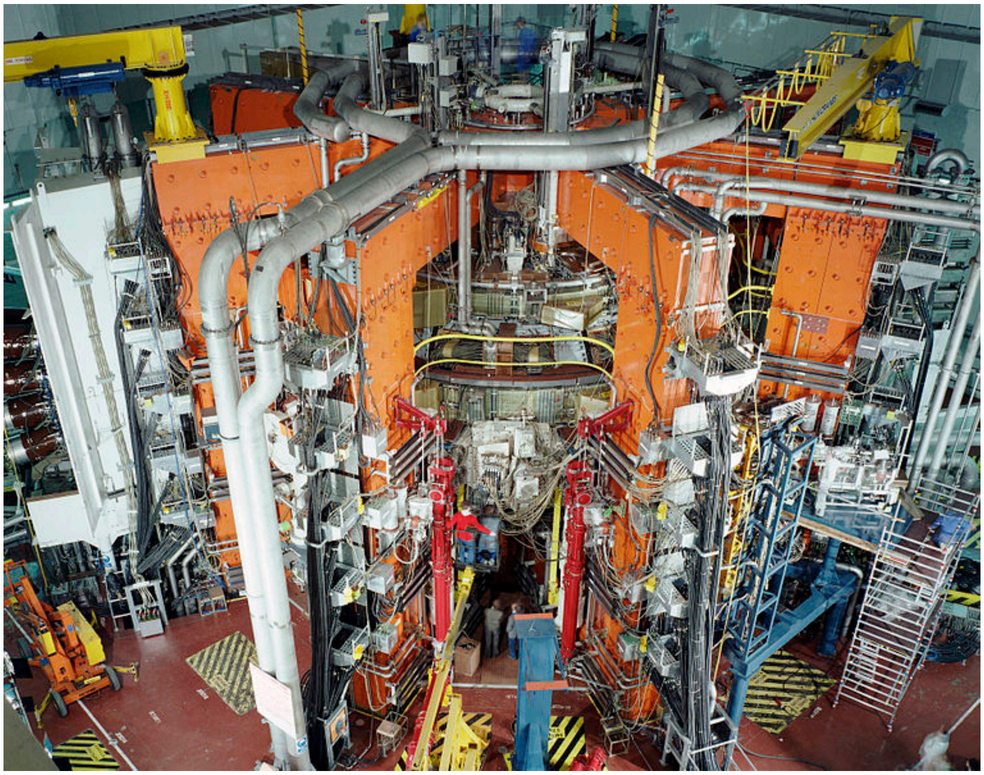

Fig. 1.14: Exterior view of JET [61].

1.7.3.2 ITER

ITER (International Thermonuclear Experimental Reactor) is the future international tokamak under construction at the Cadarache site in France. The result of decades of study and international collaborations, its first plasma is planned for 2019. ITER will be the largest tokamak ever built and one of the largest scientific collaborations in the world. Its aim is to demonstrate the technical feasibility of nuclear fusion under magnetic confinement. The main features of the device are summarised in (Table 1.3).

ITER will have superconducting coils for both the poloidal system and the toroidal system. These coils will be cooled with helium at 4K. ITER should be able to reach an amplification factor (Q) of 10 for high-performance deuterium and tritium scenarios [53].

large radius	6.20 m
small horizontal radius	2 m
small vertical radius	3.4 m
plasma form	X point
toroidal field	5.3 Tesla
plasma current	max 15 A
ICF heating	20 MW
LH heating	20 MW
Heating by neutral injection	33 MW
plasma volume	830 m^3
discharge duration	up to 1000 s
expected fusion power	500 MW
plasma heat energy	353 MJ
plasma magnetic energy	395 MJ

Table 1.3: Main features of ITER [62]

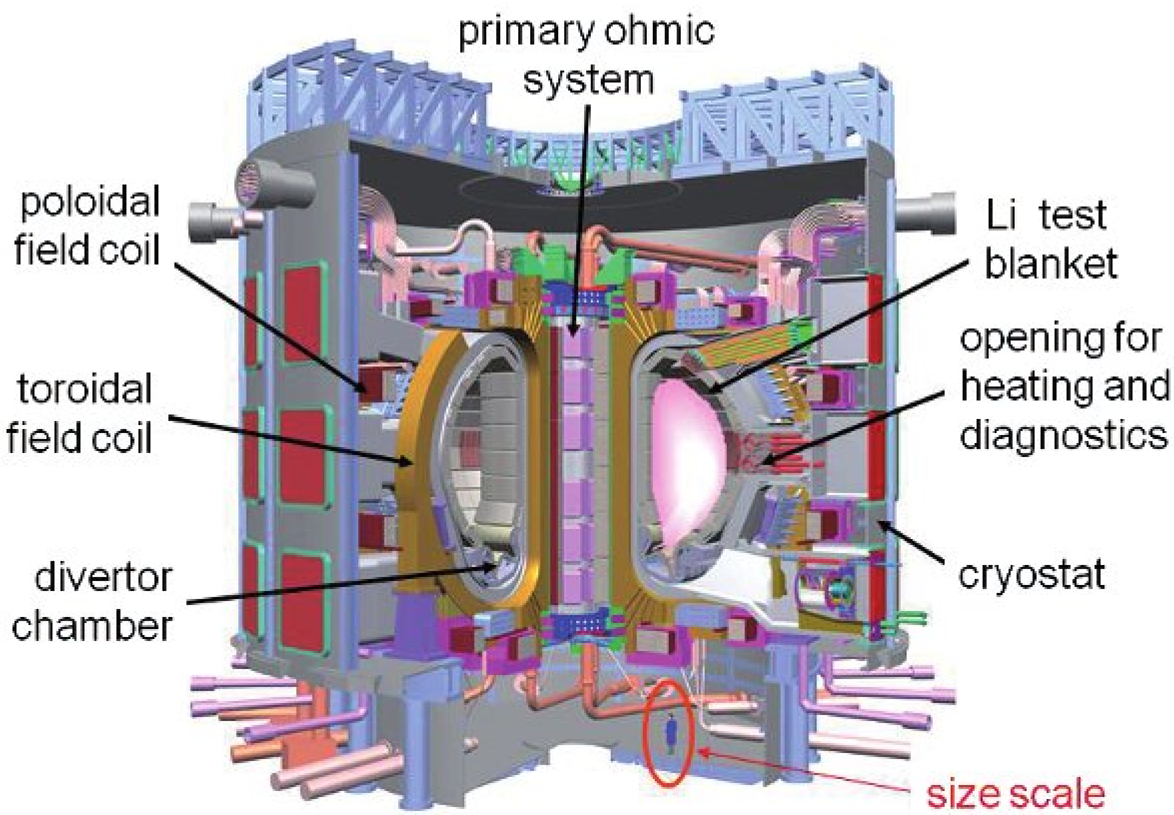

Fig. 1.15: Computer drawing of ITER [63].

1.8 Magneto-Inertial Fusion

Magneto-inertial fusion (MIF) is a modern hybrid approach that describes a class of fusion devices that combine aspects of magnetic confinement fusion and inertial confinement fusion. It can provide a simpler and less expensive scheme of controlled fusion (Figure 1.16) [64].

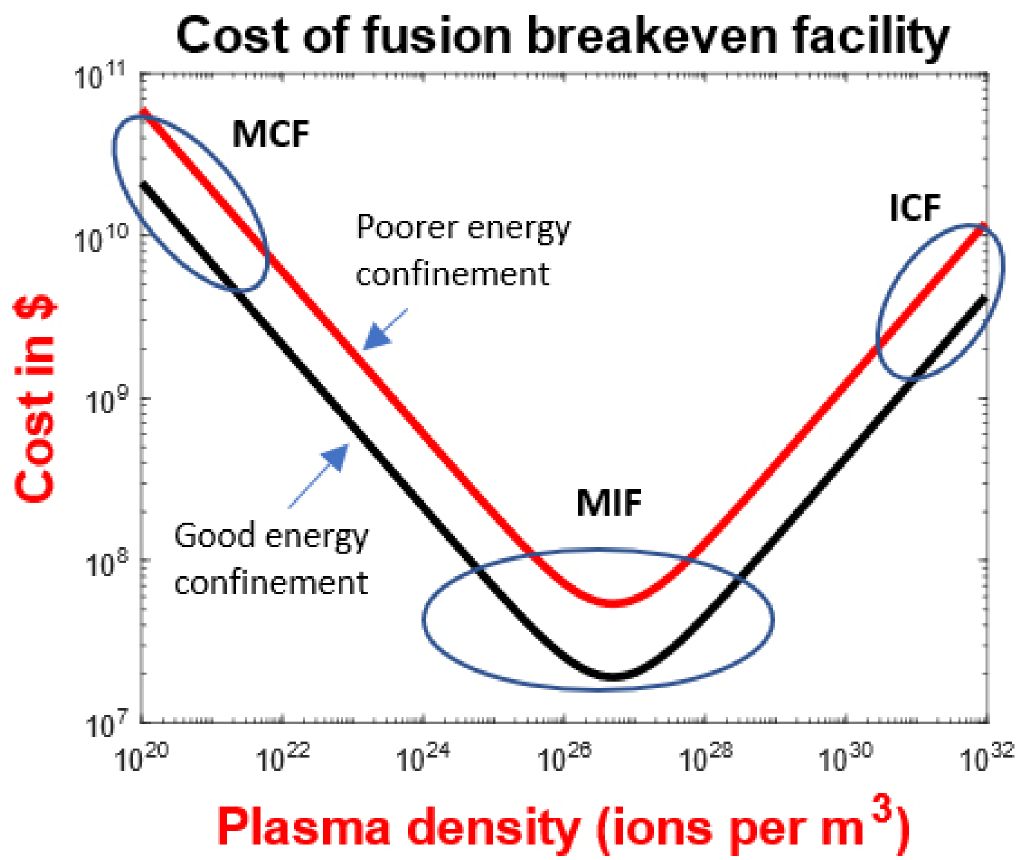

Fig. 1.16: Cost of Fusion Breakeven [65].

In inertial confinement fusion (ICF), we use powerful energy sources to heat the thermonuclear fuel to high temperatures for a time comparable to the characteristic times of hydrodynamic plasma expansion. The MIF represents the evolution of inertial confinement fusion (ICF) with magnetic confinement fusion elements; it is therefore considered a third way of fusion [66].

The main ideas of the MIF have been known for almost four decades. The concept behind this method is to deliver a strong magnetic field to the inertial fusion target and either embed the magnetic flux in the hot spot or freeze it there. The latter option is intended to act as a preservative (liner). (Figure 1.17) illustrates magnetic flux.

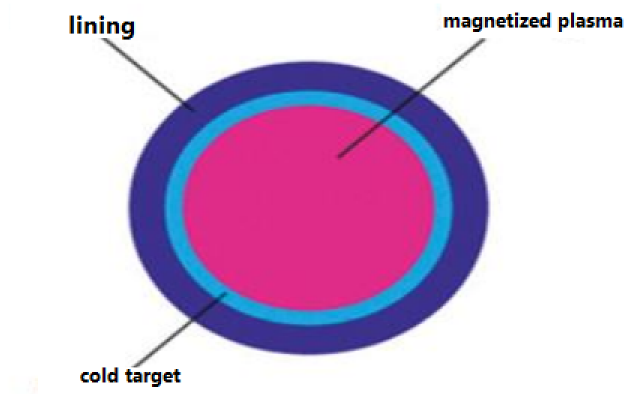

Fig. 1.17: Constituent of the MIF target, a liner is used to compress a magnetised plasma target. Sometimes a cold fuel bed is used to increase the fusion gain [64].

In a manner similar to conventional inertial fusion, the hot spot or conductive shell (liner) is imploded, the magnetic flux is compressed with it, and thus the magnetic field strength is increased. The high magnetic field used during plasma compression considerably reduces thermal diffusivity, making it simple to heat the plasma to thermonuclear melting temperatures (Fig. 1.17).

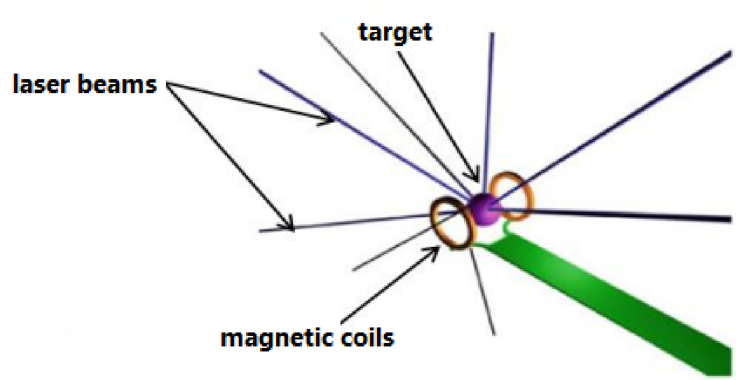

Fig. 1.18: Diagram of the spherical implosion of the magnetised target during its uniform compression with laser beams [67].

1.9 Conclusion

Controlled nuclear fusion presents significant challenge for humanity's energy development in the decades to come, offering abundant, low-polluting, and safe energy at reasonable costs.

In this chapter, a bibliographic study focuses on the main research works relating to controlled thermonuclear fusion. The technological as well as the scientific objectives of each of the pathways and of fusion remain, however, real challenges. Different magnetic and inertial research paths are currently being followed.

In the case of magnetic confinement, the idea is to compress and confine the plasma brought to very high temperatures, by magnetic fields using a tokamak. This device has benefited from considerable progress since its invention in the 1950 by Russian researchers Igor Tamm and Sakharov.

The inertial pathway involves igniting small fuel capsules using laser pulses. Compared to fusion by magnetic confinement, laser fusion has at least three advantages: it does not require a confinement device in the reaction chamber; it would allow reactions other than deuterium-tritium fusion, which produces problematic neutrons; and it assumes a lower energy investment for a greater thermonuclear gain, which promises smaller and less expensive installations.

Magneto-inertial fusion is a promising third fusion route, combining inertial fusion with magnetic confinement elements. It involves either supplying a strong magnetic field to the target and freezing the flux into a hot spot or combining the flux into a target plasma.

Research on these paths is therefore long-term, with no end in sight before the middle of the century (100 years of research in all) for production deployed in the second half of the century.

Chapter 2

Quantum Dynamics of Hydrogen-like Ions in Non-Uniform Magnetic Field

2.1 Introduction

One of the interesting problems of quantum mechanics is to find the exact solutions to the Schrodinger equation for certain systems of physical interest. Significant efforts have been made in recent years to obtain analytical solutions to plasma problems. The solution of Schrodinger's equation for electromagnetic systems has been the subject of work by many researchers using different methods; by the supersymmetric method in [68, 69], by the path integral [70] and Bessel function [71, 72].

The equations of Klein-Gordon, Dirac, and Schrodinger, the latter, are among the many physics equations that are treated in various ways; which is a second-order differential equation with respect to the position of the system and a first-order differential equation with respect to time; is of special interest in non-relativistic quantum physics. Indeed, the resolution of this equation provides us with all the information on the studied system. The term "interaction potential" renders this equation in a form that is not really easy to integrate. Schrodinger's equation can be cited (with the help of some mathematical tricks) among a set of differential equations accepting analytical solutions, this set of differential equations has been completely treated by the two known as Nikiforov and Uvarov in their book [74] and also in [73, 75–83].

The goal of this chapter is to find the Schrodinger equation solution analytically for a particle withăa vector potential and a coulomb potential. The method used to deal with this problem is that of NIKIFOROV-UVAROV (NU), which is easier to use than several other methods. In the following sections, we present in detail the different steps of this method (NU).The form of the (vector and scalar) potential treated will then be given. Following that, in the fifth section, we could write two differential equations, one radial and the other angular, with the angular part separated into two other differential equations, one dependent on ϕ,

which is easily integrated, and the other on θ.
In the sixth section, we will use the NU method to solve the radial and angular differential equations.ă This processing, which provides us with the energy spectrum and the properly orthonormalized wave functions, represents a particular case of the problem. Next We would provide some Ly-alpha spectral line shapes of three hydrogen-like ions for different magnetic fields and temperatures, and then we would discuss them. Finally, we will end with a conclusion.

2.2 Nikiforov-Uvarov Methods

The NU method [74], represents a very strong mathematical tool for solving second-order differential equations with variable coefficients in the form of polynomials. This method has been successfully used in the analytical treatment of a large number of theoretical physics problems in the stationary and relativistic case. Indeed, this method allows us to give the wave functions and the energy spectrum of the Schrodinger equation, Klein-Gordon, and Dirac in the presence of well-known potentials.

In the following, we will explain in detail the mathematical formalism of this method and how to use it to deal with problems in physics.

2.2.1 Nikiforov-Uvarov Differential Equation:

The goal of the NU method is to solve any differential equation of the form [74]:

$$\Psi''(s) + \frac{\widetilde{\tau}(s)}{\sigma(s)}\Psi'(s) + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\Psi(s) = 0 \tag{2.1}$$

Called hypergeometric type.
Here, $\sigma(s)$ and $\widetilde{\sigma}(s)$ are two polynomials with a degree of at most 2:

$$\begin{aligned} \sigma(s) &= a_1 s^2 + a_2 s + a_3 \\ \widetilde{\sigma}(s) &= b_1 s^2 + b_2 s + b_3 \end{aligned} \tag{2.2}$$

Whereas $\widetilde{\tau}(s)$ is a polynomial with a degree of at most 1:

$$\widetilde{\tau}(s) = c_1 s + c_2 \tag{2.3}$$

With: $a_1, a_2, a_3, b_1, b_2, b_3$, c_1 and c_2 are constants.

2.2.2 Solutions of the Differential Equation of (N-U): [74]

2.2.2.1 Change of Function:

The first step in this method is the following function change [74]:

$$\Psi(s) = \phi(s)Y(s) \tag{2.4}$$

Starting from equation (2.4) we have:

$$\begin{aligned}\Psi'(s) &= \phi'(s)Y(s) + \phi(s)Y'(s) &(2.5)\\ \Psi''s) &= \phi''(s)Y(s) + \phi'(s)Y'(s) + \phi'(s)Y'(s) + \phi(s)Y''(s) &(2.6)\end{aligned}$$

By inserting the equations (2.5) and (2.6) in the equation (2.1) we obtain:

$$Y''(s) + \left(2\frac{\phi'(s)}{\phi(s)} + \frac{\widetilde{\tau}(s)}{\sigma(s)}\right)Y'(s) + \left(\frac{\phi''(s)}{\phi(s)} + \frac{\widetilde{\tau}(s)}{\sigma(s)}\frac{\phi'(s)}{\phi(s)} + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\right)Y(s) = 0 \quad (2.7)$$

We choose the function $\phi(s)$ which allows us to write the coefficient of $Y'(s)$ in the form $\frac{\tau(s)}{\sigma(s)}$; where $\tau(s)$ is a polynomial with a degree of at most 1. This gives us [74]:

$$2\frac{\phi'(s)}{\phi(s)} + \frac{\widetilde{\tau}(s)}{\sigma(s)} = \frac{\tau(s)}{\sigma(s)} \quad (2.8)$$

And $\phi(s)$ is a logarithmic derivative whose solution obtained from the condition [74]:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \quad (2.9)$$

After comparing equations (2.8) and (2.9), we find that:

$$\pi(s) = \frac{1}{2}\left[\tau(s) - \widetilde{\tau}(s)\right] \quad (2.10)$$

So, we know from (2.10), the new parameter $\pi(s)$ is a polynomial of degree at most 1, from the last equation we write:

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s) \quad (2.11)$$

- $\tau(s)$ must satisfy the condition ($\tau'(s) < 0$). Which gives us the physically acceptable solutions.

In order to simplify equation (2.7) and the coefficient of $Y(s)$ in (2.9), let us first calculate $\left(\frac{\phi'(s)}{\phi(s)}\right)'$:

$$\left(\frac{\phi'(s)}{\phi(s)}\right)' = \frac{\phi''(s)}{\phi(s)} - \left(\frac{\phi'(s)}{\phi(s)}\right)^2 \quad (2.12)$$

From (2.9) and (2.12) we can write:

$$\frac{\phi''(s)}{\phi(s)} = \left(\frac{\pi(s)}{\sigma(s)}\right)' + \left(\frac{\pi(s)}{\sigma(s)}\right)^2 \quad (2.13)$$

In this case, the coefficient of $Y(s)$ transformed into a more appropriate form, by taking the equality given in (2.9).

$$\frac{\phi''(s)}{\phi(s)} + \frac{\widetilde{\tau}(s)}{\sigma(s)}\frac{\phi'(s)}{\phi(s)} + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)} = \frac{\overline{\sigma}(s)}{\sigma^2(s)} \quad (2.14)$$

where:

$$\overline{\sigma}(s) = \widetilde{\sigma}(s) + \pi^2(s) + \pi(s)\left[\widetilde{\tau}(s) - \sigma'(s)\right] + \pi'(s)\sigma(s) \tag{2.15}$$

One obtains $\overline{\sigma}(s)$ as a function of $\widetilde{\sigma}(s)$,$\pi^2(s)$ and $\sigma(s)$ i.e. $\overline{\sigma}(s)$ is a polynomial at most of degree 2.

So (2.7) is written in the same form as (2.1).

$$Y''(s) + \frac{\tau(s)}{\sigma(s)}Y'(s) + \frac{\overline{\sigma}(s)}{\sigma^2(s)}Y(s) = 0 \tag{2.16}$$

As a result of the algebraic transformations mentioned above, the functional form of equation (2.1) is systematically protected. The polynomial $\overline{\sigma}(s)$ in equation (2.15) is divisible by $\sigma(s)$ so we write [74]:

$$\overline{\sigma}(s) = \lambda\sigma(s) \tag{2.17}$$

Where λ is a constant.

So equation (2.16) is reduced to a hypergeometric type equation:

$$\sigma(s)Y''(s) + \tau(s)Y'(s) + \lambda Y(s) = 0 \tag{2.18}$$

In order to find the form of the polynomial $\pi(s)$ which is a polynomial of the first degree, we use the two equations (2.15) and (2.17).

$$\widetilde{\sigma}(s) + \pi^2(s) + \pi(s)\left[\widetilde{\tau}s) - \sigma'(s)\right] + \pi'(s)\sigma(s) = \lambda\sigma(s) \tag{2.19}$$

If we put:

$$k = \lambda - \pi'(s) \tag{2.20}$$

as a constant.

Equation (2.19) becomes:

$$\pi^2(s) + \pi(s)\left[\widetilde{\tau}(s) - \sigma'(s)\right] + \widetilde{\sigma}(s) - k\sigma(s) = 0 \tag{2.21}$$

Equation (2.21) is a second order equation with respect to $\pi(s)$, trying now to solve this equation, we have:

$$\Delta = \left[\widetilde{\tau}(s) - \sigma'(s)\right]^2 - 4\left(\widetilde{\sigma}(s) - k\sigma(s)\right)$$

then

$$\pi(s) = \frac{\sigma'(s) - \widetilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\widetilde{\tau}(s) - \sigma'(s)}{4}\right)^2 - \widetilde{\sigma}(s) + k\sigma(s)} \tag{2.22}$$

As was pointed out previously, the polynomial $\pi(s)$ is a polynomial at most of degree 1, this is true if and only if the expression under the square root of equation (2.21) must be squared by a polynomial at most degree 1 too. This requires that the discriminant of the quadratic expression under the root be zero. This condition ($\Delta = 0$) gives us the value of the constant k.

After the determination of k, the polynomial (2.21), and then $\tau(s)$ and λ also obtained by using equations (2.11) and (2.20), respectively.

2.2.2.2 Eigenvalues

In equation (2.18) looking for solutions in the form [74]:

$$Y_n(s) = \sum_{n=0}^{+\infty} a_n s^n \tag{2.23}$$

then:

$$Y'(s) = \sum_{n=1}^{+\infty} n a_n s^{n-1} \tag{2.24}$$

And also:

$$Y''(s) = \sum_{n=2}^{+\infty} n(n-1) a_n s^{n-2} \tag{2.25}$$

So (2.18) becomes:

$$\sigma(s) \sum_{n=2}^{+\infty} n(n-1) a_n s^{n-2} + \tau(s) \sum_{n=1}^{+\infty} n a_n s^{n-1} + \lambda \sum_{n=0}^{+\infty} a_n s^n = 0 \tag{2.26}$$

By replacing the expressions of $\sigma(s)$ and $\tau(s)$ we obtain:

$$a_1 n(n-1)s^n + c_1 n s^n = -\lambda_n s^n \tag{2.27}$$

with($\sigma''(s) = 2a_1$) and ($\tau'(s) = c_1$).
The Frobenius method yields the following relation:

$$\lambda = \lambda_n = -n\left(\tau'(s) + \frac{(n-1)}{2}\sigma''(s)\right) \tag{2.28}$$

Where λ_n represent the eigenvalues of the differential equation (2.18).
The new eigenvalue of the energy is obtained using equation (2.20) and (2.28).

2.2.2.3 Eigen Functions:

To generate the corresponding eigenfunctions, with the condition $\tau'(s) < 0$ which is obligatory. The solution of the second part of the wave function, $Y_n(s)$ is given by the formula of Rodrigues [80]:

$$Y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[\sigma^n(s)\rho(s)\right] \tag{2.29}$$

Where B_n is a normalization constant, and $\rho(s)$ represents the resolvent function (weight function), which was found from the following equation:

$$\sigma(s)Y''(s) + \tau(s)Y'(s) = -\lambda Y(s) = \frac{1}{\rho(s)}\left[\rho(s)\sigma(s)Y'(s)\right]' \tag{2.30}$$

And $\rho(s)$ is a logarithmic derivative whose solution obtained from the expression:

$$\frac{\rho'(s)}{\rho(s)} = \frac{\tau(s) - \sigma'(s)}{\sigma(s)} \tag{2.31}$$

Assuming that $\rho(s)$ is an analytic function on and inside a closed contour C surrounding the point ($s = z$), and making use of Cauchy's integral theorem [81].

We can write:

$$Y_n(s) = \frac{c_n}{\rho(s)} \int_C \frac{\sigma^n(z)\rho(z)}{(z-s)^{n+1}} dz \tag{2.32}$$

Where c_n is a normalization constant and $\rho(s)$ satisfies (2.31) this suggests looking for a particular solution of (2.18).

The equation of the hypergeometric type has the particular solution of the form (2.32) if:

-The derivatives of the functions $Y_n(s)$ can be computed by differentiation under the integral sign.

-The following condition is satisfied at the end points of the cut C, at $z = z_1$ and $z = z_2$:

$$\sigma(z)\rho(z)z^k|_{z_1}^{z_2} = 0, (k = 0, 1, 2, ...) \tag{2.33}$$

Where: z_1 and z_2 are the endpoints of C.

The wave function of the system is therefore obtained from equation (2.9), (2.29) and (2.31).

2.3 Magnetic Field Geometry:

The most advanced and successful method to achieve controlled fusion is plasma confinement by sufficient magnetic fields. Finding an electromagnetic field geometry that efficiently confines the plasma, which has been a significant component of the confinement problem. A group of toroidal magnets is used to do this by creating a toroidal field. It makes sure that the charged particles are contained inside the torus. It is demonstrated that this confinement is insufficient and that the field lines along the torus must be helical in an effort to further reduce the particles' leaking toward the walls. To do this, a magnetic field known as a poloidal field that is perpendicular to the toroidal field is added. Before the fusion starts, a number of devices and experiments must be ready to ensure that these conditions are met [84].

We've presummated that the strong magnetic field is the result of the addition of two magnetic fields, the most intense of which has a toroidal geometry and the other of which has a poloidal geometry, roughly a third as intense as the first [84].

$$\overrightarrow{A}(r, \theta, \phi) = -\frac{A(r)}{\sin\theta}\overrightarrow{u_r} + \varepsilon_0 y \overrightarrow{i} \tag{2.34}$$

and

$$\overrightarrow{B}(r,\theta,\phi) = \overrightarrow{\nabla} \times \overrightarrow{A}(r,\theta,\phi) = \begin{pmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \overrightarrow{u_\phi} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ -\frac{A(r)}{\sin(\theta)} & 0 & 0 \end{pmatrix} + \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\varepsilon_0 y & 0 & 0 \end{pmatrix}$$
$$= -\frac{1}{r}\frac{\partial}{\partial\theta}(-\frac{A(r)}{\sin(\theta)})\overrightarrow{u_\phi} + \varepsilon_0\overrightarrow{k} = -\frac{A(r)\cos(\theta)}{r\sin^2(\theta)}\overrightarrow{u_\phi} + \varepsilon_0\overrightarrow{k} \quad (2.35)$$

where we have used cartesian coordinates ($\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$) and spherical coordinates ($\overrightarrow{u_r}, \overrightarrow{u_\theta}, \overrightarrow{u_\phi}$). The expression of $\overrightarrow{A}$ in (2.34) is chosen so that the resulting magnetic field in the toroid has the desired geometry. ε_0 is the magnetic field's strength along the z-axis, responsible for the poloidal field. We have implicitly assumed in this form of the magnetic field (2.35), that the observation will be made in the direction parallel to the component of the magnetic field along z (the other component of the magnetic field rotates along $\overrightarrow{u_\phi}$ causing a zero average polarization). Following the symmetry of the problem, we must first express the two fields ($\overrightarrow{A}(\overrightarrow{r})$ and $\overrightarrow{B}(\overrightarrow{r})$) in the spherical coordinates before proceeding on:

$$\overrightarrow{A} = \left(-\frac{A(r)}{\sin(\theta)} - \varepsilon_0 r\sin\theta^2\sin\phi\cos\phi\right)\overrightarrow{u_r} - \varepsilon_0 r\sin\theta\cos\theta\sin\phi\cos\phi\overrightarrow{u_\theta} + \varepsilon_0 r\sin\theta\sin^2\phi\overrightarrow{u_\phi} \quad (2.36)$$

$$\overrightarrow{B}(r,\theta,\phi) = \varepsilon_0\cos\theta\overrightarrow{e_r} - \varepsilon_0\sin\theta\overrightarrow{e_\theta} - \frac{A(r)\cos(\theta)}{r\sin^2(\theta)}\overrightarrow{u_\phi} \quad (2.37)$$

2.4 Quantum Dynamics of the Confined Ions:

Let's begin by thinking about the ion's quantum dynamics that are enclosed in a strong nonuniform magnetic field. In this instance, we'll use the Schrodinger equation to model an electron with a spin of 1/2, an electric charge of -e, motions in the nuclear field (Coulomb interaction), as well as a nonuniform magnetic field.The equation for this kind of problem is known as the Pauli equation. However, it has been demonstrated in numerous papers, including [85–88], that it is a good approximation to use the Pauli equation and omit the fine structure effect when there is a strong magnetic field. It is necessary to see this problem in relativistic quantum mechanics via the Dirac equation to evaluate its dynamics. Getting the system's eigenenergy is the major goal of this section. The Pauli Hamiltonian [89] is as follows:

$$H = \frac{1}{2\mu}(\overrightarrow{P} - e\overrightarrow{A})^2 - Z\frac{e^2}{r} + \eta\overrightarrow{S}.\overrightarrow{B} \quad (2.38)$$

where

$$\eta = \frac{\mu_B}{\hbar}$$

where μ is the reduced mass, μ_B is Bohr's magneton, and $\overrightarrow{S}$ is the spin of the electron interacting with the magnetic field $\overrightarrow{B}$ (Here, the spin-orbit coupling is neglected.).

The ion's dynamics in the magnetic field is governed by the time independent Schrodinger equation given by ($H\Psi(r,\theta,\phi) = E\Psi(r,\theta,\phi)$), in other words:

$$\left[\frac{1}{2\mu}(\overrightarrow{P} - e\overrightarrow{A})^2 - Z\frac{e^2}{r} + \eta\overrightarrow{S}.\overrightarrow{B}\right]\Psi(r,\theta,\phi) = E\Psi(r,\theta,\phi) \tag{2.39}$$

or similarly

$$\left[\frac{1}{2\mu}(\overrightarrow{P}^2 - \frac{e}{c}\overrightarrow{P}.\overrightarrow{A} - \frac{e}{c}\overrightarrow{A}.\overrightarrow{P} + \frac{e^2}{c^2}A^2) - Z\frac{e^2}{r} + \eta\overrightarrow{S}.\overrightarrow{B}\right]\Psi(r,\theta,\phi) = E\Psi(r,\theta,\phi) \tag{2.40}$$

where it is also necessary to express the momentum operator $\overrightarrow{P}$ in spherical coordinates., as

$$\overrightarrow{P} = \frac{\hbar}{i}\left(\overrightarrow{u_r}\frac{\partial}{\partial r} + \overrightarrow{u_\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \overrightarrow{u_\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \tag{2.41}$$

First, we must express the products. $\overrightarrow{P}.\overrightarrow{A}$ and $\overrightarrow{A}.\overrightarrow{P}$:

$$\begin{aligned}\overrightarrow{P}.\overrightarrow{A}\Psi(r,\theta,\phi) &= \frac{\hbar}{i}\left(\overrightarrow{u_r}\frac{\partial}{\partial r} + \overrightarrow{u_\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \overrightarrow{u_\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times \\ &\quad \left(-\frac{A(r)}{\sin(\theta)} - \varepsilon_0 r\sin^2\theta\sin\phi\cos\phi\right)\overrightarrow{u_r} \\ &\quad -\varepsilon_0 r\sin\theta\cos\theta\sin\phi\cos\phi\overrightarrow{u_\theta} + \varepsilon_0 r\sin\theta\sin^2\phi\overrightarrow{u_\phi}\Psi(r,\theta,\phi). \\ &= \frac{\hbar}{i}\left[\frac{A'(r)}{\sin\theta} + 2\frac{A(r)}{r\sin\theta}\right]\Psi(r,\theta,\phi) + \frac{\hbar}{i}\overrightarrow{A}.\overrightarrow{\nabla}\Psi(r,\theta,\phi)\end{aligned} \tag{2.42}$$

then

$$\overrightarrow{P}.\overrightarrow{A} = \frac{\hbar}{i}\left[\frac{A'(r)}{\sin\theta} + 2\frac{A(r)}{r\sin\theta}\right] + \frac{\hbar}{i}\left(-\frac{A(r)}{\sin(\theta)} - \varepsilon_0 r\sin\theta^2\sin\phi\cos\phi\right)\frac{\partial}{\partial r} \tag{2.43}$$

and

$$\overrightarrow{A}.\overrightarrow{P} = \frac{\hbar}{i}\overrightarrow{A}.\overrightarrow{\nabla} = \frac{\hbar}{i}\left(-\frac{A(r)}{\sin(\theta)} - \varepsilon_0 r\sin\theta^2\sin\phi\cos\phi\right)\frac{\partial}{\partial r} \tag{2.44}$$

by substituting them into the above Pauli equation, neglecting the term of $\varepsilon_0$order:

$$\begin{aligned}&\frac{1}{2\mu}\left[\begin{array}{c}-\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \\ -\frac{e}{c}\frac{\hbar}{i}\left(-\frac{A'(r)}{\sin\theta} + 2\frac{A(r)}{r\sin\theta} + 2\overrightarrow{A}.\overrightarrow{\nabla}\right) + \frac{e^2}{c^2}\left(\frac{A^2(r)}{\sin^2\theta} - 2\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi\right)\end{array}\right]\Psi(r,\theta,\phi) \\ &= \left(E + Z\frac{e^2}{r} - \eta\overrightarrow{S}.\overrightarrow{B}\right)\Psi(r,\theta,\phi)\end{aligned} \tag{2.45}$$

On the other hand, we have

$$\overrightarrow{S}.\overrightarrow{B} = S_\phi B_\phi + S_z B_z = (-\sin\phi S_x + \cos\phi S_y).\frac{-A(r)\cos\theta}{r\sin^2\theta} + \varepsilon_0 S_z \tag{2.46}$$

where B_ϕ, B_z are the components of $\overrightarrow{B}$ along azumital angle ϕ and z, respectively, whereas Pauli Spin Matrixes are presented by

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.47}$$

We put

$$\Psi(r,\theta,\phi) = \begin{pmatrix} \Psi_1(r,\theta,\phi) \\ \Psi_2(r,\theta,\phi) \end{pmatrix} \tag{2.48}$$

$$\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \tag{2.49}$$

$$\Delta_{\theta\phi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{2.50}$$

or also:

$$\begin{aligned} r^2E\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} &= \frac{1}{2\mu}\left[\begin{array}{c} -\hbar^2\left(r^2\Delta_r + \Delta_{\theta\phi}\right) - \frac{e}{c}r^2\frac{\hbar}{i}\left(-\frac{A'(r)}{\sin\theta} - 2\frac{A(r)}{r\sin\theta} + 2\overrightarrow{A}.\overrightarrow{\nabla}\right) \\ +\frac{e^2}{c^2}r^2\left(\frac{A^2(r)}{\sin^2\theta} + 2\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi\right) \end{array}\right]\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \\ -Ze^2r\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad & -\frac{rA(r)\cos\theta}{\sin^2\theta}\left(-\sin\phi\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} + \cos\phi\begin{pmatrix} 0 & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & 0 \end{pmatrix}\right)\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \\ & +\eta\varepsilon_0 r^2\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \end{aligned} \tag{2.51}$$

$$\left\{\begin{array}{c} \left(\frac{1}{2\mu}\left[\begin{array}{c} -\hbar^2\left(r^2\Delta_r + \Delta_{\theta\phi}\right) - \frac{e}{c}\frac{\hbar}{i}r^2\left(-\frac{A'(r)}{\sin\theta} - 2\frac{A(r)}{r\sin\theta} + 2\overrightarrow{A}.\overrightarrow{\nabla}\right) \\ +\frac{e^2}{c^2}r^2\left(\frac{A^2(r)}{\sin^2\theta} + 2\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi\right) \end{array}\right] - Ze^2r\right)\Psi_1 \\ -\frac{rA(r)\cos\theta}{\sin^2\theta}\eta\left(-\frac{\hbar}{2}\sin\phi\Psi_2 - \frac{i\hbar}{2}\cos\phi\Psi_2\right) + \eta\varepsilon_0 r^2\frac{\hbar}{2}\Psi_1 = r^2E\Psi_1 \\ \left(\frac{1}{2\mu}\left[\begin{array}{c} -\hbar^2\left(r^2\Delta_r + \Delta_{\theta\phi}\right) - \frac{e}{c}\frac{\hbar}{i}r^2\left(-\frac{A'(r)}{\sin\theta} - 2\frac{A(r)}{r\sin\theta} + 2\overrightarrow{A}.\overrightarrow{\nabla}\right) \\ +\frac{e^2}{c^2}r^2\left(\frac{A^2(r)}{\sin^2\theta} + 2\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi\right) \end{array}\right] + Ze^2r\right)\Psi_2 \\ -\frac{rA(r)\cos\theta}{\sin^2\theta}\left(-\frac{\hbar}{2}\sin\phi\Psi_1 + \frac{i\hbar}{2}\cos\phi\Psi_1\right) + \eta\varepsilon_0 r^2\frac{\hbar}{2}\Psi_2 = r^2E\Psi_2 \end{array}\right\} \tag{2.52}$$

In the first equation, we separate the real part from the imaginary part.

$$\left\{\begin{array}{c}\left[\frac{-\hbar^2}{2\mu}\left(r^2\Delta_r+\Delta_{\theta\phi}\right)-Ze^2r+\frac{e^2r^2}{2\mu c^2}\left(\frac{A^2(r)}{\sin^2\theta}+2\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi\right)-r^2E+\eta\varepsilon_0 r^2\frac{\hbar}{2}\right]\Psi_1\\ +\hbar\frac{rA(r)\cos\theta}{2\sin^2\theta}\sin\phi\Psi_2=0\\ \frac{1}{2\mu}\left[\frac{e}{c}\hbar r^2\left(-\frac{A'(r)}{\sin\theta}-2\frac{A(r)}{r\sin\theta}+2\overrightarrow{A}.\overrightarrow{\nabla}\right)\right]\Psi_1+\hbar\frac{rA(r)\cos\theta}{2\sin^2\theta}\eta\cos\phi\Psi_2=0\end{array}\right\} \tag{2.53}$$

Using a member-by-member division, we get

$$\begin{aligned}&\frac{\left[\frac{-\hbar^2}{2\mu}\left(r^2\Delta_r+\Delta_{\theta\phi}\right)-Ze^2r+\frac{e^2r^2}{2\mu c^2}\left(\frac{A^2(r)}{\sin^2\theta}+2\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi\right)-r^2E+\eta\varepsilon_0 r^2\frac{\hbar}{2}\right]\Psi_1}{\frac{1}{2\mu}\left[\frac{e}{c}\hbar r^2\left(\left[-\frac{A'(r)}{\sin\theta}-2\frac{A(r)}{r\sin\theta}+2\overrightarrow{A}.\overrightarrow{\nabla}\right]\right)\right]\Psi_1}\\ =\ &\frac{\hbar\frac{rA(r)\cos\theta}{2\sin^2\theta}\sin\phi}{\hbar\frac{rA(r)\cos\theta}{2\sin^2\theta}\cos\phi}=\tan\phi\end{aligned} \tag{2.54}$$

puttig $\varepsilon=\frac{2\mu}{\hbar^2}E$ then

$$0=\left[\begin{array}{c}-\left(r^2\Delta_r+\Delta_{\theta\phi}\right)-\frac{2\mu}{\hbar^2}Ze^2r+\frac{e^2}{\hbar^2c^2}r^2\frac{A^2(r)}{\sin^2\theta}-r^2\varepsilon+\eta\frac{\mu\varepsilon_0}{\hbar}r^2\\ -\frac{e}{\hbar c}\frac{\tan\phi}{\sin\theta}\left(r^2\left[-A'(r)-2\frac{e}{\hbar c}\frac{\sin\theta}{\tan\phi}\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi-2\frac{A(r)}{r}+2\sin\theta\overrightarrow{A}.\overrightarrow{\nabla}\right]\right)\end{array}\right]\Psi_1 \tag{2.55}$$

We substitute the term in brackets with its average on the ground (without a magnetic field). (Ψ_{100})

$$\begin{aligned}&<\ \tan\phi\left(r^2\left[-A'(r)-2\frac{e}{\hbar c}\frac{\sin\theta}{\tan\phi}\varepsilon_0 rA(r)\sin\theta\sin\phi\cos\phi-2\frac{A(r)}{r}+2\sin\theta\left(\overrightarrow{A}.\overrightarrow{\nabla}\right)\right]\right)>\\ &=\ <\tan\phi\left(-A'(r)-2\frac{A(r)}{r}+2\sin\theta\left(\overrightarrow{A}.\overrightarrow{\nabla}\right)\right)\\ &=\ q_1\varepsilon_0\end{aligned} \tag{2.56}$$

where $q_1=\frac{45\pi}{32Z^2}$

and the subscript (100) stands for the principal quantum numbers (nlm). The term with ε_0 appears from the factor $\overrightarrow{A}(\overrightarrow{r})$, whereas the factor with $\cos\phi$ disappears because its average over ϕ in $[0;2\pi]$ cancels, then

$$0=\left[-\left(r^2\Delta_r+\Delta_{\theta\phi}\right)-\frac{2\mu}{\hbar^2}Ze^2r+\frac{e^2}{\hbar^2c^2}r^2\frac{A^2(r)}{\sin^2\theta}-r^2\varepsilon+\eta\frac{\mu\varepsilon_0}{\hbar}r^2-\frac{e}{\hbar c}\frac{q_1\varepsilon_0}{\sin\theta}\right]\Psi_1 \tag{2.57}$$

Now, we choose $A(r)$ as [84]:

$$A(r)=a^2\frac{B_0}{r} \tag{2.58}$$

where B_0 is a factor in the toroidal magnetic field along $\overrightarrow{u_\phi}$; it has the magnetic field unit, whereas "a" is a constant (that can be assimilated to Bohr radius); then, we obtain

$$0 = \left[-\left(r^2\Delta_r + \Delta_{\theta\phi} \right) - \frac{2\mu}{\hbar^2} Ze^2 r - r^2\varepsilon + \eta\frac{\mu\varepsilon_0}{\hbar} r^2 + \frac{e^2}{\hbar^2 c^2} a^4 \frac{B_0^2}{\sin^2\theta} - \frac{e}{\hbar c}\frac{q_1\varepsilon_0}{\sin\theta} \right] \Psi_1 \quad (2.59)$$

2.5 Separation of Variables

The Ψ wave function can be written as follows when the variables are separated according to the separation of variables method [84]:

$$\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \quad (2.60)$$

In which $R(r)$is the radial wave function and $Y(\theta,\phi)$represents the angular part.
then

$$\left[-r^2\Delta_r - \frac{2\mu}{\hbar^2} Ze^2 r - r^2(\varepsilon - \eta\frac{\mu\varepsilon_0}{\hbar}) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{e^2}{\hbar^2c^2}a^4\frac{B_0^2}{\sin^2\theta} - \frac{e}{\hbar c}\frac{q_1\varepsilon_0}{\sin\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] \\ \times R(r)Y(\theta,\phi) = 0 \quad (2.61)$$

$$\begin{aligned} 0 &= \left[\frac{\left(-r^2\Delta_r - \frac{2\mu}{\hbar^2}Ze^2 r - r^2(\varepsilon - \eta\frac{\mu\varepsilon_0}{\hbar})\right) R(r)}{R(r)} \right] \\ &- \left[\frac{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) - \frac{e^2}{\hbar^2c^2}a^4\frac{B_0^2}{\sin^2\theta} - \frac{e}{\hbar c}\frac{q_1\varepsilon_0}{\sin\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right) Y(\theta,\phi)}{Y(\theta,\phi)} \right] \end{aligned} \quad (2.62)$$

This holds true if every term is a constant ω.

$$\begin{aligned} -\omega &= \frac{\left(-r^2\Delta_r - \frac{2\mu}{\hbar^2}Ze^2 r - r^2(\varepsilon - \eta\frac{\mu\varepsilon_0}{\hbar})\right) R(r)}{R(r)} \\ &= \frac{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) - \frac{e^2}{\hbar^2c^2}a^4\frac{B_0^2}{\sin^2\theta} - \frac{e}{\hbar c}\frac{q_1\varepsilon_0}{\sin\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right) Y(\theta,\phi)}{Y(\theta,\phi)} \end{aligned} \quad (2.63)$$

Now, one can easily confirm that Schrodinger's equation(2.59), reducedto two ordinary differential equations. One is the radial equation for aparticle immersed in the Coulomb field, and the other represents the angularpart:

$$\begin{aligned}\frac{\left(-r^2\Delta_r-\frac{2\mu}{\hbar^2}Ze^2r-r^2(\varepsilon-\eta\frac{\mu\varepsilon_0}{\hbar})\right)R(r)}{R(r)} &= -\omega \qquad (2.64)\\ \frac{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{e^2}{\hbar^2c^2}a^4\frac{B_0^2}{\sin^2\theta}-\frac{e}{\hbar c}\frac{q_1\varepsilon_0}{\sin\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y(\theta,\phi)}{Y(\theta,\phi)} &= -\omega \qquad (2.65)\end{aligned}$$

We multiply (2.64) by($\frac{R(r)}{r^2}$)and (2.65) by ($\sin^2\theta Y(\theta,\phi)$), the two equations become:

$$\left(-\Delta_r-\frac{2\mu}{\hbar^2}\frac{Ze^2}{r}-(\varepsilon-\eta\frac{\mu\varepsilon_0}{\hbar})+\frac{\omega}{r^2}\right)R(r) = 0 \qquad (2.66)$$

$$\left(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{e^2}{\hbar^2c^2}a^4B_0^2-\sin\theta\frac{e}{\hbar c}q_1\varepsilon_0+\frac{\partial^2}{\partial\phi^2}+\sin^2\theta\omega\right)Y(\theta,\phi) = 0 \qquad (2.67)$$

Equation (2.66) is a second degree differential equation, it represents the final form of the radial part of the Schrodinger equation in nonuniform magnetic field.

We can use again the method of variablesvseparation to simplify the angular part (2.67), we set:

$$Y(\theta,\phi)=H(\theta)\Phi(\phi) \qquad (2.68)$$

Equation (2.67) becomes:

$$\left(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{e^2}{\hbar^2c^2}a^4B_0^2-\sin\theta\frac{e}{\hbar c}q_1\varepsilon_0+\frac{\partial^2}{\partial\phi^2}+\sin^2\theta\omega\right)H(\theta)\Phi(\phi) = 0 \qquad (2.69)$$

$$\frac{\left(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{e^2}{\hbar^2c^2}a^4B_0^2-\sin\theta\frac{e}{\hbar c}q_1\varepsilon_0+\sin^2\theta\omega\right)H(\theta)}{H(\theta)}+\frac{\frac{\partial^2}{\partial\phi^2}\Phi(\phi)}{\Phi(\phi)} = 0 \qquad (2.70)$$

$$\begin{aligned}\frac{\left(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{e^2}{\hbar^2c^2}a^4B_0^2-\sin\theta\frac{e}{\hbar c}q_1\varepsilon_0+\sin^2\theta\omega\right)H(\theta)}{H(\theta)} &= \\ -\frac{\frac{\partial^2}{\partial\phi^2}\Phi(\phi)}{\Phi(\phi)} &= m^2 \qquad (2.71)\end{aligned}$$

So we arrive at the following equations:

$$\frac{\left(\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})-\frac{e^2}{\hbar^2c^2}a^4B_0^2-\sin\theta\frac{e}{\hbar c}q_1\varepsilon_0+\sin^2\theta\omega\right)H(\theta)}{H(\theta)} = m^2 \tag{2.72}$$

$$-\frac{\frac{\partial^2}{\partial\phi^2}\Phi(\phi)}{\Phi(\phi)} = m^2 \tag{2.73}$$

where m^2 is constant.
After simplifying the last two equations, we obtain:

$$\frac{\partial^2}{\partial\theta^2}H(\theta)+\cot\theta\frac{\partial}{\partial\theta}H(\theta)+\left[\frac{\left(-\frac{e^2}{\hbar^2c^2}a^4B_0^2-m^2\right)}{\sin^2\theta}-\frac{eq_1\varepsilon_0}{\sin\theta\hbar c}+\omega\right]H(\theta) = 0 \tag{2.74}$$

$$\frac{\partial^2}{\partial\phi^2}\Phi(\phi)+m^2\Phi(\phi) = 0 \tag{2.75}$$

It is easy to integrate the azimuth part (2.75) and the well-known obtained result is:

$$\Phi(\phi)=Ce^{im\phi},(m=0,\pm1,\pm2,...) \tag{2.76}$$

where A is constant. $\left(C=\frac{1}{\sqrt{2\pi}}\right)$.

Equations (2.66) and (2.74) which are both of second degree will be solved by using the Nikiforov-Uvarov method.

2.6 Solutions of the Radial and Angular Equation Using NU Method:

2.6.1 The Radial Part:

To solve equation (2.66), suppose the following change of function:

$$R(r)=\frac{\chi(r)}{r} \tag{2.77}$$

then

$$\frac{dR(r)}{dr}=\frac{1}{r}\frac{d\chi(r)}{dr}-\frac{\chi(r)}{r^2} \tag{2.78}$$

and

$$\frac{dR(r)}{dr^2}=\frac{1}{r}\frac{d^2\chi(r)}{dr^2}-\frac{2}{r^2}\frac{d\chi(r)}{dr}+2\frac{\chi(r)}{r^3} \tag{2.79}$$

From the last two equations, equation (2.66) becomes:

$$\chi"(r) + \left(\frac{2\mu}{\hbar^2} \frac{Ze^2}{r} + (\varepsilon - \eta \frac{\mu\varepsilon_0}{\hbar}) - \frac{\omega}{r^2} \right) = 0 \tag{2.80}$$

we put
$-\epsilon^2 = \varepsilon - \eta \frac{\mu\varepsilon_0}{\hbar}$
$b^2 = -\frac{2\mu}{\hbar^2} Ze^2$
Replacing these expressions in (2.80), allows us to write:

$$\chi"(r) + \frac{1}{r^2} \left(-\epsilon^2 r^2 - b^2 r - \omega \right) = 0 \tag{2.81}$$

It is an equation of the hypergeometric type; so we can apply the NU method.
By comparing the equation (2.81) with (2.1), we get:

$$\widetilde{\tau} = 0, \sigma = r, \widetilde{\sigma} = -\epsilon^2 r^2 - b^2 r - \omega \tag{2.82}$$

2.6.1.1 Energy Spectrum:

The application of the Nikiforov-Uvarov method allowed us to define the polynomial $\pi(r)$from equation (2.22) as follows:

$$\pi(r) = \frac{\sigma' - \widetilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2} \right)^2 - \widetilde{\sigma} + k\sigma} \tag{2.83}$$

$$\pi(r) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \epsilon^2 r^2 + b^2 r + \omega + kr} \tag{2.84}$$

$$\pi(r) = \frac{1}{2} \pm \sqrt{\epsilon^2 r^2 + (b^2 + k) r + \omega + \frac{1}{4}} \tag{2.85}$$

Where the value of k can be found, we impose the condition that the expression under the square root must be a square of polynomial of degree 1 and therefore the expression under the root has zero discriminant.

$$\Delta = \left(b^2 + k \right)^2 - 4\epsilon^2 \left(\omega + \frac{1}{4} \right) \tag{2.86}$$

$$= k^2 + 2b^2 k + b^2 - 4\epsilon^2 \left(\omega + \frac{1}{4} \right) = 0 \tag{2.87}$$

$$\Delta_k = 4b^4 - 4\left(b^2 - 4\epsilon^2\left(\omega + \frac{1}{4}\right)\right) \tag{2.88}$$

$$= 16\epsilon^2\left(\omega + \frac{1}{4}\right) \text{ with } \omega = \ell(\ell+1) \tag{2.89}$$

$$\sqrt{\Delta_k} = 4\sqrt{\epsilon^2}\left(\ell + \frac{1}{2}\right) \tag{2.90}$$

There are two solutions for k

$$\begin{array}{l} k_+ = -b^2 + 2\sqrt{\epsilon^2}\left(\ell + \frac{1}{2}\right) \\ k_- = -b^2 - 2\sqrt{\epsilon^2}\left(\ell + \frac{1}{2}\right) \end{array} \tag{2.91}$$

Then the equation (2.84) becomes:

$$\pi(r) = \left\{ \begin{array}{l} \frac{1}{2} \pm \left[\sqrt{\epsilon^2}r + \left(\ell + \frac{1}{2}\right)\right], k_+ = -b^2 + 2\sqrt{\epsilon^2}\left(\ell + \frac{1}{2}\right) \\ \frac{1}{2} \pm \left[\sqrt{\epsilon^2}r - \left(\ell + \frac{1}{2}\right)\right], k_- = -b^2 - 2\sqrt{\epsilon^2}\left(\ell + \frac{1}{2}\right) \end{array} \right\} \tag{2.92}$$

Where τ is a polynomial at most of degree 1, as defined in (2.11), we find:

$$\tau = \widetilde{\tau} + 2\pi \tag{2.93}$$

And its derivative must be negative, the value of π suitable for our calculations is:

$$\pi = \frac{1}{2} - \sqrt{\epsilon^2}r - \left(\ell + \frac{1}{2}\right), k_- = -b^2 - 2\sqrt{\epsilon^2}\left(\ell + \frac{1}{2}\right) \tag{2.94}$$

$$\tau = 2\left[\ell + 1 - \sqrt{\epsilon^2}r\right] \tag{2.95}$$

To determine the eigenvalues (λ) we use the equation (2.20), we obtain:

$$\lambda = k + \pi' = -b^2 - 2\sqrt{\epsilon^2}\left(\ell + 1\right) \tag{2.96}$$

On the other hand, there is another definition of λ_N in equation (2.28):

$$\lambda_N = -N\left(-2\sqrt{\epsilon^2}\right) = 2N\sqrt{\epsilon^2} \tag{2.97}$$

$$\lambda = \lambda_N \tag{2.98}$$

By comparing the two equations (2.96) and (2.97). The Schrodinger equation's eigenenergy for the non-uniform magnetic field is given by:

$$E_N = -\frac{\mu}{\hbar^2}\frac{Z^2e^4}{2(N+\ell+1)^2} + \eta\varepsilon_0\frac{\hbar}{2} \tag{2.99}$$

Where we replaced $-\epsilon^2$ and ε by its expressions.

2.6.1.2 Wave Functions:

First we look for the function $\phi(r)$, by substituting $\pi(r)$ and $\sigma(r)$ in the expression (2.9) we obtain:

$$\frac{\phi'(r)}{\phi(r)} = \frac{\pi}{\sigma} = -\sqrt{\epsilon^2} + \frac{\ell+1}{r} \tag{2.100}$$

$$\phi(r) = r^{\ell+1}\exp(-\sqrt{\epsilon^2}r) \tag{2.101}$$

And from the expression (2.31), the weight function is written as follows:

$$\frac{\rho'}{\rho} = \frac{\tau-\sigma'}{\sigma} = -\sqrt{\epsilon^2} + \frac{2\ell+1}{r} \tag{2.102}$$

$$\rho = r^{2\ell+1}\exp(-\sqrt{\epsilon^2}r) \tag{2.103}$$

The solutions of polynomials $Y_n(r)$ in equation (2.29) are expressed in terms of the associated Laguerre polynomials [90], which are orthogonal polynomials, so:

$$Y_n(r) = \frac{B_n}{\rho}\frac{d^n}{dr^n}\left[\sigma^n\rho\right] \tag{2.104}$$

$$= B_n\left(r\sqrt{\epsilon^2}\right)^{-(2\ell+1)}\exp(\sqrt{\epsilon^2}r)\frac{d^n}{dr^n}\left[\left(r\sqrt{\epsilon^2}\right)^{n+2\ell+1}\exp(-\sqrt{\epsilon^2}r)\right] \tag{2.105}$$

where B_n is a normalisation term given by

$$B_n = \frac{1}{n!}\frac{1}{\left(\sqrt{\epsilon^2}\right)^{-(2\ell+1)}} \tag{2.106}$$

The solutions of polynomials $Y_n(r)$ in equation (2.104) are expressed in terms of the associated Laguerre polynomials [90], which are orthogonal polynomials, so:

$$Y_n(r) = L_N^{2\ell+1}(\sqrt{\epsilon^2}r) \tag{2.107}$$

By combining the related Laguerre polynomials and $\phi(r)$ in equation (2.4), the radial wave functions are given as:

$$\chi_{N\ell}(r) = \phi(r)Y_n(r) \tag{2.108}$$

Now, we write χ in terms of z; as following :

$$z = \frac{2\mu Ze^2}{\hbar^2(N+\ell+1)}r$$

that is

$$\chi_{N\ell}(z) = C_{N\ell} r^{\ell+1} \exp(-z/2) L_N^{2\ell+1}(z) \tag{2.109}$$

Where $C_{N\ell}$ is the normalization constant, determined by the condition: $\int_0^\infty \chi_{N\ell}^2(r)dr = 1$

$$C_{N\ell} = \left(\frac{\mu Z e^2}{\hbar^2 n'}\right)^{\frac{1}{2}} \left(\frac{(n'-\ell-1)!}{n'\Gamma(n'+\ell+1)}\right)^{\frac{1}{2}} \tag{2.110}$$

Thus, the corresponding normalized wave functions are revealed:

$$\chi_{N\ell}(r) = \left(\frac{\mu Z e^2}{\hbar^2 n'}\right)^{\frac{1}{2}} \left(\frac{(n'-\ell-1)!}{n'\Gamma(n'+\ell+1)}\right)^{\frac{1}{2}} \left(\frac{2\mu Z e^2}{\hbar^2 n'}\right)^{\ell+1} r^{\ell+1} \exp\left(-\frac{\mu Z e^2}{\hbar^2 n'} r\right) L_{n'-\ell-1}^{2\ell+1}\left(\frac{2\mu Z e^2}{\hbar^2 n'} r\right) \tag{2.111}$$

with $n' = N + \ell + 1$

We notice that the wave function satisfies the boundary conditions;

$$\chi(r=0) = \chi(r=\infty) = 0$$

2.6.2 The Angular Part:

Returning now to the angular part of the Schrodinger equation (2.74), we use the same procedure used to solve the radial part in the section (2.6.1). First we introduce the following change of variable: $\cos\theta = x$;

thus

$$H(\theta) \to H(\cos\theta) = H(x) \tag{2.112}$$

and

$$\begin{aligned}
\cos\theta &= x \\
\frac{d}{d\theta} &= \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} \\
1 &= \sin^2\theta + \cos^2\theta \\
&\Rightarrow \sin^2\theta = 1 - x^2 \\
\sin\theta &= (1-x^2)^{\frac{1}{2}}
\end{aligned}$$

so

$$\frac{dH(\theta)}{d\theta} = -\sqrt{1-x^2}\frac{dH(x)}{dx} \tag{2.113}$$

$$\frac{d^2H(\theta)}{d\theta^2} = (1-x^2)\frac{d^2H(x)}{dx^2} - x\frac{dH(x)}{dx} \tag{2.114}$$

We replace (2.113) and (2.114) in the equation (2.74) we obtain:

$$0=\frac{\partial^2}{\partial x^2}H(x)-\frac{2x}{(1-x^2)}\frac{\partial}{\partial x}H(x)+\frac{\Big(\alpha+\beta(1-x^2)^{\frac{1}{2}}+\omega(1-x^2)\Big)}{(1-x^2)^2}H(x) \tag{2.115}$$

where $\begin{array}{c}\alpha=-m^2-\frac{e^2}{\hbar^2c^2}a^4B_0^2\\ \beta=\frac{e}{\hbar c}q_1\varepsilon_0\end{array}$
using the Taylor series or Taylor expansion, get

$$(1-x^2)^{\frac{1}{2}} = 1-\frac{1}{2}x^2-\frac{1}{8}x^4+O(x^5) \tag{2.116}$$

We are satisfied with the second term or the second order

$$(1-x^2)^{\frac{1}{2}} = 1-\frac{1}{2}x^2 \tag{2.117}$$

$$0 = \frac{d^2}{dx^2}H(x)-\frac{2x}{(1-x^2)}\frac{d}{dx}H(x)+\frac{(\alpha+\beta\left(1-\frac{1}{2}x^2\right)+\omega(1-x^2))}{(1-x^2)^2}H(x) \tag{2.118}$$

$$0 = \frac{d^2}{dx^2}H(x)-\frac{2x}{(1-x^2)}\frac{d}{dx}H(x)+\frac{(\alpha+\beta-\beta\frac{1}{2}x^2+\omega-\omega x^2)}{(1-x^2)^2}H(x) \tag{2.119}$$

$$0 = \frac{d^2}{dx^2}H(x)-\frac{2x}{(1-x^2)}\frac{d}{dx}H(x)+\frac{(-\beta\frac{1}{2}-\omega)x^2+(\alpha+\beta+\omega)}{(1-x^2)^2}H(x) \tag{2.120}$$

By comparing eq.(2.118) with the eq. (2.1) and using the unit system ($\hbar=c=2\mu=1$); We obtain the expressions of the following polynomials:

$$\begin{aligned}\widetilde{\tau} &= -2x\\ \sigma &= 1-x^2\\ \widetilde{\sigma} &= (-\omega-\beta\frac{1}{2})x^2+(\alpha+\beta+\omega)\end{aligned}$$

2.6.2.1 Energy Spectrum:

We put the expressions of $\widetilde{\tau}$,σ and $\widetilde{\sigma}$ in the equation (2.22), the function $\pi(x)$ becomes:

$$\pi(x) = \frac{\sigma' - \widetilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + k\sigma} \tag{2.121}$$

$$\pi(x) = \pm\sqrt{-\left((-\omega - \beta\frac{1}{2})x^2 + (\alpha + \beta + \omega)\right) + k(1 - x^2)} \tag{2.122}$$

$$\pi(x) = \pm\sqrt{(\omega + \frac{\beta}{2} - k)x^2 - (\alpha + \beta + \omega) + k} \tag{2.123}$$

According to the method of NU, the expression under the square root must be a square of a first degree polynomial and to determine the constant k we set the discriminant of the expression under the root is zero:

$$\Delta = 4\left[(\omega + \frac{\beta}{2} - k)(\alpha + \beta + -k)\right] \tag{2.124}$$

$$\Delta = 0 = 4k^2 + k\left[-4(\alpha + \beta + \omega) - 4(\omega + \frac{\beta}{2})\right] + 4(\omega + \frac{\beta}{2})(\alpha + \beta + \omega) \tag{2.125}$$

So the constant (k), its dual roots are:

$$\Rightarrow \begin{array}{c} k_+ = \alpha + \beta + \omega \\ k_- = \omega + \frac{\beta}{2} \end{array} \tag{2.126}$$

Substitution $k_\pm$ into equation (2.121); the solutions are obtained for $\pi(x)$ is:

$$\pi(x) = \pm\left\{\begin{array}{c} x\sqrt{-\alpha - \frac{\beta}{2}}, k_+ = \alpha + \beta + \omega \\ \sqrt{-\alpha - \frac{\beta}{2}}, k_- = \omega + \frac{\beta}{2} \end{array}\right\} \tag{2.127}$$

The polynomial τ as defined in (2.11) by: $(\tau = \widetilde{\tau} + 2\pi)$ must be chosen such that its derivative is negative, so:

$$\pi(x) = x\sqrt{-\alpha - \frac{\beta}{2}}, k_+ = \alpha + \beta + \lambda \tag{2.128}$$

Therefore :

$$\Rightarrow \tau = -2x + 2x\sqrt{-\alpha - \frac{\beta}{2}} \tag{2.129}$$

then

$$\tau' = -2 + 2\sqrt{-\alpha - \frac{\beta}{2}} = 2\left(-1 + \sqrt{-\alpha - \frac{\beta}{2}}\right) \tag{2.130}$$

Starting from $\pi'$and $\tau'$the values of λ in (2.20) and (2.28) respectively are given by:

$$\lambda = k+\pi' = \alpha+\beta+\omega+\sqrt{-\alpha-\frac{\beta}{2}} \tag{2.131}$$

$$\lambda_n = -n\left(\tau'+\frac{n-1}{2}\sigma"\right) = n\left(n+1-2\sqrt{-\alpha-\frac{\beta}{2}}\right) \tag{2.132}$$

By assimilating eq. (2.131) with eq. (2.132), we have:

$$\begin{gathered}\lambda=\lambda_n\\ \alpha+\beta+\omega+\sqrt{-\alpha-\frac{\beta}{2}}=n\left(n+1-2\sqrt{-\alpha-\frac{\beta}{2}}\right)\\ \omega_n=-\alpha-\beta-\sqrt{-\alpha-\frac{\beta}{2}}+n\left(n+1-2\sqrt{-\alpha-\frac{\beta}{2}}\right)\\ \omega_n=\ell(\ell+1)\end{gathered} \tag{2.133}$$

And using the fact that $(\omega=\ell(\ell+1))$, we get the value of ℓin the form:

$$\ell = n+\sqrt{-\alpha-\frac{\beta}{2}} \tag{2.134}$$

$$\ell = n+\sqrt{m^2+\frac{e^2}{\hbar^2c^2}a^4B_0^2-\frac{e}{2\hbar c}q_1\varepsilon_0} \tag{2.135}$$

We substitute the expression of ℓ; in the expression of the energy spectrum of the radial part , we finally find the expression of the energy spectrum for the system under studying:

$$E_N = -\frac{\mu}{\hbar^2}\frac{Z^2e^4}{2(N+\ell+1)^2}+\eta\varepsilon_0\frac{\hbar}{2} \tag{2.136}$$

$$E_N = -\frac{\mu}{\hbar^2}\frac{Z^2e^4}{2(N+n+1+\sqrt{m^2+\frac{e^2}{\hbar^2c^2}a^4B_0^2-\frac{e}{2\hbar c}q_1\varepsilon_0})^2}+\eta\varepsilon_0\frac{\hbar}{2} \tag{2.137}$$

We can confirm that the Bohr levels of energy are recovered when the magnetic fields B_0, ε_0 are zero. So if we put $B_0=0$ and $\varepsilon_0 \prec\prec 1$our result collapses to:

$$E_N(B_0=0)=-\frac{\mu}{\hbar^2}\frac{Z^2e^4}{2\left(N+n+m+1\right)^2}$$

2.6.2.2 Wave Function:

We make the following changes $A=x\sqrt{-\alpha-\frac{\beta}{2}}$,and calculate the functions: $\phi(x),\rho(x)$ and $Y_n(x)$from the equations (2.9), (2.31) and (2.29) respectively.

The first part of the wave function is determined by the following relation:

$$\begin{aligned} \frac{\phi'}{\phi} &= \frac{\pi}{\sigma} = \frac{\sqrt{-\alpha - \frac{\beta}{2}}x}{1-x^2} = \frac{Ax}{1-x^2} && (2.138)\\ \int \frac{\phi'}{\phi}dx &= \int \frac{Ax}{1-x^2}dx && (2.139) \end{aligned}$$

then

$$\phi(x) = (1-x^2)^{-\frac{A}{2}} = (1-x)^{-\frac{A}{2}}(1+x)^{-\frac{A}{2}} \tag{2.140}$$

On the other hand, we are looking for a solution for (Y_n) . First we need to obtain the function weight ρ solution of the equation (2.31).

$$\begin{aligned} \frac{\rho'}{\rho} &= \frac{\tau - \sigma'}{\sigma} = \frac{2xA}{1-x^2} && (2.141)\\ \int \frac{\rho'}{\rho}dx &= \int \frac{2xA}{1-x^2}dx && (2.142)\\ \ln\rho &= \ln\left[(1-x^2)^{-A}\right] && (2.143) \end{aligned}$$

then

$$\rho(x) = (1-x^2)^{-A} \tag{2.144}$$

Substituting ρ in the eq. (2.29) allows us to obtain the polynomial Y_n which is written as follows:

$$\begin{aligned} Y_n &= \frac{B_n}{\rho}\frac{d^n}{dx^n}\left[\sigma^n\rho\right] = B_n(1-x^2)^A\frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n\ -A}\right] && (2.145)\\ Y_n &= P_n^{-A}(x) && (2.146) \end{aligned}$$

where B_n is a normalisation term
$P_n^{-A}(x)$ is a Jakobi polynome

$$B_n = \frac{(-1)^n}{2^n n!} \tag{2.147}$$

Using (2.140) and (2.145) in the wave function formula (2.4), we obtain the wave functions of the angular part:

$$H_n(x) = N_n(1-x^2)^{-\frac{A}{2}}P_n^{-A}(x) \tag{2.148}$$

Where the normalization constant N_n has been calculated from condition $\int_{-1}^{1}\left(H(x)\right)^2 dx = 1$ and using the orthogonality relation of Jacobi polynomials [90–92], the normalization constant N_n is:

$$N_n = \sqrt{\frac{(2n-2A+1)\Gamma(n+1)\Gamma(n-2A+1)}{2^{-2A+1}\Gamma(n-A+1)\Gamma(n-A+1)}} \tag{2.149}$$

then

$$H_n(\cos\theta) = \sqrt{\frac{(2n-2A+1)\Gamma(n+1)\Gamma(n-2A+1)}{2^{-2A+1}\Gamma(n-A+1)\Gamma(n-A+1)}}(1-\cos^2\theta)^{-\frac{A}{2}} P_n^{-A}(\cos\theta) \quad (2.150)$$

And therefore, the total wave functions appearing in (2.60) from (2.76);(2.111) and (2.150) is written as:

$$\begin{aligned}
\Psi(r,\theta,\varphi) &= \frac{1}{r}\chi(r)H(\cos\theta)F(\phi) \\
\Psi(r,\theta,\varphi) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(2n-2A+1)\Gamma(n+1)\Gamma(n-2A+1)}{2^{-2A+1}\Gamma(n-A+1)\Gamma(n-A+1)}}\left(\frac{\mu Ze^2}{\hbar^2 n'}\right)^{\frac{1}{2}}\left(\frac{(n'-\ell-1)!}{n'\Gamma\left(n'+\ell+1\right)}\right)^{\frac{1}{2}} \\
&\quad \times\left(\frac{2\mu Ze^2}{\hbar^2 n'}\right)^{\ell+1} r^{\ell}\exp\left(-\frac{\mu Ze^2}{\hbar^2 n'}r\right)L_{n'-\ell-1}^{2\ell+1}\left(\frac{2\mu Ze^2}{\hbar^2 n'}r\right) \\
&\quad \times(1-\cos^2\theta)^{-\frac{A}{2}}P_n^{-A}(\cos\theta)e^{im\phi} \qquad (2.151)
\end{aligned}$$

$$\begin{aligned}
m &= 0, \pm 1, \pm 2, ... \\
n &= 0, 1, 2, ... \\
N &= 0, 1, 2, ... \qquad (2.152)
\end{aligned}$$

Here, (N, n,m) Take on the role of quantum numbers. (n, l,m). We now have the system's eigenenergy, which we will use to analyze the fusion plasma's spectral radiative characteristics in the following section.

2.7 Results and Discussion

Before presenting the discussion results, two points need to be mentioned: The first relates to the powerful magnetic field that can be found in some stars and a white dwarf. The second point is that we considered, to make Figures (Figures 2.1-2.3) readable, that the Stark broadening is overestimated in one part and the Doppler broadening is Lorentzian in another. Only in Figure (Figure 2.4) have we considered, as we must, a convolution of the Doppler (gaussian) and the Stark (Lorentzian) broadenings. The values' choice of B_0 and ε_0 is arbitrary, but we have enlarged them so they hide the fine structural effect. and keeping B_0 substantially bigger than ε_0 (B_0/ε_0 =3) because the toroidal B_0 field is larger than the poloidal field ε_0 [84].

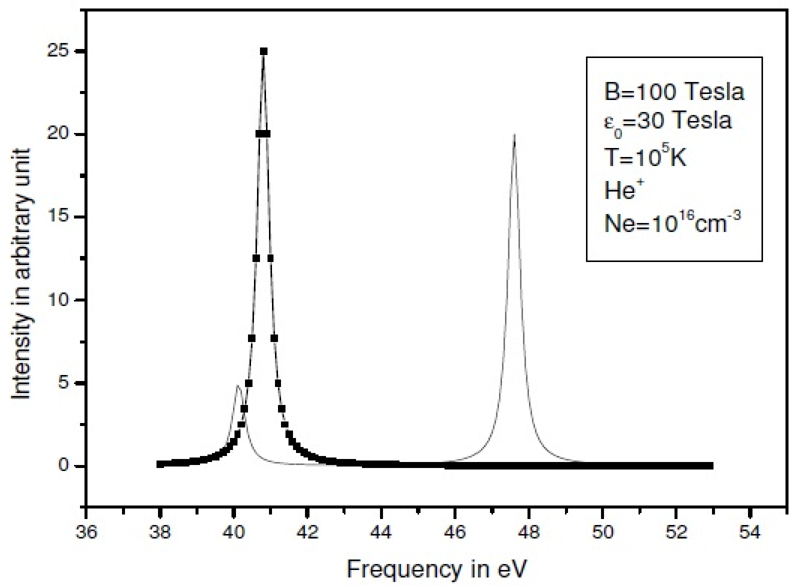

Fig. 2.1: Ly-alpha line for Hydrogen-like Helium for $Ne = 10^{16}\mathrm{cm}^{-3}$, $T = 10^5\mathrm{K}$, $B_0 = 100\mathrm{Tesla}$, and $\varepsilon_0 = 30\mathrm{Tesla}$. The line + symbol is for $B = 0\mathrm{Tesla}$ [84].

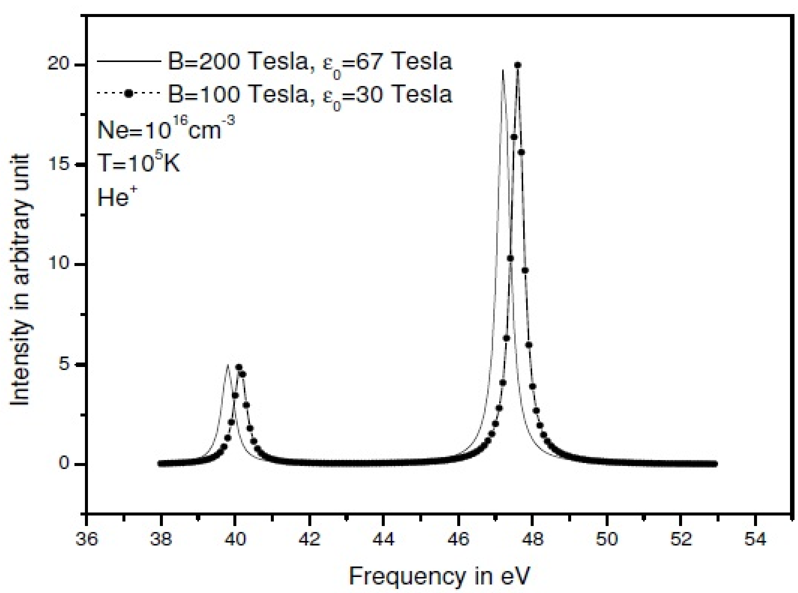

Fig. 2.2: Ly-alpha line for Hydrogen-like Helium for $Ne = 10^{16}\mathrm{cm}^{-3}$, $T = 10^5\mathrm{K}$, $B_0 = 100\mathrm{Tesla}$, and $\varepsilon_0 = 30\mathrm{Tesla}$ and for $B_0 = 200\mathrm{Tesla}$ and $\varepsilon_0 = 67\mathrm{Tesla}$ [84]

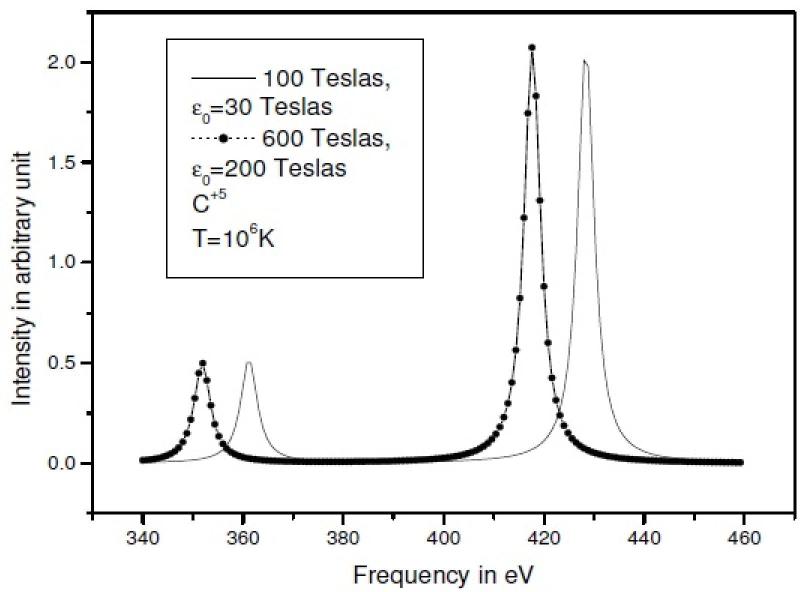

Fig. 2.3: Ly-alpha line for Hydrogen-like Carbon for $Ne = 10^{16}\mathrm{cm}^{-3}$, $T = 10^6\mathrm{K}$, $B_0 = 100\mathrm{Tesla}$, and $\varepsilon_0 = 30\mathrm{Tesla}$ and for $B_0 = 600\mathrm{Tesla}$ and $\varepsilon_0 = 200\mathrm{Tesla}$ [84]

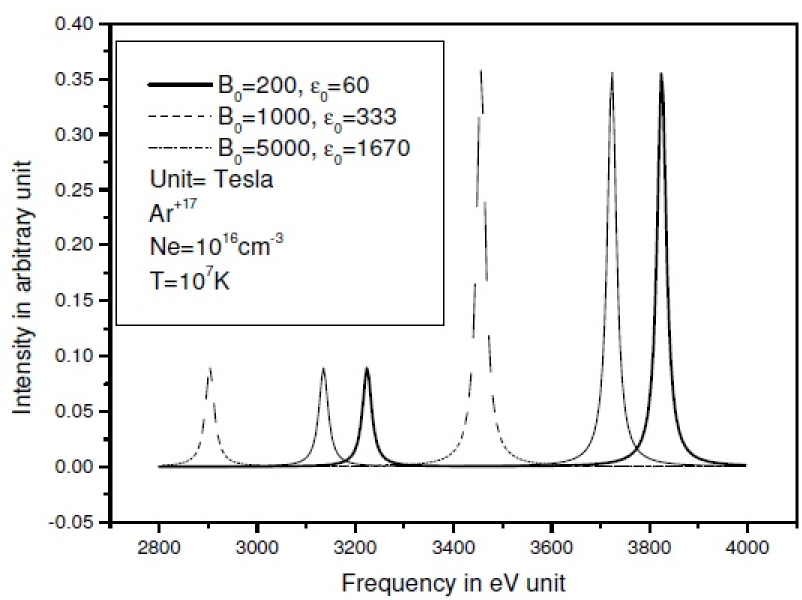

Fig. 2.4: Ly-alpha line for Hydrogen-like Argon for $Ne = 10^{16}\mathrm{cm}^{-3}$, $T = 10^7\mathrm{K}$, $B_0 = 200\mathrm{Tesla}$, and $\varepsilon_0 = 68\mathrm{Tesla}$, $B_0 = 1000\mathrm{Tesla}$, $\varepsilon_0 = 333\mathrm{Tesla}$, and for $B_0 = 5000\mathrm{Tesla}$ and $\varepsilon_0 = 1670\mathrm{Tesla}$ [84]

In Fig2.1 and Fig2.2 (Fig2.1 includes the line without magnetic field, B = 0), we present the Ly-alpha line for Hydrogen-like Helium for $Ne = 10^{16}\mathrm{cm}^{-3}$, $T = 10^{5}\mathrm{K}$, $B_0 = 100\mathrm{Tesla}$, and $\varepsilon_0 = 30\mathrm{Tesla}$ and for $B_0 = 200\mathrm{Tesla}$ and $\varepsilon_0 = 67\mathrm{Tesla}$. The magnetic field's presence improves the elimination of degeneracy and gives the profile two peaks; enhancing the field's value ($B_0 = 200\mathrm{Tesla}$ and $\varepsilon_0 = 67\mathrm{Tesla}$), we can notice that The two peaks' centers are shifting toward the low frequencies (toward the red). The shift can be interpreted as follows: from Equation (2.136), It is obvious that when the magnetic field's strength increases, the effect it has on energy levels causes these levels to move apart from each other [84].

As a result, as the magnetic field grows, the line's center, which represents the energy differential between both levels, moves. The line's center then moves in a different direction. In Fig2.3, we have the Ly-alpha line for Hydrogen-like Carbon for $Ne = 10^{16}\mathrm{cm}^{-3}$, $T = 10^{6}\mathrm{K}$, $B_0 = 100\mathrm{Tesla}$, and $\varepsilon_0 = 30\mathrm{Tesla}$ and $B_0 = 600\mathrm{Tesla}$ and $\varepsilon_0 = 200\mathrm{Tesla}$, we notice the same as the previous remarks, but the shift increases between the two peaks of heline [84]. For the Ly-alpha line for Hydrogen-like Argon for $Ne = 10^{16}\mathrm{cm}^{-3}$,$T = 10^{7}\mathrm{K}$, $B_0 = 200\mathrm{Tesla}$ and $\varepsilon_0 = 67\mathrm{Tesla}$, $B_0 = 1000\mathrm{Tesla}$ and $\varepsilon_0 = 333\mathrm{Tesla}$, and $B_0 = 5000\mathrm{Tesla}$ and $\varepsilon_0 = 1670\mathrm{Tesla}$, in this case, we see a clear shift towards the red, going from $1000\mathrm{Tesla}$ to $5000\mathrm{Tesla}$; the second peak has a shift about $267\mathrm{eV}$ (2.4). In all figures, we used a convolution product to introduce the Doppler broadening, and the final profile will be a convolution of the Gaussian and Zeeman effects rather than a pure Gaussian profile [84].

2.8 Conclusion

Finally, we have solved the quantum problem of hydrogen-like ions in a nonuniform and intense magnetic field. The most intense field has a toroidal geometry, while the less intense field (approximately the third) is poloidal. It is considered that the intense magnetic field is the sum of these two magnetic fields.

In this chapter, we have given the exact solutions of the radial and angular parts of the Schrodinger equation for a non-uniform magnetic field. The eigenenergies and the corresponding wave functions are obtained using the Nikiforov-Uvarov method,which is a very interesting mathematical model that shows successes when it is used to deal analytically with a lot of problems in theoretical physics, especially in the last few years. We have given a detailed presentation of it. The wave functions are expressed in terms of the associated Laguerre and Jacobi polynomials for the radial and angular parts, respectively. We see that the method of (NU) is an applicable tool to give the exact solutions for non-uniform magnetic field problems.

We were able to investigate the spectral line shape (Lyman-alpha) of three different types of ions using the results obtained: He^{+} , C^{5+}, and Ar^{17+} for different magnetic field magnitudes.We remark that, the Zeeman separation is as large as the magnetic field is large, and the charge number Z is great. Another feature must be noticed: the left Zeeman component is near the line without a magnetic field (see Figure 2.1).

Chapter 3

Path Integrals Approach to Pressure Broadening in Plasma

3.1 Introduction

Richard Feynman was introduced path integrals to physics in his thesis, defended in May 1942, on the formulation of quantum mechanics based on the Lagrangian. Due to the Second World War, these results would not be published until 1948. This mathematical tool quickly imposed itself in theoretical physics with its generalization to quantum field theory, allowing in particular a quantification of non-abelian gauge theories that is simpler than the canonical quantization procedure [93].

In 1948, Richard Feynman initiated a real revolution by proposing a new formulation of quantum mechanics. The starting idea, due to Dirac, is the search for a formulation of quantum mechanics directly in terms of the Lagrangian. The Hamiltonian formulation is indeed not completely satisfactory for at least two reasons:

- It supposes that the existence of a conjugate moment $p_i = \frac{\partial L}{\partial \dot{q}}$ for each variable, which is not the most general case (the difficulties of this method for the electromagnetic field).

- It is not obviously covariant by a Lorentz transformation. To better understand the motivations of Dirac and Feynman, it is useful to briefly review the different formulations of quantum mechanics [94].

The Path Integral is a functional integral, that is, the integrated is a functional and the sum is taken over functions, not real (or complex) numbers as with ordinary integrals. We are therefore dealing here with an integral in infinite dimension. Thus, we will carefully distinguish the path integral (functional integral) from an ordinary integral calculated on a path in physical space, which the mathematicians call it curvilinear integral.

The path integral is a mathematical object which can be considered as a generalization to an infinite number of variables, represented by paths and ordinary integrals. It shares the algebraic properties of ordinary integrals, but it presents new properties from the analysis's point of view. The path integral is also a powerful tool for the quantum mechanics study, because it puts in correspondence in a very explicit way between the two theories: classical and quantum mechanics. Physical quantities are obtained by takeing the average over all possible paths, but in the semi-classical limit $\hbar \to 0$, the paths dominating the integral lie in a neighborhood of the classical path.

Thus, the path integral allows an intuitive understanding and a simple calculation of semi-classical effects from both the diffusionist's perspective and spectral properties or the tunnel effect. Moreover, the formulation of quantum mechanics based on the path integral, if it may seem more complicated from a mathematical perspective, since it replaces a formalism of partial differential equations, is well suited to the study of systems with one large number of degrees of freedom or a formalism of the Schrodinger equation type is less useful.

3.1.1 Quantum Theory (Bohr-Sommerfeld, 1915)

The first theory of the energy levels's quantization in atoms due to Bohr and Sommerfeld (1915) is based in a very essential way on the Hamiltonian formulation of classical mechanics. The Steps are as follows [95]:

- Determine the Lagrangian $L\left(q_1,...q_N,\dot{q_1},...\dot{q_N}\right)$ (assumed that it is time independent).
- Deduce the Hamiltonian by a Legendre transformation [95]:

$$H\left(q_1,...,q_N,p_1,...,p_N\right)=\sum_i p_i\dot{q_i}-L\left(q_1,...q_N,\dot{q_1},...\dot{q_N}\right) \tag{3.1}$$

where $\dot{q}_i$ are functions of $\left(q_1,...,q_N,p_1,...,p_N\right)$ given by $p_i=\frac{\partial L}{\dot{q_i}}$.
- Solve the Hamilton-Jacobi equation:

$$H\left(q_1,...,q_N,\frac{\partial W}{\partial q_1},...,\frac{\partial W}{\partial q_N}\right)=E \tag{3.2}$$

and deduce therefrom a canonical transformation towards the action-angle variables I_i,ω_i as:

$$\left\{\begin{array}{l}\frac{dI_i}{dt}=0\\ \frac{d\omega_i}{dt}=\omega_i\end{array}\right\} \tag{3.3}$$

The quantization rules are then given by: $I_i=\hbar n_i$ with $n_i=0,1,2,...$. For a system with one degree of freedom, the expression of I is given by [95]:

$$I=\frac{1}{2\pi}\oint pdq=n\hbar \tag{3.4}$$

where $\hbar=\frac{h}{2\pi}$,((h is Planck's constant) = action quantum).

This theory, considered by Sommerfelde as "the royal road to quantification", in fact quickly proved insufficient. Its major limitation is to rely on a complete solution of the Hamilton-Jacobi equation. This assumes that the system is separable. However, separable systems are the exception: the vast majority of mechanical systems are not separable, and it has proven impossible to generalize this formulation without introducing new concepts [95].

3.1.2 Matrix Theory (Heisenberg, 1926)

In 1926, Heisenberg demonstrated that the same quantization rules could be obtained for a separable system if the energies were defined as the eigenvalues of the Hamiltonian operator H [96]:

$$H = \frac{p^2}{2m} + V(x) \tag{3.5}$$

for one particle.
where x and p must be considered as operators satisfying the commutation rules [96]:

$$[x, p] = i\hbar \tag{3.6}$$

But this formulation, which can be extended automatically to any system comprising several particles by introducing the Hamiltonian [96].

$$H = \sum_i \frac{p_i^2}{2m} + V(x_1, ..., x_N) \tag{3.7}$$

and the commutation rules [96]:

$$[x_i, x_j] = 0, [p_i, p_j] = 0, [x_i, p_j] = i\hbar\delta ij \tag{3.8}$$

does not require the system to be integrable. Heisenberg therefore proposed it as a general theory of quatification. This is indeed the first theory of quantum mechanics. The idea of introducing non-commuting operators was based on the properties of matrices, and Heisenberg used this representation to determine the Hamiltonian's eigenvalues.

This way of quantifying motion, known as "canonical quantification", is still essentially based on the Hamiltonian formulation of the corresponding classical mechanics problem. In particular, the temporal evolution of operators is governed by the equation [96]:

$$i\hbar\frac{d\widehat{O}}{dt} = \left[\widehat{O}, H\right] \tag{3.9}$$

3.1.3 Wave Mechanics (Schrodinger, 1927)

Independent of Heisenberg, and based on the wave properties of electrons revealed in diffraction experiments, Schrodinger formulated another approach based on the concept of wave function [94].

In this approach, the state of a system is identified by a wave function whose temporal evolution is governed by the equation [94]:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) \tag{3.10}$$

(for a non-relativistic particle).

The energy levels of an atom correspond to stationary states

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = E\Psi(x,t) \tag{3.11}$$

In this formulation, the Hamiltonian does not appear explicitly. However, it is possible to make the link with Heisenberg's formulation by noting that the equation can be rewritten [94]:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t) \tag{3.12}$$

with

$$H = \frac{\widehat{p}^2}{2m} + V(\widehat{x}), \text{ and } p = -i\hbar\frac{\partial}{\partial x} \tag{3.13}$$

Indeed, the operators $\widehat{x}$ and $\widehat{p}$ defined by

$$\widehat{x} : \phi \to x\phi, \text{ and } \widehat{p} : \phi \to -i\hbar\frac{\partial\phi}{\partial x} \tag{3.14}$$

satisfy $[\widehat{x}, \widehat{p}] = i\hbar$.

The link between the two representations in terms of bases of Hilbert space was established by Dirac.

3.1.4 Path Integral (Feynman, 1948)

The time evolution of the wave function can be described using the evolution operator [93]:

$$\Psi(t) = U(t, t_0)\Psi(t_0) \tag{3.15}$$

where$U(t;t_0)$ is an operator that satisfies the equation:

$$i\hbar\frac{\partial U}{\partial t} = HU \tag{3.16}$$

The evolution operator can also be defined by its matrix elements in a Hilbert space basis of wave functions. Two bases are particularly useful: $|x\rangle$ and $|p\rangle$ [94].

- $|p\rangle$ are defined as the eigenvectors of the operator $\widehat{p}$ with eigenvalues p as:

$$p = -i\hbar\frac{\partial}{\partial x} \tag{3.17}$$

The wave function $\phi_p(x)$ associated with p thus satisfies [93]:

$$-i\hbar\frac{\partial}{\partial x}\phi_p = p\phi_p \tag{3.18}$$

This equation is obviously satisfied by $\exp(ipx\hbar)$: The only subtlety lies in the normalization. Since $\exp(ipx/\hbar)$ is not square summable, it cannot be normalized.

1- The normalization convention is fixed by the closure relation [94]:

$$\int dp|p\rangle\langle p| = 1 \tag{3.19}$$

Let's pose

$$|p\rangle = |\frac{1}{N}\exp\left(ipx/\hbar\right)\rangle \tag{3.20}$$

$$\begin{aligned} \int dp|p\rangle\langle p|\phi(x)\rangle &= \frac{1}{N^2}\int dp \int dx' \exp\left(ipx/\hbar\right)\exp\left(-ipx'/\hbar\right)\phi(x') \\ &= \frac{1}{N^2}\int dp \exp\left(ipx/\hbar\right)\int dx' \exp\left(-ipx'/\hbar\right)\phi(x') \end{aligned} \tag{3.21}$$

Now, the Fourier transform of $\phi(x)$ is defined by [94]:

$$\phi'(k) = \frac{1}{\sqrt{2\pi}}\int dx \exp(-ikx)\phi(x) \tag{3.22}$$

and the inverse transformation is given by [94]:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int dk \exp(ikx)\phi'(k) \tag{3.23}$$

$$\begin{aligned} \int dp|p\rangle\langle p|\phi(x)\rangle &= \frac{1}{N^2}\int dp \exp\left(ipx/\hbar\right)\sqrt{2\pi}\phi'(\frac{p}{\hbar}), (k=\frac{p}{\hbar}) \\ &= \frac{\hbar\sqrt{2\pi}}{N^2}\int dk \exp(ikx)\phi'(k) \\ &= \frac{2\pi\hbar}{N^2}\phi(x) \\ \Longrightarrow N = \sqrt{2\pi\hbar} \end{aligned} \tag{3.24}$$

- $|x\rangle$ are the eigenvectors of $\widehat{x}$. Now, by definition of the operator $\widehat{x}$, we have:

$$\widehat{x}\phi(x) = x\phi(x) \tag{3.25}$$

An eigenfunction ϕ_{x_0} of operator $\widehat{x}$ with eigenvalue x_0 also satisfies:

$$\widehat{x}\phi_{x_0}(x) = x_0\phi_{x_0}(x) \tag{3.26}$$

must therefore satisfy the equation: ϕ_{x_0}

$$x\phi_{x_0}(x) = x_0\phi_{x_0}(x) \tag{3.27}$$

for all x.

We deduce that $\phi_{x_0}(x) = 0$ if $x \neq x_0$ and that $\phi_{x_0}(x)$ can take any value. But for this function not to have a zero integral, it must be proportional to the δ Dirac function $\delta(x - x_0)$ wich defined by:

$$\int \delta(x - x_0) f(x) dx = f(x_0) \tag{3.28}$$

for any function f.

For normalization, we will still impose a closure relation. However, the choice:

$$|x_0\rangle = |\delta(x - x_0)\rangle \tag{3.29}$$

leads to the closure relation

$$\int |x\rangle\langle x| = 1 \tag{3.30}$$

In effect,$\langle\phi(x)|x\rangle = \phi^*(x)$ and $\langle x|\phi(x)\rangle = \phi(x)$ thus

$$\langle\phi(x)|x\rangle\langle x|\phi'(x)\rangle = \phi^*(x)\phi'(x) \tag{3.31}$$

$$\Longrightarrow \int dx \langle\phi(x)|x\rangle\langle x|\phi'(x)\rangle = \int dx \phi^*(x)\phi'(x) = \langle\phi|\phi'\rangle \tag{3.32}$$

Moreover, the Fourier transform of the function $\delta(x - x_0)$ is a plane wave:

$$\int \delta(x - x_0) \exp(-ipx/\hbar) dx = \exp(-ipx_0/\hbar) \tag{3.33}$$

The inverse transformation leads to the following relation:

$$\int \exp(ipx/\hbar) \exp(-ipx_0/\hbar) dp = \frac{1}{2\pi\hbar} \delta(x - x_0) \tag{3.34}$$

The propagator (or Green's function) is defined by [93]:

$$G(x, t, x_0, t_0) = \langle x|U(t, t_0)|x_0\rangle \tag{3.35}$$

If we know the propagator for all x and all x_0; we know the evolution operator, i.e. the solution of the Schrodinger equation.

Feynman's goal was to relate the propagator to the Lagrangian of the corresponding classical system. Now, The motion equation's solution for the evolution operator is written as [94]:

$$U(t, t_0) = \exp\left(-\frac{i(t - t_0)}{\hbar} H\right) \tag{3.36}$$

an expression that involves the Hamiltonian and not the Lagrangian.

The basic idea comes from the following remark: The Hamiltonian H is the sum of two terms: $H = T + V$

- $V(\widehat{x})$ has eigenstates $|x\rangle$: $V(\widehat{x})|x\rangle = V(x)|x\rangle$

- $T = \frac{\widehat{p}^2}{2m}$ has eigenstates $|p\rangle : T|p\rangle = \frac{p^2}{2m}|p\rangle$

On the other hand, an average value of type $\langle x|e^{-\lambda T}e^{-\lambda V}|x_0\rangle$ is easy to calculate. It is enough to insert the relation of closure $1 = \int dp|p\rangle\langle p|$ (we put $\lambda = \frac{i(t-t_0)}{\hbar}$)

$$\begin{aligned} \Longrightarrow \quad \langle x|e^{-\lambda T}e^{-\lambda V}|x_0\rangle &= \int dp\langle x|e^{-\lambda T}|p\rangle\langle p|e^{-\lambda V}|x_0\rangle \\ = \quad & \int dp\langle x|p\rangle e^{-\lambda\frac{p^2}{2m}}\langle p|x_0\rangle e^{-\lambda V(x_0)} \end{aligned} \tag{3.37}$$

Unfortunately, since T and V do not commute,

$$e^{-\lambda(T+V)} \neq e^{-\lambda T}e^{-\lambda V} \tag{3.38}$$

How to do?

Feynman's idea is to use what is called Trotter's formula which states that:

$$\langle x|e^{-\lambda(T+V)}|x_0\rangle = \lim_{N\to+\infty}\langle x|e^{-\lambda(T/N)}e^{-\lambda(V/N)}...e^{-\lambda(T/N)}e^{-\lambda(V/N)}|x_0\rangle \tag{3.39}$$

The demonstration proceeds in 3 steps:

$$\begin{aligned} e^{-\lambda(T+V)} &= \left(e^{-\frac{\lambda(T+V)}{N}}\right)^N \\ e^{-\frac{\lambda(T+V)}{N}} &= e^{-\frac{\lambda T}{N}}e^{-\frac{\lambda V}{N}} + 0\left(\frac{1}{N^2}\right) \end{aligned} \tag{3.40}$$

in effect

$$\begin{aligned} e^{\varepsilon(A+B)} &= I + \varepsilon\left(A+B\right) + \frac{\varepsilon^2}{2}\left(A+B\right)^2 + ... \\ e^{\varepsilon A}e^{\varepsilon B} &= \left(I + \varepsilon A + \frac{\varepsilon^2}{2}A^2 + ...\right)\left(I + \varepsilon B + \frac{\varepsilon^2}{2}B^2 + ...\right) \\ &= I + \varepsilon A + \varepsilon B + 0\left(\varepsilon^2\right) \end{aligned} \tag{3.41}$$

then

$$e^{\varepsilon(A+B)} = e^{\varepsilon A}e^{\varepsilon B} + 0\left(\varepsilon^2\right) \tag{3.42}$$

$$\left(e^{-\frac{\lambda T}{N}}e^{-\frac{\lambda V}{N}}\right)^N - \left(e^{-\frac{\lambda(T+V)}{N}}\right)^N = 0\left(\frac{1}{N}\right) \tag{3.43}$$

For this, we first notice that the identity:

$$x^N - y^N = (x-y)\left(x^{N-1} + x^{N-2}y + ... + y^{N-1}\right) \tag{3.44}$$

for $x, y \in \mathbb{R}$ must be rearranged for two operators that do not commute:

$$x^N - y^N = x^{N-1}\left(x-y\right) + x^{N-2}\left(x-y\right)y + ... + \left(x-y\right)y^{N-1} \tag{3.45}$$

In the right-hand side of this equality, all the terms are of the form $x^p y^q$; and they cancel 2 by 2, as in the usual identity, except x^N and y^N.

therefore, $\left(e^{-\frac{\lambda T}{N}}e^{-\frac{\lambda V}{N}}\right)^N - \left(e^{-\frac{\lambda(T+V)}{N}}\right)^N$ is the sum of N terms that all contain $e^{-\frac{\lambda T}{N}}e^{-\frac{\lambda V}{N}} - e^{-\frac{\lambda(T+V)}{N}}$.: Since this factor is $0\left(\frac{1}{N^2}\right)$; the sum is of order $\left(\frac{1}{N}\right)$.

Let's go back to the expression of the propagator:

$$G(x,t;x_0,t_0) = \lim_{N\to+\infty}\langle x|e^{-\lambda T/N}e^{-\lambda V/N}...e^{-\lambda T/N}e^{-\lambda V/N}|x_0\rangle \tag{3.46}$$

and insert $\int dp|p\rangle\langle p|$-type closure relations between $e^{-\lambda T/N}$ and$e^{-\lambda V/N}$; and $\int |x\rangle\langle x|$ type between $e^{-\lambda V/N}$ and $e^{-\lambda T/N}$: It comes:

$$\begin{aligned}
G(x,t;x_0,t_0) \quad &= \lim_{N\to+\infty}\int dp_N e^{\frac{-\lambda p^2{}_N}{2mN}}\langle x|p_N\rangle\int dx_{N-1}e^{\frac{-\lambda(Vx_{N-1})}{N}}\langle p_N|x_{N-1}\rangle\times\\
&\times \int dp_{N-1}e^{\frac{-\lambda p^2{}_{N-1}}{2mN}}\langle x_{N-1}|p_{N-1}\rangle\int dx_{N-2}e^{\frac{-\lambda V(x_{N-2})}{N}}\langle p_{N-1}|x_{N-2}\rangle\times\\
&\times \quad ...\int dp_1 e^{\frac{-\lambda p^2{}_1}{2mN}}\langle x_1|p_1\rangle e^{\frac{-\lambda V(x_0)}{N}}\langle p_1|x_0\rangle
\end{aligned}\tag{3.47}$$

The expression therefore becomes the integral of the product of factors of the form:

$$\int dp\langle x|e^{-\lambda T/N}|p\rangle\langle p|e^{-\lambda V/N}|y\rangle = \int dpe^{\frac{-\lambda p^2}{2mN}}\langle x|p\rangle e^{\frac{-\lambda V(y)}{N}}\langle p|y\rangle \tag{3.48}$$

but $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$ the integral becomes:

$$\frac{e^{\frac{-\lambda V(y)}{N}}}{2\pi\hbar}\int dpe^{\frac{-\lambda p^2}{2mN}}exp\left(\frac{i}{\hbar}p(x-y)\right) \tag{3.49}$$

It is a Gaussian integral.
Now

$$\begin{aligned}
z(j) \quad &= \int dxe^{-\frac{1}{2}Ax^2+jx} = \sqrt{\frac{2\pi}{A}}e^{\frac{j^2}{2A}}\\
A \quad &\Leftrightarrow \frac{\lambda}{mN},\quad j \Longleftrightarrow \frac{i}{\hbar}(x-y)\\
&\Rightarrow \frac{e^{-\frac{\lambda V(y)}{N}}}{2\pi\hbar}\int dpe^{-\frac{\lambda p^2}{2mN}}e^{\frac{i}{\hbar}(x-y)} = \frac{e^{-\frac{\lambda V(y)}{N}}}{2\pi\hbar}\sqrt{\frac{2\pi mN}{\lambda}}e^{\frac{(x-y)^2mN}{2\lambda\hbar^2}}
\end{aligned}\tag{3.50}$$

This expression remains valid if λ is pure imaginary on condition that $\sqrt{i}$ is interpreted as $exp(i\pi/4)$: We deduce the following expression for G:

$$\begin{aligned} G(x,t;x_0,t_0) &= \lim_{N\to+\infty}\int dx_1...dx_{N-1}\left(\frac{mN}{2\pi\lambda\hbar^2}\right)^{\frac{N}{2}}\times \\ &\times \prod_{j=0}^{N-1}\exp\left[\frac{-m(x_{j+1}-x_j)^2N}{2\lambda\hbar^2}-\frac{\lambda V(x_j)}{N}\right] \end{aligned} \tag{3.51}$$

with $x_N \equiv x$
Putting

$$\begin{aligned} \varepsilon &= \frac{(t-t_0)}{N}=\frac{\lambda\hbar}{iN}\Rightarrow\lambda=\frac{\varepsilon iN}{\hbar} \\ \frac{-N}{\lambda\hbar^2} &= \frac{i}{\hbar\varepsilon} \\ \frac{-\lambda}{N} &= -\frac{i\varepsilon}{\hbar} \end{aligned} \tag{3.52}$$

The propagator can therefore be rewritten:

$$\begin{aligned} G(x,t;x_0,t_0) &= \lim_{N\to+\infty}\int dx_1...dx_{N-1}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{\frac{N}{2}}\times \\ &\times\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=0}^{N-1}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2-V(x_j)\right]\right\} \end{aligned} \tag{3.53}$$

the expression $\varepsilon\sum_{j=0}^{N-1}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2-V(x_j)\right]$ called a Riemann sum. Indeed, $\varepsilon = \frac{(t-t_0)}{N}$: Consider a differentiable function $x(s)$ defined between t_0 and t; and express the integral

$$\int_{t_0}^{t} ds\left[\frac{m}{2}\left(\frac{dx}{ds}\right)^2-V(x(s))\right] \tag{3.54}$$

in the form of a Riemann sum:

$$\int_{t_0}^{t} ds\left[\frac{m}{2}\dot{x}^2-V(x)\right]=\lim_{N\to+\infty}\frac{t-t_0}{N}\sum_{j=0}^{N-1}\left[\frac{m}{2}\dot{x}_j^2-V(x_j)\right] \tag{3.55}$$

but

$$\begin{aligned} \dot{x}_j &\approx \frac{x_{j+1}-x_j}{(t-t_0)/N} \\ &\Rightarrow \int_{t_0}^{t} ds\left[\frac{m}{2}\dot{x}^2-V(x)\right]= \\ &\lim_{N\to+\infty}\left(\frac{t-t_0}{N}\right)\sum_{j=0}^{N-1}\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{(t-t_0)/N}\right)^2-V(x_j)\right] \end{aligned} \tag{3.56}$$

Now, $\frac{m}{2}\dot{x}^2 - V(x)$is merely a Lagrangian, and $\int_{t_0}^{t} dsL(x, \dot{x}, s)$ is the action along the trajectory $x(s)$:

This suggests writing the propagator in the form:

$$G(x, t; x_0, t_0) = \int Dxe^{\frac{i}{\hbar}S[x]} \tag{3.57}$$

where the symbol $\int Dx$ designates the sum over all possible paths, modulo a measure which remains to be defined.The propagator thus appears as the sum of possible $e^{\frac{i}{\hbar}S[x]}$ of the system (see Fig.3.1) . This object is called a functional integral.

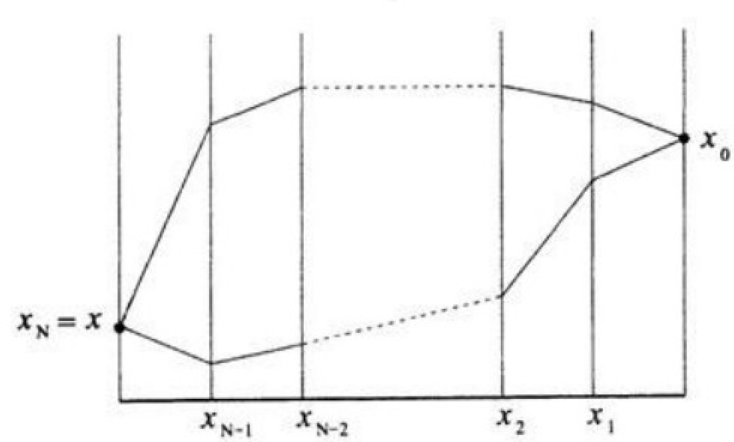

Fig. 3.1: Trajectories' Examples which intervene in the sum on the paths [97].

This writing is very suggestive, but it is purely formal, and to give it a meaning, in other words to define the measure $\int Dx$, it is imperative to return to the explicit form. This is all the more true since the typical path is indistinguishable: since x_j and x_{j+1} take all the values between $-\infty$ and $+\infty$, $\frac{x_{j+1}-x_j}{\varepsilon}$, does not tend to a constant but diverges when $N \to +\infty$.

This being said, this formulation answers the question that Feynman (and Dirac before him) asked himself about the possibility of formulating quantum mechanics using the Lagrangian. With the definition of the functional integral given by the correspondence between the two previous expressions of the propagator, quantum mechanics can be based on this definition of the propagator. In other words, we can do the Economy of canonical quantification and take this expression of the propagator to quantify the movement of a particle in a potential [93].

The essential difficulty of this method is that it does not internally provide the definition of the functional integral. It is necessary for example to add a prescription to obtain an expression which restores the Schrodinger equation in the presence of a magnetic field. Similarly, in quantum field theory, the meaning that must be given to the integral refers to canonical quantification [94].

Note that the generalization to N degrees of freedom is without problem: the intermediate integrals are volume integrals in N dimensions: $dx_i \to d^N xi_.$

ex: free particle ($V = 0$)

In this case, the propagator is easily calculated:

$$\begin{aligned} \langle x|e^{-\lambda T}|x_0\rangle &= \int dp \langle x|e^{-\lambda T}|p\rangle\langle p|x_0\rangle = \int dp e^{-\frac{\lambda p^2}{2m}} \frac{e^{ipx/\hbar}e^{-ipx_0/\hbar}}{2\pi\hbar} \\ &= \frac{1}{2\pi\hbar}\int dp e^{-\frac{\lambda p^2}{2m}} e^{ip(x-x_0)/\hbar} \\ &= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m}{\lambda}} exp\left[-\frac{(x-x_0)^2 m}{\hbar^2 2\lambda}\right] \end{aligned} \tag{3.58}$$

$$\left(A = \frac{\lambda}{m}, j = \frac{i(x-x_0)}{\hbar}\right)$$

whether

$$\langle x|e^{-\frac{i(t-t_0)}{\hbar}\frac{p^2}{2m}}|x_0\rangle = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} exp\left[\frac{im}{2\hbar(t-t_0)}(x-x_0)^2\right] \tag{3.59}$$

Now, for a free particle, the action is easily calculated. The classical trajectory is given by:

$$\begin{aligned} x(\tau) &= x_0 + \frac{\tau - t_0}{t - t_0}(x - x_0) \\ &\Rightarrow S_{cl} = \int_{t_0}^{t} d\tau L(x, \dot{x}) = \int_{t_0}^{t} d\tau \frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 \\ &= \int_{t_0}^{t} d\tau \frac{1}{2} m \left(\frac{x - x_0}{t - t_0}\right)^2 = \frac{1}{2} m \frac{(x-x_0)^2}{t - t_0} \\ &\Rightarrow G(x, t; x_0, t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\left(\frac{iS_{cl}}{\hbar}\right) \end{aligned} \tag{3.60}$$

The purpose of this work is to solve some problems by approaching path integrals using the technique of spatio-temporal transformations as well as the essential mathematical tools to solve them as simply as possible. Whenever possible, the wave functions and the corresponding spectra are compared with those obtained within the framework of classical mechanics and quantum mechanics [93].

This section constitutes a historical reminder of the usual formulations of quantum mechanics (principle of action, Heisenberg matrix mechanics, Schrodinger's wave mechanics) with the successes achieved but also the difficulties encountered. The path integral is presented as a new formulation which offers an alternative point of view to quantum mechanics and which has turned out to be a very fruitful mathematical formalism for the study of all the disciplines of theoretical physics [94].

3.2 Path Integral Formalism

The path integral approach is based on the Lagrange formalism of classical mechanics with the action as a central concept [93].For these and other reasons, path integrals have assumed a central role in most areas of modern quantum physics, including particle physics, condensed matter physics, and statistical mechanics.Instead of finding eigenfunctions of a Hamiltonian one can evaluate a functional integral which directly yields the propagator required to determine the dynamics of a quantum system [98].In addition, path integrals simplify certain theoretical problems, such as the quantization of gauge fields and the development of perturbation expansions in field theory.

3.2.1 Properties of Amplitude

3.2.1.1 Case of Two Events

Consider an instant s_c belonging to the time interval $[S_a; S_b]$ [97].

The action along any path between a and b can be written as:

$$S(a, b) = S(b, c) + S(c, a) \tag{3.61}$$

This is a direct consequence of the definition of the action as an integral over time. We can write the amplitude as:

$$K(b, a) = \int D[x(s)] \exp\left(\frac{i}{\hbar} S(b, c) + \frac{i}{\hbar} S(c, a)\right) \tag{3.62}$$

Where $D[x(s)]$ is the measure of integration in the functional space of trajectories.

It is possible to split each path into two parts. The first part has the end points x_a and x_c , the second the points x_c and x_b. We can integrate over all paths from a to c , then over all paths from c to b, and finally integrate over all possible values of x_c . The result of the first step for which $S(c; a)$ is constant written as:

$$K(b, a) = \int_{X_c} \int_c^b \exp\left\{\frac{i}{\hbar} S(b, c)\right\} K(c, a) D[x(s)] dx_c \tag{3.63}$$

We then perform the integral between an arbitrary point x_c and the point b , only the integral remains on x_c:

$$K(b, a) = \int_{X_c} K(b, c) K(c, a) dx_c \tag{3.64}$$

By examining this last equation we can notice the following rule applies: the amplitudes for events which follow one another multiply between them.

3.2.1.2 Extension to Multiple Events

It is quite possible to make two divisions on the paths: one at S_c and one at S_d.

The probability amplitude is then written as [97]:

$$K(b,a) = \int_{X_c} K(b,c)K(c,d)K(d,a)dx_c dx_d \tag{3.65}$$

The particle that follows the path from a to b is considered to pass first through d , then through c to finally arrive at b . The probability amplitude for the events to take place in this order is equal to the product of the relative amplitudes every part of the way. The amplitude to go from a to b is obtained by integrating this product over all possible values of x_c and x_d . If we divide the time scale into N intervals, we can generalize the previous equation:

$$K(b,a) = \int_{x_1}\int_{x_2} ... \int_{x_{N-1}} K(b,N-1)K(N-1,N-2)...K(i+1,i)...K(1,a)dx_1 dx_2 ... dx_{N-1} \tag{3.66}$$

In this alternate definition, the amplitude for a particle moving between two points separated by an infinitesimal time ε can be written as follows:

$$K(i+1,i) = \left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{\frac{1}{2}} \exp\left\{\frac{i\varepsilon}{\hbar}L\left(\frac{x_{i+1}-x_i}{\varepsilon}, \frac{x_{i+1}+x_i}{\varepsilon}, \frac{s_{i+1}+s_i}{\varepsilon}\right)\right\} \tag{3.67}$$

Expression which is valid to the first order in ε . Using the rule of multiplication of the amplitudes for events which follow one another one obtains multiplication of the amplitudes for events which follow one another one obtains:

$$\Phi[x(s)] = \lim_{\varepsilon \to 0} \prod_{i=0}^{N-1} K(i+1,i) \tag{3.68}$$

For the amplitude of a complete path, from this expression we can find $K(b,a)$.

3.2.2 The Perturbation Approach to Path Integral Formalism

In the case when a quantum system is exposed to a potential that solely includes quadratic terms in the action, then the treatment of the problem can be done in the formalization of the path integral. However , many problems encountered in quantum mechanics have non - quadratic potentials . This then requires the use of an Approximation method [97].

One possible method is perturbative expansion which is particularly useful when the potential energy is small compared to the kinetic energy of the system. Suppose particle is subjected to a one-dimensional potential $V(x;t)$. The amplitude of motion between two points a and b in this potential is [97]:

$$K_V(b,a) = \int_a^b Dx \exp\left[\frac{i}{\hbar}\int_{s_1}^{s_2}\left(\frac{m}{2}\dot{x}^2 - V(x,t)\right)dt\right] \tag{3.69}$$

The index V in the amplitude means that the particle is subjected to the potential V . We denote $K_0(a;b)$ the amplitude of a free particle ($V=0$). If the integral of the potential along a path is small compared to $\hbar$, according to the inequality:

$$\int_{s_1}^{s_2} V(x,s)ds \ll \hbar \tag{3.70}$$

It is possible to expand the exponential in (3.70):

$$\exp\left\{-\frac{i}{\hbar}\int_{s_1}^{s_2} dsV(x,s)\right\} = 1 - \frac{i}{\hbar}\int_{s_1}^{s_2} dsV(x,s) + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2\left(\int_{s_1}^{s_2} dsV(x,s)\right)^2 \tag{3.71}$$

This allows us to write (3.70) in the form:

$$K_V(b,a) = K_0(b,a) + K^{(1)}(b,a) + K^{(2)}(b,a) + ... \tag{3.72}$$

Where the terms of the expansion are written as:

$$K_0(b,a) = \int_a^b D\left[x(s)\right]\exp\left\{\frac{i}{\hbar}\int_{s_1}^{s_2}\frac{m}{2}\dot{x}^2 ds\right\} \tag{3.73}$$

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_a^b D\left[x(s)\right]\exp\left(\frac{i}{\hbar}\int_{s_1}^{s_2}\frac{m}{2}\dot{x}^2\right)ds\int_{s_1}^{s_2} d\tau V\left(x(\tau),\tau\right) \tag{3.74}$$

$$K^{(2)}(b,a) = -\frac{1}{2\hbar^2}\int_a^b D\left[x(s)\right]\exp\left(\frac{i}{\hbar}\int_{s_1}^{s_2}\frac{m}{2}\dot{x}^2\right)ds\int_{s_1}^{s_2} d\tau V\left(x(\tau),\tau\right)\int_{s_1}^{s_2} d\tau' V\left(x(\tau'),\tau'\right) \tag{3.75}$$

To evaluate the terms of the expansion let us first consider the term $K^{(1)}\left(b,a\right)$ and reverse the order of integration between time and path $x(s)$

$$\begin{aligned} K^{(1)}(b,a) &= -\frac{i}{\hbar}\int_{s_1}^{s_2} d\tau \int_a^b D\left[x(s)\right]\exp\left(\frac{i}{\hbar}\int_{s_1}^{s_2}\frac{m}{2}\dot{x}^2\right)dsV\left(x(\tau),\tau\right) \\ &= -\frac{i}{\hbar}\int_{s_1}^{s_2} d\tau F(\tau) \end{aligned} \tag{3.76}$$

where:

$$F(\tau) = \int_a^b D\left[x(s)\right] \exp\left(\frac{i}{\hbar}\int_{s_1}^{s_2} \frac{m}{2}\dot{x}^2\right) dsV\left(x(\tau),\tau\right) \tag{3.77}$$

The path integral $F(\tau)$ can be interpreted as the sum over all paths of the amplitude of the free particle. Each path is assigned a weight which is the potential $V(x(\tau);\tau)$ evaluated at time τ.

Each path can therefore be divided into two parts: one before the time $s = \tau$, the other after. For $s = \tau$ we assume that each path passes through the point $x_C(\tau)$, point on which we then integrate . If we call C the point $x_C(\tau)$, we can schematically trace the path in space-time.

We can now use the amplitude multiplication rule for successive events, which allows us to write:

$$F(\tau) = \int_{-\infty}^{+\infty} K_0(b,c)V\left(x_C,\tau'\right)K_0(c,a)dx_C \tag{3.78}$$

Which leads to the first order amplitude:

$$K^{(1)}(b,a) = -\frac{i}{\hbar}\int_{s_1}^{s_2}\int_{-\infty}^{+\infty} K_0(b,c)V\left(x_C,\tau'\right)K_0(c,a)dx_C d\tau \tag{3.79}$$

We can characterize the interaction between the potential and the particle in this way as scattering. A single scattering particle is included in the first order's amplitude, but the amplitude of the k^{th} order indicates that the particle may scatter k times.

3.3 Quantum Coulomb Problem in Time Gaussian Electric Field within Path Integral Formalism

In this section, the Coulomb problem in non-relativistic quantum mechanics is addressed with a charge that depends on a parameter that, in the formalism, represents time. The choice of this dependence, for instance, is met when studying the interaction of a "small" system (quantal sub-system) with a "big" one, like a bath, following specific spatio-temporal transformations. The route integral and these spatio-temporal changes enable us to identify the Feynman propagator of the quantal sub-system. Then, we have determined the pure Coulomb Green's function as a limit of our outcome to evaluate our methodology [99].

3.3.1 The Dipole Moment Correlation Function

Nonlocal quadratic actions appear in several physical applications as a result of the context of path integral theory; the general form of these actions is [100].

$$S = \frac{1}{2}\int_0^T dt\dot{x}^2 - \frac{1}{2}\int_0^T dt\int_0^T dsC(t,s)x(t)x(s) \tag{3.80}$$

where C(t, s) is a symmetric function of t and s called the correlation function, it characterizes the memory effects that appear when the system in question is in interaction with a larger system (for example thermal bath or a reservoir). It turns out in our problem that C(t,s) comes from the fulfillment of the statistics on the electric field.

Moreover, the action of our problem contains a Coulomb term in addition to a non-local term, i.e. (3.80) is written in the form [101]:

$$S = \int_0^T \left(\frac{m\dot{x}^2}{2} - \frac{Ze^2}{\|\vec{x}\|} \right) dt - \frac{1}{2} \int_0^T dt \int_0^T ds C(t,s) x(t) x(s) \tag{3.81}$$

Historically, Feynman was the first to introduce the nonlocal action of the form (3.80) into his path integral theory for the polaron problem.

Then we will use the canonical and unitary transformations to solve this problem for the autocorrelation function of the electric field.

3.3.2 Feynman Propagator Formulation

we start with the path integral formulation of the Feynman's propagator wich related to our system [97].

$$K\left(\vec{x}, \vec{y}, t\right) = \int_{\vec{x}_0=\vec{x}}^{\vec{x}(t)=\vec{y}} D\left[\vec{x}\right] \exp \left\{ \begin{array}{c} \int_0^t d\tau \left[\frac{m}{2} \left(\dot{\vec{x}}(\tau) \right)^2 + \frac{Ze^2}{x} \right] \\ -\frac{ie^2}{3\hbar} \int_0^t d\tau \int_0^\tau d\tau' \vec{x}(\tau).\vec{x}(\tau') C_{EE}(\tau - \tau') \end{array} \right\} \tag{3.82}$$

where $C_{EE}(\tau)$ is the electric field's auto-correlation function as a result of the medium on our system (hydrogenic ion). Additionally, if the electric field contains statistics of white noise [99],

$$C_{EE}(\tau - \tau') = C_{EE}(\tau) \delta(\tau - \tau') \tag{3.83}$$

Then:

$$\begin{aligned}
K\left(\vec{x}_f,\tau_f;\vec{x}_i,\tau_i\right) &= \int D\left[\vec{x}\right]\exp\left\{\begin{array}{c}\int_0^t d\tau\left[\frac{m}{2}\left(\dot{\vec{x}}(\tau)\right)^2+\frac{Ze^2}{x}\right]\\ -\frac{ie^2}{3\hbar}\int_0^t d\tau\int_0^\tau d\tau'\vec{x}(\tau).\vec{x}(\tau')C_{EE}(\tau)\delta(\tau-\tau')\end{array}\right\}\\
&= \int D\left[\vec{x}\right]\exp\left\{\int_0^t d\tau\left[\frac{m}{2}\left(\dot{\vec{x}}(\tau)\right)^2+\frac{Ze^2}{x}\right]-\frac{ie^2}{3\hbar}\int_0^t d\tau\vec{x}^2(\tau)C_{EE}(\tau)\right\}\\
&= \int D\left[\vec{x}\right]\exp\left\{\int_0^t d\tau\frac{m}{2}\left(\dot{\vec{x}}(\tau)\right)^2+\frac{Ze^2}{x}-\frac{ie^2}{3\hbar}\vec{x}^2(\tau)C_{EE}(\tau)\right\}\\
&= \int D\left[\vec{x}\right]\exp\left\{\int_0^t d\tau\frac{m}{2}\left(\dot{\vec{x}}(\tau)\right)^2-V\left(\vec{x}(\tau),\tau\right)\right\} \qquad (3.84)
\end{aligned}$$

Where

$$\begin{aligned}
V\left(\vec{x}(\tau),\tau\right) &= -\frac{Ze^2}{x}+\frac{ie^2}{3\hbar}\vec{x}^2(\tau)C_{EE}(\tau)=-\frac{Ze^2}{r}+\frac{ie^2}{3\hbar}C_{EE}(\tau)r^2(\tau)\\
&= -\frac{Ze^2}{x}+i\alpha(\tau)r^2(\tau) \qquad (3.85)
\end{aligned}$$

and $\alpha=\frac{e^2}{3\hbar}C_{EE}(\tau)$
is a Coulomb potential perturbed by a quadratic term in $r(\tau)$ multiplied by a time dependent factor ($\frac{ie^2}{3\hbar}C_{EE}(\tau)$) representing the surrounding medium. Let's do the canonical transformation, [100]:

$$\begin{aligned}
\vec{x} &= Q\rho\left(\tau\right)\\
\vec{p} &= \frac{P}{\rho\left(\tau\right)} \qquad (3.86)
\end{aligned}$$

So we can write:

$$\vec{p}d\vec{x}+\vec{Q}d\vec{P}+\left(\widetilde{H}-H\right)d\tau=d\left(F_1+\vec{P}\vec{Q}\right)=dF_2 \qquad (3.87)$$

Where $F_2\left(\vec{x},\vec{P},\tau\right)=F_1+\vec{P}\vec{Q}$ is called the generating function.
According to (3.87), we write:

$$dF_2=\frac{\partial F_2}{\partial\vec{x}}d\vec{x}+\frac{\partial F_2}{\partial\vec{P}}d\vec{P}+\frac{\partial F_2}{\partial\tau}d\tau \qquad (3.88)$$

Comparing (3.87) with (3.88), we find ,

$$\frac{\partial F_2}{\partial\vec{x}}=\vec{p}\ ,\ \frac{\partial F_2}{\partial\vec{P}}=\vec{Q}\ ,\ \frac{\partial F_2}{\partial\tau}=\widetilde{H}-H \qquad (3.89)$$

From (3.89), F_2 takes the form:

$$F_2 = \overrightarrow{P}\overrightarrow{Q} = \overrightarrow{P}\frac{\overrightarrow{x}}{\rho} \tag{3.90}$$

$$\frac{\partial F_2}{\partial \tau} = \overrightarrow{P}\overrightarrow{x}\left(-\frac{\dot{\rho}}{\rho^2}\right) = -\frac{\overrightarrow{P}\overrightarrow{Q}\dot{\rho}}{\rho} \tag{3.91}$$

The Hamiltonian of the system is written:

$$\begin{aligned}\widetilde{H}\left(\overrightarrow{P},\overrightarrow{Q},\tau\right) &= H\left(\overrightarrow{x}\left(\overrightarrow{P},\overrightarrow{Q}\right),\overrightarrow{p}\left(\overrightarrow{P},\overrightarrow{Q}\right),\tau\right) + \frac{\partial F_2}{\partial \tau} \\ &= \frac{\overrightarrow{P}^2}{2m\rho^2} - \frac{\overrightarrow{P}\overrightarrow{Q}\dot{\rho}}{\rho} + V\left(\overrightarrow{Q}\rho\left(\tau\right),\tau\right)\end{aligned} \tag{3.92}$$

Where:

$$V\left(\overrightarrow{Q}\rho\left(\tau\right),\tau\right) = -\frac{Ze^2}{\rho\left\|\overrightarrow{Q}\right\|} + i\alpha\left(\tau\right)\rho^2\left(\tau\right)Q^2 \tag{3.93}$$

The measure transforms as,

$$D\left[\overrightarrow{x}\left(\tau\right)\right]D\left[\overrightarrow{p}\left(\tau\right)\right] = \left(\frac{1}{\rho_f\rho_i}\right)^{3/2} D\left[\overrightarrow{Q}\left(\tau\right)\right]D\left[\overrightarrow{P}\left(\tau\right)\right] \tag{3.94}$$

According to (3.86), the new propagator is written as,

$$\begin{aligned}K\left(\overrightarrow{Q_f},\tau_f/\overrightarrow{Q_i},\tau_i\right) &= \left(\rho_f\rho_i\right)^{-3/2}\int D\left[\overrightarrow{Q}\left(\tau\right)\right]D\left[\overrightarrow{P}\left(\tau\right)\right] \\ &\quad \exp\left\{i\int_{\tau_i}^{\tau_f}\left[\overrightarrow{P}\dot{\overrightarrow{Q}} - \left(\frac{\overrightarrow{P}^2}{2m\rho^2} - \frac{\overrightarrow{P}\overrightarrow{Q}\dot{\rho}}{\rho} + V\left(\overrightarrow{Q}\rho\left(\tau\right),\tau\right)\right)\right]d\tau\right\}\end{aligned} \tag{3.95}$$

Equation (3.91) is written,

$$\begin{aligned}\widetilde{H}\left(\overrightarrow{P},\overrightarrow{Q},\tau\right) &= \frac{1}{2m\rho^2}\left(\overrightarrow{P}^2 - 2m\dot{\rho}\rho\overrightarrow{P}\overrightarrow{Q}\right) + V\left(\overrightarrow{Q}\rho\left(\tau\right),\tau\right) \\ &= \frac{1}{2m\rho^2}\left(\overrightarrow{P} - m\dot{\rho}\rho\overrightarrow{Q}\right)^2 - \frac{m}{2}\dot{\rho}^2\overrightarrow{Q}^2 + V\left(\overrightarrow{Q}\rho\left(\tau\right),\tau\right)\end{aligned} \tag{3.96}$$

Let's take another canonical transformation,

$$\begin{aligned}\overrightarrow{B} &= \overrightarrow{P} - m\dot{\rho}\rho\overrightarrow{Q} \\ \overrightarrow{Q} &= \overrightarrow{Q}\end{aligned} \tag{3.97}$$

In the same way we find,

$$\begin{aligned}
\frac{\partial \overline{F}_2}{\partial \overrightarrow{B}} &= \overrightarrow{Q} \\
\frac{\partial \overline{F}_2}{\partial \overrightarrow{Q}} &= \overrightarrow{P} = \overrightarrow{B} + m\dot{\rho}\rho\overrightarrow{Q}
\end{aligned} \tag{3.98}$$

what gives us,

$$\begin{aligned}
\overline{F}_2 &= \overrightarrow{B}\overrightarrow{Q} + \frac{m}{2}\dot{\rho}\rho\overrightarrow{Q}^2 \\
\frac{\partial \overline{F}_2}{\partial \tau} &= \frac{m}{2}\left(\dot{\rho}^2 + \ddot{\rho}\rho\right)\overrightarrow{Q}^2
\end{aligned} \tag{3.99}$$

The new Hamiltonian is written,

$$\begin{aligned}
\overline{H}\left(\overrightarrow{B}, \overrightarrow{Q}, \tau\right) &= \frac{\overrightarrow{B}^2}{2m\rho^2} + \frac{m}{2}\ddot{\rho}\rho\overrightarrow{Q}^2 + V\left(\overrightarrow{Q}\rho\left(\tau\right), \tau\right) \\
&= \frac{\overrightarrow{B}^2}{2m\rho^2} + \frac{m}{2}\ddot{\rho}\rho\overrightarrow{Q}^2 - \frac{Ze^2}{\rho\left\|\overrightarrow{Q}\right\|} + i\alpha\left(\tau\right)\rho^2\overrightarrow{Q}^2 \\
&= \frac{\overrightarrow{B}^2}{2m\rho^2} - \frac{Ze^2}{\rho\left\|\overrightarrow{Q}\right\|} + \left(\frac{m}{2}\ddot{\rho} + i\alpha\left(\tau\right)\rho\right)\overrightarrow{Q}^2
\end{aligned} \tag{3.100}$$

We take the following condition $\rho(\tau)$ which must be satisfied:

$$\frac{m}{2}\ddot{\rho} + i\alpha\left(\tau\right)\rho = 0 \tag{3.101}$$

It is clear that $\rho(\tau)$ is complex:

$$\rho\left(\tau\right) = \rho_1\left(\tau\right) + i\rho_2\left(\tau\right) \tag{3.102}$$

Substituting (3.102) into (3.101), we find two identical equations:

$$\rho_1^{''''}\left(\tau\right) + \frac{4\alpha^2\left(\tau\right)}{m}\rho_1\left(\tau\right) = 0 \tag{3.103}$$

$$\rho_2^{''''}\left(\tau\right) + \frac{4\alpha^2\left(\tau\right)}{m}\rho_2\left(\tau\right) = 0 \tag{3.104}$$

So, the function $\rho(\tau)$ takes the following form:

$$\rho\left(\tau\right) = (1+i)\rho_1\left(\tau\right) \tag{3.105}$$

Our Hamiltonian then becomes:

$$\overline{H}\left(\overrightarrow{B}, \overrightarrow{Q}, \tau\right) = \frac{\overrightarrow{B}^2}{2m\rho^2} - \frac{Ze^2}{\rho\left\|\overrightarrow{Q}\right\|} \tag{3.106}$$

The action relative to this Hamiltonian is,

$$\begin{aligned} S &= \int_0^T \left(\overrightarrow{B}\dot{\overrightarrow{Q}} - \overline{H}\left(\overrightarrow{B}, \overrightarrow{Q}, \tau\right)\right) d\tau \\ &= \int_0^T \left(\overrightarrow{B}\dot{\overrightarrow{Q}} - \frac{\overrightarrow{B}^2}{2m\rho^2} + \frac{Ze^2}{\rho\left\|\overrightarrow{Q}\right\|}\right) d\tau \end{aligned} \tag{3.107}$$

To bring the mass constant, we take the following transformation over time:

$$\frac{ds}{d\tau} = \frac{1}{\rho^2(\tau)} = \frac{1}{\rho^2(\tau(s))} \equiv \frac{1}{\overline{\rho}^2(s)} \tag{3.108}$$

Taking into consideration in this last canonical transformation (3.97) the invariance of the measure, the propagator is written after the integration on B as:

$$\begin{aligned} K\left(\overrightarrow{Q_f}, \tau_f / \overrightarrow{Q_i}, \tau_i\right) &= (\rho_f\rho_i)^{-3/2} \exp\left[\frac{im}{2\hbar}\left[\frac{\dot{\overline{\rho}}_f}{\rho_f}\overrightarrow{\overline{Q}}_f^2 - \frac{\dot{\overline{\rho}}_i}{\rho_i}\overrightarrow{\overline{Q}}_i^2\right]\right] \\ & \int D\left[\overrightarrow{Q}(s)\right] \exp\left\{ i\int_{s_i}^{s_f} \left[\frac{m}{2}\dot{\overrightarrow{\overline{Q}}}^2 + \frac{\overline{\rho}Ze^2}{\left\|\overrightarrow{\overline{Q}}\right\|}\right] d\tau \right\} \\ & \equiv (\rho_f\rho_i)^{-3/2} \exp\left[\frac{im}{2\hbar}\left[\frac{\dot{\overline{\rho}}_f}{\rho_f}\overrightarrow{\overline{Q}}_f^2 - \frac{\dot{\overline{\rho}}_i}{\rho_i}\overrightarrow{\overline{Q}}_i^2\right]\right] \mathcal{K}\left(\overrightarrow{q_f}, s_f / \overrightarrow{q_i}, s_i\right) \end{aligned} \tag{3.109}$$

Where $s_i = \int^{\tau_i} \frac{d\sigma}{\rho^2(\sigma)}$ and $s_f = \int^{\tau_f} \frac{d\sigma}{\rho^2(\sigma)}$.
Note that the bar indicates a pseudo-time s.

3.3.3 Perturbative Approach of the Propagator

We want to calculate the propagator of the hydrogen atom with variable charge of a parameter s where the Lagrangian is [97]:

$$L\left(\overrightarrow{\overline{Q}}, \dot{\overrightarrow{\overline{Q}}}, s\right) = \frac{m}{2}\dot{\overrightarrow{\overline{Q}}}^2 + \frac{\overline{\rho}e^2}{\left\|\overrightarrow{\overline{Q}}\right\|} \tag{3.110}$$

We can write it with less complexity:

$$L\left(\dot{\overrightarrow{x}}, \overrightarrow{x}, s\right) = \frac{m}{2}\dot{\overrightarrow{x}} + \frac{e^2\overline{\rho}(s)}{r} \tag{3.111}$$

Where $r = \|\overrightarrow{x}\|$

where the derivative is with respect to the parameter s, and $\overline{\rho}(s)$ is an analytic function depending on the parameter s.

Then this Lagrangian's propagator can be written as follows [97]:

$$\begin{aligned}\mathcal{K}\left(q_f, S/q_i, 0\right) &= \int_{\overrightarrow{q}(0)=\overrightarrow{q}_i}^{\overrightarrow{q}(S)=\overrightarrow{q}_f} D\left[\overrightarrow{q}(s)\right] \exp\left[\frac{i}{\hbar}\int_0^S \left(\frac{m}{2}\dot{\overrightarrow{q}}^2 + \frac{\overline{\rho}e^2}{q}\right) ds\right] \\ &= \int_{\overrightarrow{q}(0)=\overrightarrow{q}_i}^{\overrightarrow{q}(S)=\overrightarrow{q}_f} D\left[\overrightarrow{q}(s)\right] \exp\left(\frac{i}{\hbar}\int_0^S \frac{m}{2}\dot{\overrightarrow{q}}^2 ds\right) \exp\left(\frac{i}{\hbar}\int_0^S \frac{\overline{\rho}e^2}{q} ds\right)\end{aligned} \tag{3.112}$$

Because we are dealing with classical quantities, we can expand the second exponential factor:

$$\begin{aligned}\mathcal{K}\left(q_f, S/q_i, 0\right) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \frac{1}{n!} \times \\ &\int_{\overrightarrow{q}(0)=\overrightarrow{q}_i}^{\overrightarrow{q}(S)=\overrightarrow{q}_f} \left[\overrightarrow{q}(s)\right] \exp\left(\frac{i}{\hbar}\int_0^S \frac{m}{2}\dot{\overrightarrow{q}}^2 ds\right)\left(\int_0^S \frac{\overline{\rho}e^2}{q} ds\right)^n\end{aligned} \tag{3.113}$$

Observe that:

$$\left(\int_0^S \frac{\overline{\rho}e^2}{q} ds\right)^n = \left(e^2\right)^n n! \int_0^S ds_n \frac{\overline{\rho}(s_n)}{q_n} \int_0^{s_n} ds_{n-1} \frac{\overline{\rho}(s_{n-1})}{q_{n-1}} \int \ldots \int_0^{s_1} ds_0 \frac{\overline{\rho}(s_0)}{q_0} \tag{3.114}$$

It is stated that in some situations when studying open quantum systems, we can choose:

$$\overline{\rho}(s) = \exp(ias) \tag{3.115}$$

Replacing Eq.(3.114) and Eq.(3.115) in Eq.(3.113), we get:

$$\begin{aligned}\mathcal{K}\left(q_f, S/q_i, 0\right) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int_0^S ds_n e^{ias_n} \int_0^{s_n} ds_{n-1} e^{ias_{n-1}} \ldots \int ds_1 e^{ias_1} \\ &\times \int \ldots \int \prod_{j=0}^{n} \mathcal{K}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right) \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j}\end{aligned} \tag{3.116}$$

Where $\mathcal{K}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right)$ represents the free particle propagator. then

$$\begin{aligned}\mathcal{K}\left(q_f, S/q_i, 0\right) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int_0^S ds_n \int_0^{s_n} ds_{n-1}... \int ds_1 e^{ias_1+ias_2+...+ias_n} \\ &\times \int ... \int \prod_{j=0}^{n} \mathcal{K}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right) \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \end{aligned} \tag{3.117}$$

If $s_0 = 0$ and $s_{n+1} = S$, We lead to a form like:

$$e^{ias_1+ias_2+...+ias_n} = e^{-ia(s_1-s_0)-2ia(s_2-s_1)-3ia(s_3-s_2)-....-(n+1)ia(S-s_n)+(n+1)iaS} \tag{3.118}$$

Substitute (3.118) in (3.117), we find:

$$\begin{aligned}\mathcal{K}\left(\overrightarrow{q}_f, S/\overrightarrow{q}_i, 0\right) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int_0^S ds_n \int_0^{s_n} ds_{n-1}... \int_0^{s_2} ds_1 \\ & \exp\left[(n+1)\, iaS - ia\sum_{j=1}^{n+1} j\left(s_j - s_{j-1}\right)\right] \times \\ & \int ... \int \prod_{j=0}^{n} \mathcal{K}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right) \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \end{aligned} \tag{3.119}$$

Green's function is defined as:

$$\begin{aligned} G\left(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon\right) &= \int_0^{\infty} e^{i\varepsilon S} \mathcal{K}\left(\overrightarrow{q}_f, S/\overrightarrow{q}_i, 0\right) dS \\ &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int_0^{\infty} e^{i(\varepsilon+(n+1))S} dS \int_0^S ds_n \int_0^{s_n} ds_{n-1}... \int_0^{s_2} ds_1 \\ &\times \int ... \int \prod_{j=0}^{n} \overline{\mathcal{K}}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right) \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \end{aligned} \tag{3.120}$$

Note that ε does not represent the energy but it is the conjugate variable of the pseudo time s.

Let $p_n = \varepsilon + (n+1)a$, we find:

$$\begin{aligned} G\left(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon\right) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int_0^{\infty} e^{ip_n S} dS \int_0^S ds_n \int_0^{s_n} ds_{n-1}... \int_0^{s_2} ds_1 \\ &\times \int ... \int \prod_{j=0}^{n} \overline{\mathcal{K}}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right) \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \end{aligned} \tag{3.121}$$

Where:

$$\overline{\mathcal{K}}_0\left(\overrightarrow{q}_{j+1}, s_{j+1}/\overrightarrow{q}_j, s_j\right) = \left(\frac{m}{2\pi i\hbar\left(s_{j+1} - s_j\right)}\right)^{3/2} e^{-ia(j+1)(s_{j+1}-s_j)} e^{\frac{im\left(\overrightarrow{q}_{j+1} - \overrightarrow{q}_j\right)^2}{2\hbar\left(s_{j+1}-s_j\right)}} \tag{3.122}$$

.

Therefore, the formula (3.121) can be rewritten as

$$\begin{aligned} G(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon) &= \sum_{n=0}^{\infty} \left(\frac{ie^2}{\hbar}\right)^n \int ... \int \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \int_0^{\infty} e^{ip_n S} dS \times \\ & \int_0^S ds_n \left(\frac{m}{2\pi i\hbar (S - s_n)}\right)^{3/2} H(s_n) e^{\frac{im(\overrightarrow{q}_{n+1} - \overrightarrow{q}_n)^2}{2\hbar(S-s_n)}} \\ & \times e^{-ia(n+1)(S-s_n)} \end{aligned} \tag{3.123}$$

Where

$$\begin{aligned} H(s_n) &= \int_0^{s_n} ds_{n-1} \left(\frac{m}{2\pi i\hbar (s_n - s_{n-1})}\right)^{3/2} e^{-ian(s_n - s_{n-1})} \times e^{\frac{im(\overrightarrow{q}_n - \overrightarrow{q}_{n-1})^2}{2\hbar(s_n - s_{n-1})}} \times ... \\ & \times \int_0^{s_2} ds_1 \left(\frac{m}{2\pi i\hbar (s_2 - s_1)}\right)^{3/2} e^{-ia(s_1 - s_0)} e^{\frac{im(\overrightarrow{q}_2 - \overrightarrow{q}_1)^2}{2\hbar(s_2 - s_1)}} \times \\ & \times \left(\frac{m}{2\pi i\hbar s_1}\right)^{3/2} e^{-ias_1} e^{\frac{im(\overrightarrow{q}_1 - \overrightarrow{q}_{0=i})^2}{2\hbar s_1}} \end{aligned} \tag{3.124}$$

Applying now the convolution theorem by carrying out Laplace transform with respect to the variable S in formula (3.123) yields:

$$G(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon) = \sum_{n=0}^{\infty} \left(\frac{ie^2}{\hbar}\right)^n \int ... \int \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \widetilde{\widetilde{\mathcal{K}}}_{(n+1)a}(p_n; \overrightarrow{q}_{n+1}, \overrightarrow{q}_n) \widetilde{H}_n(p_n) \tag{3.125}$$

Where

$$\widetilde{\widetilde{\mathcal{K}}}_{(n+1)a}(p_n; \overrightarrow{q}_{n+1}, \overrightarrow{q}_n) = \int_0^{\infty} e^{ip_n S} dS \left(\frac{m}{2\pi i\hbar S}\right)^{3/2} \exp\left[\frac{im}{2\hbar S}\left(\overrightarrow{q}_n - \overrightarrow{q}_{n-1}\right)^2 - ia(n+1)S\right] \tag{3.126}$$

and

$$\begin{aligned} \widetilde{H}_n(p_n) &= \int_0^S e^{ip_n s_n} H_n(s_n) ds_n \\ &= \int_0^{\infty} e^{ip_n s_n} ds_n \int_0^{s_n} ds_{n-1} h_{n-1}(s_{n-1}) \left(\frac{m}{2\pi i\hbar (s_n - s_{n-1})}\right)^{3/2} \\ & \exp\left[\frac{im}{2\hbar (s_n - s_{n-1})}\left(\overrightarrow{q}_n - \overrightarrow{q}_{n-1}\right)^2\right] \times \\ & \exp\left[-ian(s_n - s_{n-1})\right] \end{aligned} \tag{3.127}$$

Applying the convolution theory once more, we get:

$$\widetilde{H}_n(p_n) = \widetilde{\mathcal{K}}_{na}(p_n) \widetilde{H}_{n-1}(p_n) \tag{3.128}$$

Where

$$\widetilde{\mathcal{K}}_{na}(p_n) = \int_0^\infty e^{ip_n s_n} ds_n \left(\frac{m}{2\pi i\hbar s_n}\right)^{3/2} \times \exp\left[\frac{im}{2\hbar s_n}\left(\overrightarrow{q}_n - \overrightarrow{q}_{n-1}\right)^2 + inas_n\right] \tag{3.129}$$

and

$$\widetilde{H}_{n-1}(p_n) = \int_0^\infty e^{ip_n s_{n-1}} H_{n-1}\left(s_{n-1}\right) ds_{n-1} \tag{3.130}$$

The result of using the convolution theorem repeatedly is

$$\begin{aligned} G\left(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon\right) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int ... \int \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \widetilde{\mathcal{K}}_{(n+1)a}\left(p_n\right) \widetilde{\mathcal{K}}_{na}\left(p_n\right) ... \widetilde{\mathcal{K}}_{a}\left(p_n\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int ... \int \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \prod_{j=0}^{n} \widetilde{\mathcal{K}}_{(j+1)a}\left(p_n\right) \end{aligned} \tag{3.131}$$

Where

$$\widetilde{\mathcal{K}}_{(j+1)a}\left(p_n\right) = \int_0^\infty e^{ip_n s_{j+1}} \left(\frac{m}{2\pi i\hbar s_{j+1}}\right)^{3/2} \exp\left[\frac{im}{2\hbar s_{j+1}}\left(\overrightarrow{q}_{j+1} - \overrightarrow{q}_j\right)^2 - i\left(j+1\right)as_{j+1}\right] ds_{j+1} \tag{3.132}$$

Put $\varepsilon_j^n = p_n - (j+1)a = \varepsilon + (n-j)a$ then

$$\begin{aligned} \widetilde{\mathcal{K}}_{(j+1)a}\left(p_n\right) &= \widetilde{\mathcal{K}}_{(j+1)a}\left(\varepsilon_j^n\right) \\ &= \int_0^\infty e^{i\varepsilon_j^n s_j} ds_j \left(\frac{m}{2\pi i\hbar s_j}\right)^{3/2} \exp\left[\frac{im}{2\hbar s_j}\left(\overrightarrow{q}_{j+1} - \overrightarrow{q}_j\right)^2\right] \\ &= G_0(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon_j^n) \end{aligned} \tag{3.133}$$

then Eq.(3.131) becomes:

$$\begin{aligned} G(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon) &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int ... \int \prod_{j=1}^{n} \frac{d\overrightarrow{q}_j}{q_j} \prod_{j=0}^{n} G_0(\overrightarrow{q}_f, \overrightarrow{q}_i, \varepsilon_j^n) \\ &= \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n \int ... \int \prod_{j=1}^{n} q_j dq_j d\Omega_j \prod_{j=0}^{n} ds_j \exp\left(i\varepsilon_j^n s_j\right) \times \\ &\quad \mathcal{K}_0\left(\overrightarrow{q}_{j+1}, \overrightarrow{q}_j, s_j\right) \end{aligned} \tag{3.134}$$

referencing that

$$(\overrightarrow{q}_{j+1}-\overrightarrow{q}_{j})^{2}=\overrightarrow{q}_{j+1}^{2}+\overrightarrow{q}_{j}^{2}-2\overrightarrow{q}_{j+1}\overrightarrow{q}_{j}\cos\Omega_{j+1,j}$$

and use

$$\exp\left(z\cos\Omega_{j+1,j}\right)=\sqrt{\frac{\pi}{2z}}\sum_{l_j=0}^{\infty}\left(2l_j+1\right)I_{l_j+\frac{1}{2}}\left(z\right)P_{l_j}\left(\cos\Omega_{j+1,j}\right) \tag{3.135}$$

$$\sum_{m_j=-l_j}^{+l_j}Y_{l_j}^{m_j}\left(\theta_j,\varphi_j\right)\left[Y_{l_j}^{m_j}\left(\theta_{j+1},\varphi_{j+1}\right)\right]^{*}=\frac{2\left(l_j+1\right)}{4\pi}P_{l_j}\left(\cos\Omega_j\right) \tag{3.136}$$

and

$$\int Y_l^m\left(\theta_j,\varphi_j\right)\left[Y_{l'}^{m'}\left(\theta_{j+1},\varphi_{j+1}\right)\right]^{*}d\Omega_j=\delta_{mm'}\delta_{ll'} \tag{3.137}$$

Substituting into (3.134), we find:

$$\begin{aligned}G(\overrightarrow{q}_f,\overrightarrow{q}_i,\varepsilon) \;=\;& \sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n\int\ldots\int\prod_{j=1}^{n}q_jdq_jd\Omega_j\prod_{j=0}^{n}\sum_{l_j=0}^{\infty}\sum_{m_j=-l_j}^{+l_j}\int_0^{\infty}ds_j\\ & \exp\left[i\varepsilon_j^n s_j+\frac{i\alpha}{s_j}\left(\overrightarrow{q}_{j+1}^2-\overrightarrow{q}_j^2\right)\right]\left(\frac{\alpha\left(4\pi\right)^{2/3}}{i\pi s_j}\right)^{3/2}\left(\frac{i\pi s_j}{\alpha\overrightarrow{q}_{j+1}\overrightarrow{q}_j}\right)^{1/2}\\ & I_{l_j+\frac{1}{2}}\left(\frac{-2i\alpha\overrightarrow{q}_{j+1}\overrightarrow{q}_j}{s_j}\right)Y_{l_j}^{m_j}\left(\theta_j,\varphi_j\right)\left[Y_{l_j}^{m_j}\left(\theta_{j+1},\varphi_{j+1}\right)\right]^{*}d\Omega_j\end{aligned} \tag{3.138}$$

where $\alpha=\frac{m}{2\hbar}$. since $\prod_{j=1}^{n}q_j=\frac{1}{\sqrt{q_iq_f}}\prod_{j=0}^{n}\sqrt{q_{j+1}q_j}$

Now let's integrate on the angles therefor we have:

$$G(\overrightarrow{q}_f,\overrightarrow{q}_i,\varepsilon)=\sum_{lm}Y_l^m\left(\theta_j,\varphi_j\right)\left[Y_l^m\left(\theta_{j+1},\varphi_{j+1}\right)\right]^{*}G_l(q_f,q_i,\varepsilon) \tag{3.139}$$

where $G_l(q_f,q_i,\varepsilon)$ is the radial Green's function can be written as:

$$G_l(q_f,q_i,\varepsilon)=\left(\frac{4\alpha}{i}\right)\frac{1}{\sqrt{q_iq_f}}\sum_{n=0}^{\infty}\left(\frac{ie^2}{\hbar}\right)^n\int\ldots\int\prod_{j=1}^{n}dq_j\prod_{j=0}^{n}A_l(q_f,q_i,\varepsilon_j^n) \tag{3.140}$$

where the definition of the A_l coefficients is,

$$A_l(q_f, q_i, \varepsilon_j^n) = \frac{1}{2}\int_0^\infty \exp\left[i\alpha\frac{q_{j+1}^2+q_j^2}{s_j}+\frac{i\varepsilon_j^n s_j}{\hbar}\right] I_{l+\frac{1}{2}}\left(\frac{-2i\alpha q_{j+1}q_j}{s_j}\right)\frac{ds_j}{s_j} \tag{3.141}$$

Formulas (3.134) and (3.141) give a radial Green's function $G_l(q_f, q_i, \varepsilon)$ of the following form,

$$G_l(q_f, q_i, \varepsilon) = \left(\frac{4\alpha}{i}\right)(q_i q_f)^{\frac{-1}{2}}\sum_{n=0}^{\infty}\left(\frac{4\alpha e^2}{\hbar}\right)^n F_l^{(n)}(q_f, q_i, \varepsilon) \tag{3.142}$$

where the coefficients $F_l^{(n)}$ are defined as:

$$\begin{aligned} F_l^{(0)}(q_f, q_i, \varepsilon) &= A_l(q_f, q_i, \varepsilon) \\ F_l^{(n)}(q_f, q_i, \varepsilon) &= \int \ldots \int \prod_{j=0}^{n} A_l(\overrightarrow{q}_{j+1}, \overrightarrow{q}_j, \varepsilon_j^n)\prod_{j=1}^{n} dq_j \end{aligned} \tag{3.143}$$

and

$$A_l(q_{j+1}, q_j, \varepsilon_j^n) = \int_0^\infty G_l(\varpi)\, d\varpi \tag{3.144}$$

where

$$G_l(\varpi) = \frac{\exp\left[-2i(\alpha\epsilon_j)^{1/2}(q_{j+1}+q_j)\coth\varpi\right]}{\sinh\varpi} I_{l+\frac{1}{2}}\left(\frac{-4i(\alpha\epsilon_j q_{j+1}q_j)^{1/2}}{\sinh\varpi}\right) \tag{3.145}$$

where $\epsilon_j = \frac{\varepsilon_j^n}{\hbar}$, then

$$\begin{aligned} F_l^{(1)}(q_f, q_i, \varepsilon) &= \int_0^\infty A_l(q_2, q_1, \varepsilon_1^n) A_l(q_1, q_0, \varepsilon_0^n) dq_1 \\ &= \frac{i}{2}(\alpha\epsilon_0)^{\frac{-1}{2}}(\alpha\epsilon_1)^{\frac{-1}{2}}\int_0^\infty \varpi G_l(\varpi) d\varpi \end{aligned} \tag{3.146}$$

and

$$\begin{aligned} F_l^{(n)}(q_f, q_i, \varepsilon) &= \frac{\left(\frac{i}{2}\right)^n}{n!}\prod_{j=0}^{n}(\alpha\epsilon_j)^{-1/2}\int_0^\infty \varpi^n G_l(\varpi) d\varpi \\ &= \frac{\left(\frac{i}{2}\right)^n}{n!}\exp\left[-\frac{1}{2}\sum_j \log(\alpha\epsilon_j)\right]\int_0^\infty \varpi^n G_l(\varpi) d\varpi \end{aligned} \tag{3.147}$$

Replacing (3.147) in (3.142) one find

$$\begin{aligned} G_l(q_f,q_i,\varepsilon) &= \left(\frac{4\alpha}{i}\right)(q_iq_f)^{\frac{-1}{2}}\int_0^\infty \sum_{n=0}^{\infty}\left(\frac{2i\alpha e^2}{\hbar}\varpi\right)^n \times \\ &\quad \frac{1}{n!}\exp\left[-\frac{1}{2}\sum_j \log\left(\alpha\epsilon_j\right)\right] G_l(\varpi)d\varpi \\ &= \left(\frac{4\alpha}{i}\right)(q_iq_f)^{\frac{-1}{2}}\int_0^\infty F(\varpi,a)G_l(\varpi)d\varpi \end{aligned} \tag{3.148}$$

where

$$\begin{aligned} F(\varpi,a) &= \sum_{n=0}^{\infty}\left(\frac{2i\alpha e^2}{\hbar}\varpi\right)^n \frac{1}{n!}\exp\left[-\frac{1}{2}\sum_j \log\left(\alpha\epsilon_j\right)\right] \\ &= \sum_{n=0}^{\infty}\left(\frac{2i\alpha e^2}{\hbar}\varpi\right)^n \frac{1}{n!}\times \\ &\quad \exp\left(-\frac{1}{2}\left\{\log\frac{\alpha}{\hbar}\left(\varepsilon+na\right)+\log\frac{\alpha}{\hbar}\left(\varepsilon+\left(n-1\right)a\right)+...+\log\frac{\alpha}{\hbar}\varepsilon\right\}\right) \end{aligned} \tag{3.149}$$

We can check if we set a = 0, we find the Green's function of the Coulomb potential [97]. Indeed, and according to the time transformation formula (3.108) τ and s play the same role of time and ε represents in this case the energy E, let us then write:

$$\begin{aligned} F(\varpi,a &= 0) = \sum_{n=0}^{\infty}\left(\frac{2i\alpha e^2}{\hbar}\varpi\right)^n \frac{1}{n!}\exp\left[\frac{-n}{2}\log(-i\alpha\epsilon)\right] \\ &= \sum_{n=0}^{\infty}\left(\frac{2i\alpha e^2}{\hbar}\varpi\right)^n \frac{1}{n!}(-i\alpha\epsilon)^{\frac{-n}{2}} \\ &= \exp\left[\left(\frac{2ie^2}{\hbar}\right)(-i\alpha\epsilon)^{1/2}\varpi\right] \end{aligned} \tag{3.150}$$

The equation (3.148) becomes:

$$G_l(q_f,q_i,\varepsilon=E) = \left(\frac{4\alpha}{i}\right)(q_fq_i)^{-1/2}\int_0^\infty \exp\left[\left(\frac{2ie^2}{\hbar}\right)(-i\alpha\epsilon)^{1/2}\varpi\right]G_l(\varpi)d\varpi \tag{3.151}$$

Using the following formula [102]

$$\int_0^\infty \frac{\exp\left[2\mu x-t(a+b)\coth x\right]}{\sinh x}I_{2\nu}\left(\frac{2t(ab)^{1/2}}{\sinh x}\right)dx = \frac{m\Gamma(\nu-\mu+\frac{1}{2})}{2t(ab)^{1/2}\Gamma(2\nu+1)}W_{\mu,\nu}(2ta)M_{\mu,\nu}(2tb) \tag{3.152}$$

where $W_{\mu,\nu}$and $M_{\mu,\nu}$ are Whittaker functions and the result is valid for $a > b, Re(\nu) > 0, Re(\nu - \mu + 1/2) > 0$. Then, the radial Green's function is exactly the
same for the Coulomb problem [103].

$$G_l(q_f, q_i, E) = \frac{m\Gamma\left(l+1-u\right)}{\hbar k q_f q_i (2l+1)!} W_{u,l+1/2}(-2ikq_f) M_{u,l+1/2}(-2ikq_i) \tag{3.153}$$

where $u = i(me^4/2\hbar^2 E)^{1/2}$, $k = (2mE/\hbar^2)^{1/2}$, and $q_f > q_i$. The poles of the radial Green's function are used to generate the discrete spectra (3.152): $l + 1 + u = -n$, and $E = -me^4/(2\hbar(n + l + 1)^2)$

Finally, the radial propagator can be expressed as:

$$\mathcal{K}_l(q_f, q_i, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_l(q_f, q_i, \varepsilon) e^{i\varepsilon S(T)} d\varepsilon \tag{3.154}$$

The formulae (3.108) and (3.115) also suggest the following integration

$$\int^S ds = \int^T \frac{d\tau}{2ia\tau} \tag{3.155}$$

Then

$$S = S_0 + \int^T \frac{d\tau}{2ia\tau} \equiv S(T) \tag{3.156}$$

leading with (3.154) to

$$\begin{aligned}
\mathcal{K}_l(q_f, q_i, T) \;\; &= \;\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_l(q_f, q_i, \varepsilon) T^{i\varepsilon/a} e^{i\varepsilon S_0} d\varepsilon \\
&= \;\; \frac{1}{2\pi} \left(\frac{4\alpha}{i}\right) \left(q_f q_i\right)^{-1/2} \int_{-\infty}^{+\infty} d\varepsilon \int_0^\infty \left\{ \sum_{n=0}^{\infty} \left(\frac{2i\alpha e^2}{\hbar}\varpi\right)^n \frac{1}{n!} \right\} \times \\
&\quad \exp\left[-\frac{1}{2}\sum_j \log\left(\alpha\epsilon_j\right)\right] T^{i\varepsilon/a} e^{i\varepsilon S_0} G_l(\varpi) d\varpi
\end{aligned} \tag{3.157}$$

which is what is wanted. It's interesting to note that if we set $a = 0$, our conclusion is reduced to the pure Coulomb case. [103].

3.4 Conclusion

In this chapter, first, we have introduced an analytical method equivalent to Schrodinger's operational method to calculate the line profiles. This method is based on the concept of the path integral of Richard Feynman. He stated that the amplitude of transition between two quantum states corresponds to the exponential of the classical action by summing over all possible trajectories between these states. The propagator in the dipole autocorrelation function is relative to a nonlocal quadratic action. The quadratic term is proportional to the electric field correlation function. This approach led us to deal with time-dependent problems with form of the field correlation function, the form is taken as white noise. Then we have solved the problem with the canonical and units transformations etc. next we started with the historical presentation of the application of perturbation theory. We presented the technique to evaluate the propagator (Green's function) relative to the Lagrangian describing the Coulomb problem whose charge depends on a parameter function.The Feynman propagator for the hydrogen atom (or hydrogenic ion) in the time-dependent field obeying Gaussian white noise statistics has been derived in the non-relativistic limit. When applying the average to these statistics, it becomes clear that our problem is a brand-new Coulomb problem with a quadratic term in the Lagrangian. This time-dependent term is compounded by a complex analytical factor. The waste of energy in the surrounding medium is encoded by this complicated component. We have thus obtained a concise expression of the system's Feynman propagator using the proper canonical transformations.

General Conclusion

Most current research in the plasma spectroscopy field is based on standard quantum mechanics tools and focuses mainly on the numerical resolution of the Schrodinger equation of the radiation emitter. In the first part of this work, We pointed out that the confinement of plasmas by adequate magnetic fields is the most highly developed technique to obtain a controlled fusion.This is achieved by adding to the toroidal magnetic field a poloidal one perpendicular to it. These conditions can be guaranteed by setting up several devices and experiments before the startup of the fusion. In our study, limiting ourselves to hydrogen-like ions, we are interested, firstly, in the influence of a nonuniform magnetic field on the quantum dynamics of ions located in fusion plasma. We have presented the quantum mechanical equations that describe the ions in the presence of the nonuniform magnetic field.Secondly we have used the obtained eigenenergy to determine the spectral line shape (Lyman-alpha) of three types of ions: He^{+}, C^{5+}, and Ar^{17+}. we have presented some spectral line shapes for Ly-alpha of the three hydrogen-like ions, for different magnetic field intensities and temperatures. The geometry of the chosen magnetic field is very important in terms of simulating reality for the tokamak device, is a new addition to previous studies, and enables the use of the results obtained in applications in fusion plasma.

In the second part of this thesis, we introduced an analytical method equivalent to Schrodinger's operational method to calculate the line profiles. This method is based on the path integrals concept, which was developed by Richard Feynman in 1948. He stated that the transition amplitude between two quantum states corresponds to the exponential of the classical action by summing over all possible trajectories between these states. Since this work deals with the hydrogen atom in the non-relativistic limit interacting with a plasma environment, it presents a more complicated issue. The latter is represented by a fluctuating electric field that obeys a gaussian white noise statistics. The electric field is due to the presence of all the charged components (ions and electrons) of the plasma that are in perpetual motion. Since the electrons are too light with respect to the ions can be replaced by a collision operator. When the ions are represented by an electric field that follows a Gaussian white noise statistic. If we consider our system as a hydrogen ion in a medium that acts instantly by a time-dependent electric field, the response would be obviously a radiation.Then, since this scheme is repeated at each medium point, in this work We only have discussed the interaction with the medium's ions because the interaction of our system with the medium's free electrons is replaced by an collesion operator. When performing the statistical average over the ionic electric field, we encounter a Coulomb problem perturbed by a quadratic term multiplied by a time dependent complex factor. Technically, this scheme is achieved by transforming the initial problem from(x, t) into a novel Coulomb problem (q, s) with a pseudo-time-dependent charge, $Ze\overline{\rho}(s)$. The purposed choice for $\rho(s)$ is indicated by a few models that can be found in the literature. This behavior of $C_{EE}(\tau)$ reveals the exact calculation of the Green's function. But a few works have treated time-dependent problems in the perturbative approach of the path integral formalism, and our work is among them. First we have set the propagator relative to a hydrogen ion submerged in a Gaussian electric field, and have used a generalized canonical transformation to get a pseudo Lagrangian with pseudo-time-dependent potential, then, we have calculated the propagator relative to the formulated Lagrangian using the exact summation of the perturbation series.

In future studies, this model of magnetic field geometry can be studied in relativistic quantum mechanics via the Dirac equation, and the thermodynamic properties of the system can be analysed. Then, for the second part we can present some spectral line shapes for our system and its optical properties.

Appendices

Appendix A

Calculation of Gaussian Functional Integrals

To calculate the average value of:

$$\exp\left(i\overrightarrow{k}\int_0^t \overrightarrow{v}(\tau)d\tau\right) \tag{A.1}$$

In other words, we want to calculate:

$$\left\langle \exp\left(i\overrightarrow{k}\int_0^t \overrightarrow{v}(\tau)d\tau\right)\right\rangle \tag{A.2}$$

using a Gaussian distribution $P(\overrightarrow{v},t)$ on speeds,

$$P(\overrightarrow{v},t) = \exp\left[-\frac{1}{2}\int_0^t d\tau\int_0^t \overrightarrow{v}(\tau)C^{-1}(\tau-\tau')\overrightarrow{v}(\tau')d\tau'\right] \tag{A.3}$$

$$\equiv \exp\left[-\frac{1}{2}\int_0^t d\tau\int_0^t \overrightarrow{v}(\tau)M(\tau-\tau')\overrightarrow{v}(\tau')d\tau'\right] \tag{A.4}$$

Where

$$M = C^{-1}$$
$$C = M^{-1}$$

For simplicity, if we consider the one-dimensional case, the previous average is expressed as:

$$S(k,t) = \left\langle \exp\left(i\overrightarrow{k}\int_0^t \overrightarrow{v}(\tau)d\tau\right)\right\rangle = \int D[v(t)]P(v,t)\exp\left(ik\int_0^t v(\tau)d\tau\right) \tag{A.5}$$

$$S(k,t) = \int D[v(t)]\exp\left[-\frac{1}{2}\int_0^t d\tau\int_0^t v(\tau)M(\tau-\tau')v(\tau')d\tau' + ik\int_0^t v(\tau)d\tau\right] \tag{A.6}$$

To keep things simple, we'll only look at the discretization in two points.ă

$$S(k,t) = \int\int dv_1 dv_2 \exp\left[-\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} v_i M_{ij} v_j + ik\sum_{i=1}^{2} v_i\right] \tag{A.7}$$

$$S(k,t) = \int\int dv_1 dv_2 \exp\left[-\frac{1}{2}v_1 M_{11} v_1 - \frac{1}{2}v_1 M_{12} v_2 - \frac{1}{2}v_2 M_{21} v_1 - \frac{1}{2}v_2 M_{22} v_2 + ikv_1 + ikv_2\right] \tag{A.8}$$

note that $M_{12} = M_{21}$

$$S(k,t) = \int\int dv_1 dv_2 \exp\left[-\frac{1}{2}v_1^2 M_{11} - v_1 M_{12} v_2 + ikv_1\right] \exp\left[-\frac{1}{2}v_2^2 M_{22} + ikv_2\right] \tag{A.9}$$

$$= \int dv_2 exp\left[-\frac{1}{2}v_2^2 M_{22} + ikv_2\right] \int dv_1 \exp\left[-\frac{1}{2}v_1^2 M_{11} - v_1 M_{12} v_2 + ikv_1\right] \tag{A.10}$$

on the other hand

$$\int dv_1 \exp\left[-\frac{1}{2}v_1^2 M_{11} - v_1 M_{12} v_2 + ikv_1\right] = \int dv_1 \exp\left[-\frac{1}{2}M_{11}\left(v_1^2 + 2\frac{M_{12}}{M_{11}}v_1 v_2 - \frac{2ik}{M_{11}}v_1\right)\right] \tag{A.11}$$

$$= \int dv_1 \exp\left[-\frac{1}{2}M_{11}\left(v_1^2 + v_1\left(2\frac{M_{12}}{M_{11}}v_2 - \frac{2ik}{M_{11}}\right)\right)\right] \tag{A.12}$$

we pose

$$2A = 2\frac{M_{12}}{M_{11}}v_2 - \frac{2ik}{M_{11}} \Longrightarrow A = \frac{M_{12}}{M_{11}}v_2 - \frac{ik}{M_{11}} \tag{A.13}$$

$$\int dv_1 \exp\left[-\frac{1}{2}M_{11}\left(v_1^2 + v_1\left(2\frac{M_{12}}{M_{11}}v_2 - \frac{2ik}{M_{11}}\right)\right)\right] = \int dv_1 \exp\left[-\frac{1}{2}M_{11}\left((v_1+A)^2 - A^2\right)\right] \tag{A.14}$$

$$= \exp\left(\frac{1}{2}M_{11}A^2\right)\int dv_1 \exp\left[-\frac{1}{2}M_{11}(v_1+A)^2\right] \tag{A.15}$$

$$= \left(\frac{2\pi}{M_{11}}\right)^{1/2} \exp\left(\frac{1}{2}M_{11}A^2\right) \tag{A.16}$$

$$= \left(\frac{2\pi}{M_{11}}\right)^{1/2} \exp\left(\frac{1}{2}M_{11}\left(\frac{M_{12}}{M_{11}}v_2 - \frac{ik}{M_{11}}\right)^2\right) \tag{A.17}$$

then

$$\begin{aligned} S(k,t) &= \left(\frac{2\pi}{M_{11}}\right)^{1/2} \int dv_2 \exp\left[-\frac{1}{2}v_2^2 M_{22} + ikv_2 + \frac{1}{2}M_{11}\left(\frac{M_{12}}{M_{11}}v_2 - \frac{ik}{M_{11}}\right)^2\right] && \text{(A.18)} \\ &= \left(\frac{2\pi}{M_{11}}\right)^{1/2} \int dv_2 \exp\left[-\frac{1}{2}v_2^2 M_{22} + ikv_2 + \frac{1}{2}\left(\frac{M_{12}^2}{M_{11}}v_2^2 - \frac{2ikM_{12}}{M11}v_2 - \frac{k^2}{M_{11}}\right)\right] && \text{(A.19)} \\ &= \left(\frac{2\pi}{M_{11}}\right)^{1/2} \int dv_2 \exp\left[-\frac{1}{2}v_2^2\left(M_{22} + \frac{M_{12}^2}{M_{11}}\right) + v_2\left(ik - \frac{ikM_{12}}{M11}\right) - \frac{k^2}{M_{11}}\right] && \text{(A.20)} \end{aligned}$$

putting $B = \left(M_{22} + \frac{M_{12}^2}{M_{11}}\right)$ and $C = \left(ik - \frac{ikM_{12}}{M11}\right)$ we obtained:

$$\begin{aligned} S(k,t) &= \left(\frac{2\pi}{M_{11}}\right)^{1/2} \int dv_2 \exp\left[-\frac{1}{2}B\left(\left(v_2 - \frac{2C}{B}\right)^2 - \frac{4C^2}{B^2}\right)\right] \exp\left(-\frac{k^2}{2M_{11}}\right) && \text{(A.21)} \\ &= \left(\frac{2\pi}{M_{11}}\right)^{1/2}\left(\frac{2\pi}{B}\right)^{1/2} \exp\left(-\frac{k^2}{2M_{11}}\right)\exp\left(\frac{2C^2}{B}\right) && \text{(A.22)} \\ &= \frac{2\pi}{(M_{11}M_{22} - M_{12}^2)^{1/2}} \exp\left[\left(-\frac{k^2}{2M_{11}}\right) + \frac{2}{M_{22} - \frac{M_{12}^2}{M_{11}}}\left(\frac{ik}{2} - \frac{ikM_{12}}{M_{11}}\right)^2\right] && \text{(A.23)} \\ &= \frac{2\pi}{(\det M)^{1/2}} \exp\left(-\frac{k^2}{2}\left[\frac{M_{22}}{\det M} + \frac{M_{11}}{\det M} - \frac{M_{12}}{\det M} - \frac{M_{21}}{\det M}\right]\right) && \text{(A.24)} \\ &= \exp\left[-\frac{k^2}{2}\left(C_{11} + C_{22} + C_{12} + C_{21}\right)\right] && \text{(A.25)} \\ &= \exp\left[-\frac{k^2}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} C_{ij}\right] && \text{(A.26)} \end{aligned}$$

We find the double integral on this last form so that we can write:

$$S(k,t) = \exp\left[-\frac{k^2}{2}\int_0^t d\tau \int_0^t d\tau' C(\tau - \tau')\right] \tag{A.27}$$

Using the symmetry property:

$$C(\tau - \tau') = C(\tau' - \tau) \tag{A.28}$$

we can say that the function $C(\tau-\tau')$ takes the same value on either side of the first bisector of the first quadrant.
The integral over the square (0, t)*(0, t) is thus equal to twice the integral over a triangle, say the upper triangle:ă

$$\int_0^t d\tau \int_0^t d\tau' C(\tau-\tau') = 2\int_0^t d\tau \int_\tau^t d\tau' C(\tau-\tau') \tag{A.29}$$

putting $u=\tau-\tau'$ then

$$2\int_0^t d\tau \int_\tau^t d\tau' C(\tau-\tau') = -2\int_0^t d\tau \int_0^{\tau-t} duC(u) = -2\int_0^t d\tau \int_0^{t-\tau} duC(u) \tag{A.30}$$

we integrate by parts overτ

$$\begin{aligned} 2\int_0^t d\tau \int_0^t d\tau' C(\tau-\tau') &= -2\left[\tau \int_0^{t-\tau} duC(u)\right] - 2\int_0^t \tau d\tau C(t-\tau) \\ &= -2\int_0^t \tau d\tau C(t-\tau) \end{aligned} \tag{A.31}$$

we set the change of variable:

$$\begin{aligned} t-\tau &= q \\ d\tau &= -dq \end{aligned}$$

then

$$2\int_0^t d\tau \int_0^t d\tau' C(\tau-\tau') = 2\int_0^t (t-q)dqC(q) \tag{A.32}$$

In conclusion:

$$S(k,t) = \exp\left[-\frac{k^2}{2}\int_0^t d\tau \int_0^t d\tau' C_{VV}(\tau-\tau')\right] \tag{A.33}$$

$$S(k,t) = \exp\left[-k^2 \int_0^t C_{VV}(\tau) d\tau\right] \tag{A.34}$$

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ملخص

في هذه الأطروحة، قمنا بحل مسألة كمومية للأيونات الشبيهة بالهيدروجين تحت مجال مغناطيسي قوي وغير منتظم.المجال المغناطيسي الكثيف هو مجموع حقلين ذي هندسة حلقية وبولويدية بشدة تساوي ثلث الحقل الحلقي. كشفت النتائج عن شكل الخط الطيفي(Lyman alpha) لثلاثة أنواع من الأيونات He^{+}، C^{5+}وAr^{17+} لشدات مجال مغناطيسي مختلفة. يكون فصل زيمان كبيرا كلما كان المجال المغناطيسي كبيرا والعدد الشحني مع كون خط زيمان الأيسر قريب من خط الطيف في غياب الحقل المغناطيسي.

بعد ذلك، قمنا بحساب ناشر فاينمان لذرة الهيدروجين في ميكانيكا الكم غير النسبي، حيث تخضع ذرات الهيدروجين لحقل كهربائي يتعلق بالزمن يتوافق مع احصائيات الضوضاء البيضاء الغاوسية. تبين أن مسألتنا هي مسألة كولومب جديدة بحد تربيعي في عبارة لاغرانج وعامل تحليلي يتعلق بالزمن. من خلال تطبيق تحولات قانونية مناسبة، تم اشتقاق صيغة موجزة لناشر فاينمان للنظام.

الكلمات المفتاحية: بلازما، مجال مغناطيسي، فصل زيمان، ناشر فاينمان، لاغرانج، مسألة كولومب.

Abstract

In this thesis, we have found a quantum problem solution of hydrogen-like ions under a strong, nonuniform magnetic field. The intense magnetic field is the sum of two fields with toroidal and poloidal geometry, with an intensity equal to the third of the toroidal one. The results reveal the spectral line shape (Lyman-alpha) of three types of ions, He^{+}, C^{5+}, and Ar^{17+}, for different magnetic field intensities. The Zeeman separation is as large as the magnetic field and charge number Z, with the left Zeeman component close to the line without a magnetic field.

Then, we have calculated the hydrogen atom Feynman propagator in the non-relativistic limit where the hydrogen atoms are obeing a time-dependent electric field that complies with the Gaussian white noise statistics. Our problem turns out to be a novel Coulomb problem with a quadratic term in the lagrangian and a time-dependent coefficient. By applying some canonical transformations, a concise formula for the system's Feynman propagator was derived.

Key words: plasma, magnetic field, Zeeman separation, Feynman propagator, Lagrangian, Coulomb problem.

Résumé

Dans cette thèse, nous avons trouvé une solution au problème quantique des ions de type hydrogène sous un champ magnétique fort et non uniforme. Ce champ magnétique intense est la somme de deux champs à géométrie toroïdale et poloidale d'intensité égale au tiers du toroïdal. Les résultats révèlent les profils de raies (Lyman-alpha) de trois types d'ions, He^{+}, C^{5+}, et Ar^{17+}, pour différentes intensités du champ magnétique. la séparation Zeeman augmente avec le champ magnétique et le numéro de charge Z et on s'aperçoit que la composante Zeeman gauche ressemble au profil de raies sans champ magnétique.

Ensuite, nous avons calculé le propagateur de Feynman de l'atome d'hydrogène dans la limite non relativiste où les atomes d'hydrogène obéissent à un champ électrique dépendant du temps qui respecte la statistique de bruit blanc gaussien. Notre problème s'avère être un nouveau problème de Coulomb avec un terme quadratique dans le lagrangien avec un coefficient dépendant du temps. En appliquant les transformations canoniques appropriées, une formule concise pour le propagateur de Feynman du système a été trouvée.

Mots clés: plasma, champ magnétique, séparation de Zeeman, propagateur de Feynman, Lagrangien, problème de Coulomb.