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Les Méthodes Multigrilles pour Les Inéquations Quasi Variationnelles Elliptiques

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DEDICATED TO

MY

BELOVED PARENTS

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Abstracts

Abstract in English

In this dissertation, multigrid methods have been investigated for solving certain classes of obstacle problems. In the first part of this thesis, we investigated the solution of elliptic quasi-variational inequalities arising in discretization by the finite element method, in this case, we have chosen a linear right-hand side and the variational form associated with linear operator. For these problems, we have proposed a standard multigrid approach for solving a linear system obtained. Otherwise, we have proposed a nonlinear multigrid method for elliptic quasi-variational inequalities with nonlinear source terms and nonlinear variational form.

The L_∞ norm convergence of these two approaches has been constructed which demonstrates that the multigrid method has a contraction number with respect to the L_∞ norm. Numerical results which demonstrate the high efficiency of these methods are given for a quasi-variational inequality arising from impulse control problem on a domain with nonpolygonal boundaries. From these numerical results, we have seen that the multigrid method proves to be more efficient than the other iterative methods.

Keywords and phrases: *Finite element method, Quasi-Variational Inequalities, HJB Equation, Newton's method, Multigrid Method.*

Abstract in French

Dans cette thèse, nous avons étudiées une classe des inéquations quasi- variationnelles, dans le sens ou l'obstacle dépend de la solution, en appliquant les méthodes multigrilles.

En premier lieu, nous avons étudiées la solution des inéquations quasi-variationnelles elliptiques apparaissant dans la discrétisation par la méthode des éléments finis, dans le cas, ou le second membre et l'opérateur sont linéaires.

Après avoir établir la convergence uniforme de la méthode multigrille linéaire, nous avons généralisées ce résultat de convergence pour les inéquations quasi-variationnelles elliptiques dans le cas ou le second membre et l'opérateur sont non-linéaires, en appliquant une methode multigrille non linéaire.

Les résultats numériques qui démontre la grande efficacité de ces méthodes sont donnés pour une inéquations quasi-variationnelle issue d'un problème de contrôle impulsionnel sur un domaine à frontières non polygonales. A partir de ces résultats numériques, ile est evident que la méthode multigrille s'avère plus efficace que les autres méthodes itératives.

Mots-clés: *inégalité quasi-variationnelle, méthode des éléments finis, équation HJB, méthode multigrille, méthode de Newton.*

الملخص بالعربية

في هذه الأطروحة، تمت دراسة الطريقة متعددة الشبكات لحل المتراجحات الشبه تغايرية الناقصية اين يتعلق الحل بالحاجز, في هاته المسألة اخترنا شكل تغايري يتعلق بمؤثر خطي, ثم اعتمدنا على طريقة العناصر المنتهية لاستخراج الشكل المتقطع لهاته المسألة. إستعملنا الطريقة متعدد الشبكات لحل النظام الخطي الذي تم الحصول عليه. ثم اقترحنا طريقة غير خطية متعددة الشبكات لحل المتراجحات الشبه تغايرية الناقصية الغير خطية.

تم إنشاء التقارب المنتظم للتقديرين المُقترحين مما يثبت أن الطرق متعددة الشبكات لها رقم تقلص مُتعلق بالمعيار المنتظم. النتائج العددية التي تأكد ما أثبتناه نظرياً مطبقة على متراجحة شبه تغايرية ناقصية متعلقة بمشاكل التحكم. من هذه النتائج العددية ، رأينا أن الطريقة متعددة الشبكات أثبتت أنها أكثر كفاءة من الطرق التكرارية الأخرى.

الكلمات المفتاحية: متراجحات شبه تغايرية ناقصية، طريقة العناصر المنتهية، معادلة HJB، الطريقة متعددة الشبكات، طريقة نيوتن.

Contents

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Symbols and Notations

Description	Notation
A domain (bounded and open set)	$\Omega \subset \mathbb{R}^d,\ d = 2,3$
The real numbers	$\mathbb{R}$
Lipschitz continuous boundary domain	$\partial\Omega$
Absolute value	$\|.\|$
A vector norm	$\|\| . \|\|$
The inner product	$(.,.)$
The space of continuous functions which posses at least k continuous derivatives	$C^k(\Omega)$
The space of square integrable functions	$L^2(\Omega)$
A sobolev spaces	$W^{k,p}(\Omega), H^k(\Omega)$
A gradient of a vector g	∇g
The Jacobian matrix of a function g	$\mathbb{J}_g$
A square identity matrix	I_n
The transpose of matrix M	M^T
The inverse of matrix M	M^{-1}
Restriction operator (interpolating from fine to coarse levels)	R_k
prolongation operator (interpolating from coarse to fine levels)	P_k
The inverse of matrix M	M^{-1}

1 Introduction

1.1 Research hypothesis and objectives

Multigrid methods is a framework for solving problems with many variables and multiscales. It have been developed for elliptic PDEs, but have also found application among various problems such as parabolic and hyperbolic PDEs, integral equations, evolution problems, Image reconstruction, etc.

In a brief history of multigrid methods, in the 1960s Fedorenko [24, 25] and Bakhvalov [1] first conceived the multigrid methods for the Poisson equation. While A.Brandt [14] realized the practical utility of these techniques for more general problems in the 1973. Since then a vast amount of literature on multigrid methods has been introduced. For detailed introductions to multigrid, we refer to the publications [54, 51, 17] and the overview provided in [28]. A detailed presentation of the multigrid idea is further given in [50, 15]. Also available is a set of videoed multigrid lectures [52, 53]. The objective of the Multigrid method was to accelerate the convergence of the relaxation iterative methods. So far have been established that many iterative methods have the smoothing property. This property makes these methods very effective to eliminate the high-frequency components of the error while the low-frequency components are relatively unchanged. The multigrid idea is to modify the iterative methods in some way to make them more effective on all error components.

The main aim of this thesis is to extend the application of multigrid to one of the most important free boundary problems, which are called the obstacle problems. As seen below, *the obstacle problems may be written as variational and quasi-variational inequalities. in studying these problems, we begin by investigating the existence, uniqueness and the regularity of the defined solution, which were proposed by C.Dumont in [22]. To solve these problems numerically, we use the finite element method (FEM), the reason for choosing to use the finite element method rather than the finite difference method (FDM) is due to several advantages: A most attractive advantage of the FEM is its ability to handling complex geometries, boundaries, and differential operators efficiently. In addition, its ability to enforce the obstacle constraints because the FEM approaches the underlying function space while the finite difference method approaches the differential equation. Moreover, since we cannot calculate the exact error of the solution, FEM provides an adaptive algorithms for deriving an efficient prioir and posteriori error estimates. For a more detailed discussion about the construction and the error analysis of FEM we refer to the literature [35, 18, 26].*

In this work, once the discrete system has been constructed and justified, we use an iterative procedure developed by Hoppe in [33] for solving this system, this iterative scheme is based on the reformulation of the quasi-variational inequalities as a Hamilton-Jacobi-Bellman (HJB)-equation, which can be interpreted as an optimal stochastic switching problem see [36, 39, 40, 4].

This thesis is briefly outlined as follows:

In chapter 2, a general formulation of variational and quasi-variational inequalities is introduced, then we recall the existence, uniqueness and the regularity results of the solutions of the obstacle problems, which are particular cases of variational inequalities, the readers are referred to [35] to get more detail about the results that are briefly explained here.

Chapter 3, is an essential chapter concerned to applying multigrid methods for solving linear and nonlinear systems, it is essential for understanding how multigrid methods can be extended to obstacl problems. In the beginning of this chapter we give a short introduction to the basic concepts of the splitting iterative methods, then the multigrid process has been developed in three steps: two-grid iteration, multigrid iteration, and nested iteration. Finaly, a numerical example for solving a linear model problem is given to test the efficiency of the multigrid method proposed.

In chapter 4, we treat multigrid method for the solution of one-sided obstacle problems based on restating quasi-variational inequalities (QVI) as a Hamilton-Jacobi-Bellman (HJB)-equations. a great deal of work for the analysis of elliptic quasi-variational inequalities related to the HJB-equation is given in [23, 12, 10], to get more details about the regularity of the continuous problem and the discretization error bound we refer to [2, 12, 10, 9, 11, 8, 7]. In Multigrid methods, we try to reduce the amount of geometric information (i.e, make these methods more algebraic), for a thorough treatment of these methods we refer to the monograph [28]. For the discretization, we use the finite element method to get a large sparse linear system of equations and then use an iterative scheme *established by Hoppe in [33] to solve this system. However, In contrast to Hoppe [33] we apply a multigrid method directly to solve the algebraic systems obtained by the HJB reformulation 4.3. The convergence results proposed by Arnold for elliptic PDEs [48] yield the multigrid L^∞-convergence result described in this work. The proof of these results is based on the smoothing property and the approximation property which introduced by Hackbusch [28]. The analysis of the smoothing property presented here is the same as for the elliptic PDEs in [48], and as is shown in [29], the main points in the proof of the approximation property are the finite element L^∞-error estimate given in the theorem 4 and the results of the lemma 2, which do not depend on the bilinear form, but only on the triangulation.*

In chapter 5, Newton-multigrid method is applied for solving obstacle problems based on restating the nonlinear quasi-variational inequalities as an H.J.B-equation. For a discretization, finite element method has been used to derive a discret system, as in chapter (4) the iterative procedure proposed by Hoppe is used to solve the obtained system. Then we apply Newton-multigrid method for solving the nonlinear algebraic systems obtained, we do this by appling Newton's method to linearizing the nonlinear system and then we use multigrid method for the solution of Jacobian system in each iteration step. For the inner iteration, the maximum norm convergence of the multigrid method established by Arnold for elliptic PDEs [48] is used this problem. According to Hackbusch [28], the demonstration of these results is based on the combinaison of the smoothing property and the approximation property. For the outer iteration, It have been shown that Newton's method converges quadratically if the approximate solution is close to the actual solution of the nonlinear system. Finally, the dissertation is concluded and we give an outlook to future work.

1.2 Mathematical preliminaries

Here we recall some essential definitions which are needed in this study for more details see [35, 45, 20].

Let Ω be an open in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$, and let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d$ be a multi-index such that $|\alpha| = \sum_{i=1}^{d} \alpha_i$, we set $D^\alpha f$ as the partial derivative written as

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}.$$

Definition 1 *The space $C^k(\Omega)$ is defined as a set of continuous functions possessing at least k continuous derivatives.*

In particular, when $k = 0$, C^0 denote the space of continuous functions. And when $k = 1$, C^1 denote the space of continuously differentiable functions. And when $k := \infty$, C^∞ denote the space of functions which are continuous and its derivatives of all orders are continuous.

Definition 2 *$L^p(\Omega)$ for $(1 \leq p \leq \infty)$ is a Banach space (i.e., normed vector space that is complete with respect to its norm) of the measurable functions f defined on Ω, such that, $|f(x)|^p$ is integrable.*

In particular, when $p = 2$, we get a Hilbert space $L^2(\Omega)$ (i.e., a complete inner product space) of the square integrable functions.

Definition 3 *$L^\infty(\Omega)$ is a Banach space of all measurable functions that are bounded almost everywhere in Ω.*

Definition 4 *An operator $a : V \times V \longrightarrow \mathbb{R}$ is called bilinear form on a real valued vector Hilbert space V, if it is linear for each argument, i.e. $\forall u, v \in V$ and $\lambda \in \mathbb{R}$*

$$a(u + v, w) = a(u, w) + a(v, w), \tag{1.1}$$

$$a(u, v + w) = a(u, v) + a(u, w), \tag{1.2}$$

$$a(\lambda u, v) = a(u, \lambda v) = \lambda a(u, v), \tag{1.3}$$

are hold.

- *A bilinear form $a : V \times V \longrightarrow \mathbb{R}$ is called positive definite, if for all $u, v \in V$*

$$a(u, u) \geq 0 \tag{1.4}$$

$$a(u, u) = 0 \Leftrightarrow u = 0. \tag{1.5}$$

- *A bilinear form $a : V \times V \longrightarrow \mathbb{R}$ is called symmetric, if*

$$a(u, v) = a(v, u), \quad \forall u, v \in V \tag{1.6}$$

Definition 5 *The scalar product* $(.,.) : V \times V \longrightarrow \mathbb{R}$ *is a symmetric and positive definite bilinear form on the real valued vector Hilbert space V.*

Definition 6 *The Sobolev space* $W^{k,p}(\Omega)$ *is a Banach space defined as*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) | D^\alpha f \in L^p(\Omega),\ |\alpha| \leq k\}, \tag{1.7}$$

when is equipped with the norm

$$\| f \|_{W^{k,p}(\Omega)} = (\textstyle\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha f|^p d\Omega)^{\frac{1}{p}}.$$

In particular, when $p = 2$, *we define the Sobolev space* $H^k(\Omega)$ *as*

$$H^k(\Omega) = \left\{f \in L^2(\Omega) | D^\alpha f \in L^2(\Omega),\ |\alpha| \leq k\right\}, \tag{1.8}$$

$H^k(\Omega)$ *is the inner product space if is equipped with the inner product* $(.,.)_{H^k}$

$$(.,.)_{H^k} = \textstyle\int_\Omega (\sum_{|\alpha|\leq k} (D^\alpha f)(D^\alpha g) d\Omega), \quad f, g \in H^k(\Omega).$$

From this, we get the associated Sobolev norm

$$\| f \|^2_{H^k(\Omega)} = \textstyle\int_\Omega (\sum_{|\alpha|\leq k} (D^\alpha f)^2 dx).$$

Definition 7 *Let H be a Hilbert space, then the bilinear form* $a : H \times H \longrightarrow \mathbb{R}$ *is called H-elliptic if the following inequality*

$$a(u,u) \geq C \| u \|^2_H, \quad \forall u \in H, \tag{1.9}$$

holds.

Definition 8 (Matrix Properties) *Let* $A = (a_{sl})_{s,l\in\mathbb{N}} \in \mathbb{R}^{N\times N}$ *be a square matrix, we say that:*

- A^T *is the transposed matrix if* $A^T = (a_{ls})_{s,l\in\mathbb{N}}$.
- $\bar{A}$ *is the complex conjugate matrix if* $\bar{A} = (\bar{a}_{sl})_{s,l\in\mathbb{N}}$.
- $\bar{A}^T$ *is the adjoint matrix if* $\bar{A}^T = (\bar{a}_{sl})_{s,l\in\mathbb{N}}$.
- *The matrix A is symmetric if* $A^T = A$.
- *The matrix A is Hermitian if* $\bar{A}^T = A$.
- *The matrix A is invertible if there exists* $B \in \mathbb{R}^{N\times N}$, *such that*

$$BA = AB = I \quad (B = A^{-1}),$$

where I is the identity matrix $I = \{\delta_{sl}\}_{s,l\in\mathbb{N}}$ *(* δ_{sl} *define the Kronecker symbol).*

- *The matrix A is regular if A^{-1} exists.*
- *The matrix A is said to be positive definite if:* $\forall X \in \mathbb{R}^N, \quad X^T A X > 0, \quad \forall X \neq 0.$
- *The matrix A is said to be diagonally dominant if:*

$$|a_{ss}| \geq \sum_{s=1,\ l\neq s}^{s=N} |a_{sl}|, \quad l = 1, ..., N.$$

- *The matrix A is said to be strictly diagonally dominant if:*

$$|a_{ss}| > \sum_{s=1,\ l\neq s}^{s=N} |a_{sl}|, \quad l = 1, ..., N.$$

- *The matrix A is said to be sparse if the most of its entries are zeros.*
- *The matrix A is said to be banded matrix if the matrix diagonal bands contain all non-zero elements.*
- *The matrix A is said to be M-matrix if A^{-1} exists, such that:*

$$a_{ss} > 0 \text{ and } a_{ls} \leq 0 \quad (l \neq s).$$

Definition 9 *A real or complex λ is called eigenvalue of A if $AX = \lambda X \quad (X \in \mathbb{K}^N (\mathbb{K} = \mathbb{R}$ or $\mathbb{C}))$.*

The eigenvalues of A are the roots of the polynomial $\det(A - \lambda I)$.
$\sigma(A)$ denote the set of eigenvalues of the matrix A, which is called the spectrum of the matrix A.

$$\sigma(A) = \{\lambda \in \mathbb{K} | \quad \det(A - \lambda I) = 0\}.$$

The largest absolute value of the spectrum of A is called spectral radius of A, it is denoted by $\rho(A)$

$$\rho(A) = \max\{|\lambda| : \quad \lambda \in \sigma(A)\}.$$

The vector $X \in \mathbb{K}^N$ is said to be eigenvector of the matrix A if $AX = \lambda X$, with $X \neq 0$.

2

Variational Inequalities and Quasi-Variational Inequalities

In the last fifty years, variational inequalities have become a relevant tool in the study of nonlinear problems in physics and mechanics. In this chapter, we focus on a particular class of variational inequalities, namely obstacle problems. Obstacle problems were one of the main motivations of the theory of variational inequalities and have many important applications in various areas. We recall here some essential results of variational inequalities introduced by G. Stampacchia [35].

2.1 Variational inequalities (Continuous problem)

2.1.1 Definitions and hypotheses

Definition 10 *In what follow, we assume that $\Omega \in \mathbb{R}^N$ is a bounded open set, and we denote by $\partial\Omega$ for its sufficiently smooth boundary. Let V and W two Hilbert spaces associated with the norm $\| . \|$, K and Q two closed convex subsets of V and W respectively. V^* the dual of V and $\langle .,. \rangle$ the dual product between V and V^* and $f \in V^*$. Let $a(u,v) : V \times V \longrightarrow \mathbb{R}$ be a continuous V-elliptic (coercive) bilinear form of $V \times V$, i.e*

$$\exists M > 0 \quad \textit{such that} \quad a(u,v) \leq M \| u \| \| v \|, \quad \forall u, v \in V \qquad \textit{(continuous)}$$
$$\exists \alpha > 0 \quad \textit{such that} \quad a(u,u) \geq \alpha \| u \|^2, \quad \forall u \in V \qquad \textit{(coercive)}$$

The following problem:

$$\begin{cases} Find\ u \in K\ such\ that: \\ a(u, v-u) \geq \langle f, v-u \rangle, \quad \forall v \in K \end{cases} \tag{2.1}$$

is called variational inequality. By the Riesz representation theorem for Hilbert spaces [45], there exist a differential operator $A \in \ell(V, V)$ such that

$$a(u, v) = \langle Au, v \rangle, \quad \forall u, v \in K$$

Thus, the problem (2.1) is equivalent to

$$\begin{cases} Find\ u \in K\ such\ that: \\ \langle Au, v-u \rangle \geq \langle f, v-u \rangle, \quad \forall v \in K. \end{cases} \tag{2.2}$$

Remark 11 *If $K = V$, the problem (2.1) reduces to the classic variational equation:*

$$\begin{cases} Find\ u \in K\ such\ that: \\ a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \end{cases} \tag{2.3}$$

Definition 12 *An operator $A : V \longrightarrow V^*$ is called monotone if*

$$\langle Au - Av, u - v \rangle \geq 0, \qquad \forall u, v \in V.$$

The monotone operator A is called strictly monotone if

$$\langle Au - Av, u - v \rangle = 0, \quad implies \quad u = v.$$

Definition 13 *The operator $A : V \longrightarrow V^*$ is called continuous if there exists a constant $\beta > 0$ such that*

$$\langle Au - Av, w \rangle \leq \beta \| u - v \|_V \| w \|_V, \quad \forall u, v, w \in V.$$

The operator A is called Lipschitz operator if there exists a constant $\eta > 0$ such that

$$\| Au - Av \|_{V^*} \leq \eta \| u - v \|_V, \qquad \forall u, v \in V.$$

The problem (2.1) can be rewritten as the following energy minimization problem:

$$\begin{cases} Find\ u \in K\ such\ that: \\ \min_{v \in K} \varrho(v) := \frac{1}{2} a(v, v) - \langle f, v \rangle. \end{cases} \tag{2.4}$$

In this part, we recall the regularity results of the obstacle problem, which is a special case of variational inequalities.

Definition 14 (Elliptic Obstacle Problems) *Suppose in Problem (2.1), the convex set K has the following structure*

$$K = \{w \in V,\ w \leq \Psi,\ in\ \Omega,\ w = 0\ on\ \partial\Omega\}, \quad V = H_0^1(\Omega).$$

The corresponding variational inequality:

$$\begin{cases} Find\ u \in K\ such\ that: \\ a(u, w-u) \geq \langle f, w-u\rangle \qquad w \in K, \end{cases} \tag{2.5}$$

is called elliptic obstacle problem. Where $f \in L^\infty(\Omega)$, and the function $\Psi \in W^{2,\infty}$ is the so-called obstacle.

Suppose $u \in K$ is the solution of the obstacle problem (2.5), the set of points

$$\Re(u) := \{x \in \Omega | u(x) = \Psi(x)\}.$$

is called the contact set or coincidence set, and its complement

$$\Im(u) = \Omega \setminus \Re(u),$$

called a noncontact set or non-coincidence set. The boundary $\Gamma(u)$ between the two sets is called the free boundary or free interface.

2.1.2 Existence and regularity of the solution of the obstacle problem

We next present a theorem which ensure the existence and the uniqueness of the solution of the problem (2.5).

Theorem 15 ([35]) *According to the preceding conditions the problem (2.5) has a unique solution $u \in K$ Moreover we have*

$$u \in W^{2,p}(\Omega),\ 2 \leq p \leq \infty. \tag{2.6}$$

Proof.

1-The uniqueness

Let u_1 and u_2 are two solutions to the problem (2.5), then we have

$$a(u_1, w-u_1) \geq \langle f, w-u_1\rangle, \qquad w \in K, \tag{2.7}$$

$$a(u_2, w-u_2) \geq \langle f, w-u_2\rangle, \qquad w \in K. \tag{2.8}$$

Put $w = u_2$ in (2.8) and $w = u_1$ in (2.8), summing the two inequalities, we get

$$a(u_2 - u_1, u_2 - u_1) \leq 0. \tag{2.9}$$

Using the coercivity of $a(.,.)$, we get

$$\alpha \| u_2 - u_1 \|^2 \leq a(u_2 - u_1, u_2 - u_1) \leq 0, \tag{2.10}$$

hence

$$u_2 = u_1.$$

2-The existence

The idea is to transform the problem (2.5) to a fixed point problem. Indeed, we have

$$a(u, w-u) = \langle Au, w-u \rangle, \qquad \forall w \in K,$$

also, the problem (2.5) is equivalent to

$$\begin{cases} Find\ u \in K\ such\ that : \\ \langle Au - f, v-u \rangle \geq 0, \quad \forall w \in K. \end{cases} \tag{2.11}$$

Let t be such that $t > 0$, then

$$\langle -t(Au - f), v-u \rangle \leq 0, \quad \forall w \in K.$$

which implies

$$\langle u - t(Au - f) - u, v-u \rangle \leq 0, \quad \forall w \in K.$$

Let π_k be the orthogonal projection of V onto K, then problem (2.5) is equivalent to the following fixed point problem

$$\begin{cases} Find\ u \in K\ such\ that : \\ u = \pi_k(u - t(Au - f)), \quad for\ \ t{>}0. \end{cases} \tag{2.12}$$

One can easily prove that π_k is a contraction on$V$.

Let w_1, $w_2 \in V$, then

$$\begin{aligned} \| \pi_k(w_1 - t(Aw_1 - f)) - \pi_k(w_2 - t(Aw_2 - f)) \|^2 &\leq \| (w_1 - t(Aw_1 - f)) - (w_2 - t(Aw_2 - f)) \|^2 \\ &= \| (w_1 - w_2) - t(A(w_1 - w_2)) \|^2 \\ &= \| w_1 - w_2 \|^2 - 2ta(w_1 - w_2, w_1 - w_2) + t^2 \| A(w_1 - w_2) \|^2 \\ &\leq (1 - 2t\gamma + t^2\Lambda) \| w_1 - w_2 \|^2 \end{aligned}$$

Choose t such that $0 < t < \frac{2\gamma}{\Lambda^2}$ Therefore π_k is a contraction, and by the theorem of fixed point of Banach, there exists a unique solution $u \in K$ ■

2.2 Variational inequalities (Discrete problem)

2.2.1 Notations and hypotheses

Consider a conform finite elements space constructed from polynomials of degree 1. We introduce a family of regular, quasi-uniform triangulations T_h of Ω, and consider the conform finite elements space V_h.

$$V_h = \left\{ v_h \in C(\Omega) \cap H^1(\Omega) \mid v_{h|\Omega_h} \in P_1 \right\}. \tag{2.13}$$

Let $X_s,\ s = 1, 2, ..., m(h)$ be the vertices of the triangulation which do not belong to $\partial\Omega$. The standar basis functions $\varphi_s,\ s = 1, 2, ..., m(h)$ are defined by

$$\varphi_h^i\left(X_h^j\right) = \delta_{ij}, \quad (\ \delta \text{ is the Kronecker symbol}).$$

We also define the restriction operator r_h by:

$$r_h v(X) = \sum_{i=1}^{m(h)} v\left(X_h^i\right)\varphi_h^i(X), \quad \forall v \in C\left(\bar{\Omega}\right) \cap H^1(\Omega). \tag{2.14}$$

The order on V_h will be that induced by $\mathbb{R}^{m_k}$.

We naturally introduce the discretization matrix A_h of generic coefficients

$$a(\varphi_k^l(x), \varphi_k^s(x)), \quad s \in (1, ..., m(h_k)).$$

Consider the discrete problem associated with the problem 2.5: Find $u_h \in V_h$ such that

$$\begin{cases} \langle A_h u_h, v_h - u_h \rangle \geq \langle f, v_h - u_h \rangle, & \forall v_h \in V_h, \\ u_h \leq \Psi_h, \quad v_h \leq \Psi_h. \end{cases} \tag{2.15}$$

2.2.2 Existence and regularity of the discrete solution

Assume that A_h is an M-matrix, it means the discrete maximum principle holds.

In [20], we find simple geometric conditions related to the triangulation, for which the A_h is an M-matrix.

Theorem 16 ([26]) *According to the preceding notations and assumptions, the problem 2.15 admits a unique solution.*

As in the continuous case, the regularity of the discrete solution of the V.I 2.15 repose on the following theorem.

Theorem 17 **([22])** *There exists a constant C independent of h such that:*

$$|a(u_h,\ \varphi_s)| \leq C\|\varphi_s\|, \quad \forall \varphi_s,\ s = 1,2,...,m(h).$$

The following theorem due to Cortey-Dumont [22], establishes the uniform convergence of the discrete solution u_h to the continuous solution u.

Theorem 18 **([22])** *Under the preceding notations and hypotheses and the (d.m.p), there exists a constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^2 \left|\ln h\right|^2. \tag{2.16}$$

2.3 Quasi-variational inequalities (Continuous problem)

2.3.1 Notations and hypotheses

We are interested in the study of the quasi-variational inequality (I.Q.V) when the obstacle depends on the solution u. More precisely let $K_g(u)$ an implicit convex and non empty set defined as

$$K_g(u) = \{v \in V,\ v = g\ on\ \partial\Omega,\ v \leq Mu,\ in\ \Omega\}, \quad V = H^1(\Omega). \tag{2.17}$$

where g is a regular function defined on $\partial\Omega$. And the operator

$$M : V \cap L^\infty(\Omega) \longrightarrow V \cap L^\infty(\Omega)$$

is defined as

$$\begin{aligned} Mu &= \kappa + \inf_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u(x+\varepsilon), \quad \kappa\ \text{is a positive constant,} \\ M &\in W^{2,\infty}(\Omega),\ Mu \geq 0,\ on\ \partial\Omega : 0 \leq g \leq Mu. \end{aligned} \tag{2.18}$$

Let $a(,)$ be a continuous, coercive and strictly T-monotone bilinear form, consider the following quasi-variational inequality

$$\begin{cases} Find\ u \in K_g(u)\ such\ that : \\ a(u, v-u) \geq \langle f, v-u \rangle, \quad \forall v \in K_g(u) \end{cases} \tag{2.19}$$

2.3.2 Existence and regularity of the solution of Q.V.I

We present a theorem which ensure the existence and the uniqueness of the solution of the problem (2.19). The existence and uniqueness of the solution of Q.V.I is deduced from the *construction of a sequence of solutions of variational inequalities which converges to the solution of Q.V.I.*

Theorem 19 **([31])** *Under the preceding conditions the problem (2.19) has a unique solution $u \in K_g(u)$ Moreover we have*

$$u \in W^{2,p}(\Omega),\ 2 \le p \le \infty. \tag{2.20}$$

Lemma 20 **([41])** *For all $u, v \in K_g(u)$ the operator M satisfies*

$$\|Mu - Mv\|_{L^\infty(\Omega)} \le \|u-v\|_{L^\infty(\Omega)}, \forall u, v \in L^\infty(\Omega), \tag{2.21a}$$
$$u \le v \Longrightarrow Mu \le Mv, \tag{2.21b}$$
$$M(u+\lambda) = Mu + \lambda, \forall \lambda \in \mathbb{R}. \tag{2.21c}$$

In practice, this type of inequality (2.19) appears in the field of inventory management, which is related to impulse control problems.

2.4 Quasi-variational inequalities (Discrete problem)

2.4.1 Notations and hypotheses

We use the same triangulation of the previous paragraph 2.2. Let

$$f \in L^\infty(\Omega), \tag{2.22}$$
$$M_h u_h = K + \inf_{\epsilon \ge 0,\ (x+\epsilon) \in \bar{\Omega}} u_h(x+\epsilon),\ k \text{ is a positive constant}, \tag{2.23}$$
$$K_{g,h} = \{v \in V_h : v = \pi_h g \text{ on } \partial\Omega, v \le M_h u_h \text{ in } \Omega\}. \tag{2.24}$$

Denoted by π_h the interpolation operator on $\partial\Omega$. The analogous discretization of the problem (2.19) by finite element method leads to the solution of the following problem

$$\begin{cases} \text{Find } u_h \in K_{g,h}(u_h) \text{ such that :} \\ a(u_h, v_h - u_h) \ge \langle f, v_h - u_h \rangle, \quad \forall v_h \in K_{g,h}(u_h) \end{cases} \tag{2.25}$$

2.4.2 Existence and regularity of the discrete solution of Q.V.Is

Theorem 21 **([21])** *Assume that the discrete maximum principle holds. Then under the preceding conditions the problem (2.25) has a unique solution.*

Theorem 22 **([22])** *Let u and u_h be the solutions of problems (2.19) and (2.25) respectively, then there exists a constant C independent of h_h such that:*

$$\|u - u_h\|_{L^\infty(\Omega)} \le Ch^2 \left|\log h\right|^2. \tag{2.26}$$

Similarly, as in the continuous case, we will establish the discrete version of the lemma 20.

Lemma 23 ([41]) *For all $u_k, v_k \in K_{g,k}(u)$ the operator M_k satisfies*

$$\|M_k u_k - M_k v_k\|_{L^\infty(\Omega)} \leq \|u_k - v_k\|_{L^\infty(\Omega)}, \forall u, v \in L^\infty(\Omega), \tag{2.27a}$$

$$u_k \leq v_k \implies M_k u_k \leq M_k v_k, \tag{2.27b}$$

$$M_k(u_k + \lambda) = M_k u_k + \lambda, \forall \lambda \in \mathbb{R}. \tag{2.27c}$$

3
Multigrid Methods

The methods for solving linear systems are classified into two main families: direct methods and iterative methods. The choice of a method for solving linear systems is often guided by the properties of the sparse matrix of the system to be solved (structure, symmetry, density, etc.), speed and the desired accuracy of the resolution. Since direct methods are not suitable for solving very large sparse linear systems, especially those associated with three-dimensional systems (direct solvers are very expensive in terms of resolution time).

	TRIDIAGONAL MATRIX	FULL MATRIX	
		Factorization	**Solving system**
Gauss elimination method		$\frac{2n^3}{3}$	n^2
Thomas method	2n		
LU factorization method		$\frac{2n^3}{3}$	$2n^2$
Cholesky factorization method		$\frac{n^3}{3}$	$2n^2$

Table 3.1: The number of iterations needed for solving a systems that possess n unknowns. e.g. Lay (2006) and Gander (1997).

Although iterative methods are more useful than direct methods. However, it's well-

known that iterative methods could converge very slowly, especially in large systems(see for example 3.12 and 3.11). In this chapter, we use one of the most powerful tool to accelerate the convergence of these methods, which is called Multigrid method. In the second section of this chapter, we have seen that multigrid methods can be successfully applied to a large class of problems since the residual equation obtained (in the coarse-grid correction step) is linear. But, this is not true when dealing with nonlinear problems. Therefore, two different basic approaches are required in this case. The first one is to apply a standard linearization and use multigrid to solve the Jacobian system on each linearization step (it has been shown in 5). The second one, which called the Full Approximation Scheme, it is briefly described in the third section. Finally, a numerical example of a simple model problem is used to demonstrate the performance of this method.

3.1 Splitting iterative methods

Direct methods are good for dense matrices with full row rank. However, for a large sparse systems become unacceptable. That's why the iterative methods are applied, we do this here to illustrate a smoothing property that is important for multigrid methods. We consider a system of linear equations

$$Au = f, \tag{3.1}$$

where $A \in \mathbb{R}^N \times \mathbb{R}^N$ *is an invertible matrix,* $f \in \mathbb{R}^N$ *is the right hand side and* $u \in \mathbb{R}^N$ *is the unknown vector. The iterative method starts with a start guess* u_0 *for the solution, which is improved at every step, until we get the desired solution. To measure the accuracy of the solution, there is two important measures: One way is the solution error*

$$e_k := u - u_k. \tag{3.2}$$

However, when the exact solution u *is not known a priori, in practice we use the residual*

$$d_k := f - Au_k. \tag{3.3}$$

Since $f = Au$*, we get the important residual equation*

$$d_k = A(u - u_k) = Ae_k.$$

In the iterative methods, we replace the matrix A *with a non-singular preconditioning matrix* M*, which should be close to the matrix* A *in the sense that* $M^{-1}d_k$ *is close to* e_k*.*

Replace e_k *by* $M^{-1}d_k$ *in (3.2), we get*

$$u_{k+1} := u_k + M^{-1}d_k, \tag{3.4}$$

which gives

$$u_{k+1} = u_k + M^{-1}(f - Au_k) = (I - M^{-1}A)u_k + M^{-1}f, \tag{3.5}$$

this can be rewritten as

$$Mu_{k+1} = Nu_k + f, \tag{3.6}$$

where $N = M - A$ and $M^{-1}N$ is the iteration matrix. If we decompose the matrix A as $A = U + D + L$, where U is the strictly upper part of A, D is the diagonal of A and L is the strictly lower part of the matrix A. With the choice of the matrix M, the iterative methods (Jacobi method and Gauss-Seidel method) can be defined:
When $M = D$, we get the following Jacobi method:

$$u_{k+1} = (-D^{-1}(L + U))u_k + D^{-1}f. \tag{3.7}$$

And if we put $M = D + L$, we get the following Gauss-Seidel method:

$$u_{k+1} = -(D + L)^{-1}Uu_k + (D + L)^{-1}f. \tag{3.8}$$

(3.8) is called forward Gauss-Seidel method, if we replace U by L and L by U in (3.8) we obtain the following backward Gauss-Seidel method

$$u_{k+1} = -(D + U)^{-1}Lu_k + (D + U)^{-1}f. \tag{3.9}$$

Usually, this iterative methods are linearly convergent

$$\|u_{k+1} - u\| \leq C\,\|u_k - u\|, \quad k = 0, 1, ... \tag{3.10}$$

where C is a constant $(0 < C < 1)$, using the error (3.2), it follow that

$$\|e_{k+1}\| \leq C\,\|e_k\|, \quad k = 0, 1, ... \tag{3.11}$$

Inserting the error (3.2) into (3.5), we get the error propagation

$$e_{k+1} = M^{-1}Ne_k, \quad k = 0, 1, ... \tag{3.12}$$

where $M^{-1}N$ is called the error operator. These methods are convergent if the spectral radius (convergence factor) of the error operator satisfies

$$\rho(M^{-1}N) = \max_{1 \leq i \leq N} |\lambda_i| < 1, \tag{3.13}$$

where λ_i are eigenvalues of $M^{-1}N$. In most cases, these methods cannot be used as an iterative solver because they are too slow. However as we will see in the next section, they *can be used as a smoother in the multigrid methods.*

It is very important to determine how the individual multigrid components should be selected for concrete situations, for treating this question we will use the local Fourier analysis

In our analysis, we use Jacobi method because its analysis is the most transparent among all basic iterative methods. To simplify the idea, let's start with the Dirichlet boundary value problem in one dimension.

$$-u''(x) = f(x), \qquad for\ x \in]0,1[\,,\ and \quad u(x) = 0 \ \ at\ x \in \{0,1\}. \tag{3.14}$$

Choose Fourier modes as a start iterate for solving the system (3.1) derived from 3.14 (with N =128).

$$u_m = \sin\frac{im\pi}{N}, \qquad 1 \le m \le N-1, \quad 1 \le i \le N-1. \tag{3.15}$$

The figure 3.1 show the modes $u_m = \sin\frac{im\pi}{N}$, with diffrent wavenembers m=1, 3, 12.

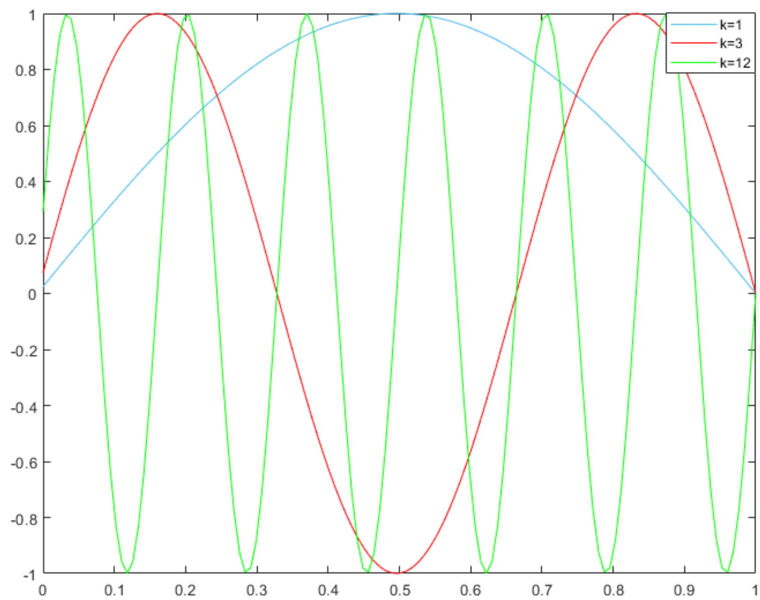

Figure 3.1: Fourier modes with wave nember $m = 1,\ 3,\ 12$

Choose Jacobi matrix $\mathbb{S}_J$ in the error propagation (3.12), we get

$$e^{k+1} = \mathbb{S}_J e^k, \quad k = 0,1,... \tag{3.16}$$

Choosing $e^0 = u_m$, $(m = 1,2,3,4)$ as an initial error, then after n iterations (3.16) becomes

$$e^n = \mathbb{S}_J^n e^0, \quad k = 0,1,... \tag{3.17}$$

Figures (3.2, 3.3, 3.4, 3.5) illustrate the smoothing effect of Jacobi iteration with differents modes.

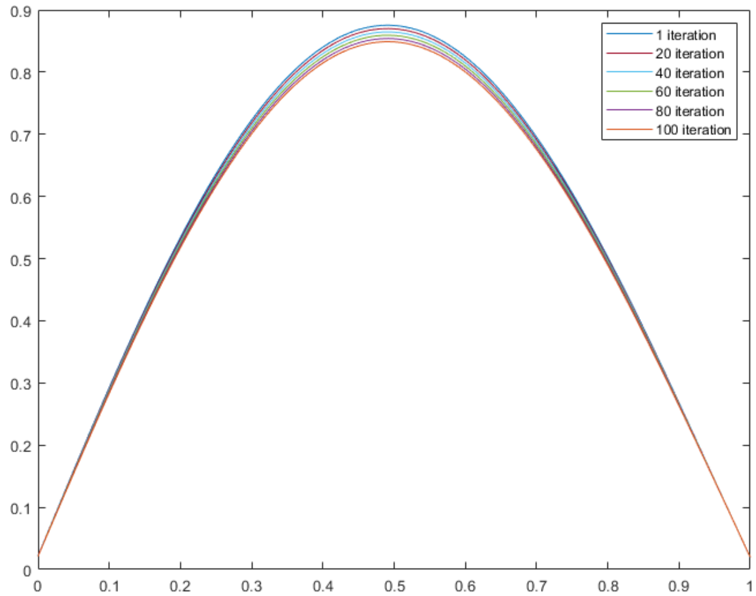

Figure 3.2: Smoothing effect of Jacobi method applied to (3.14) using Fourier mode as a start iterate with k=1, and N=128

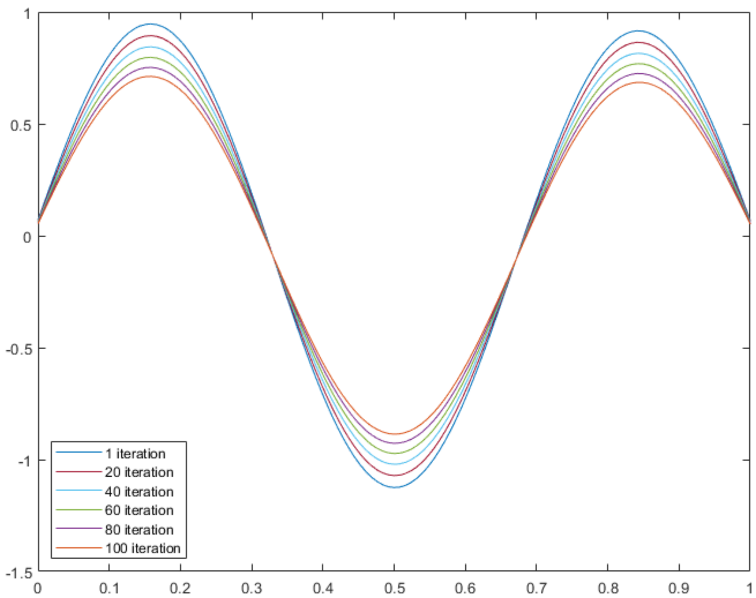

Figure 3.3: Smoothing effect of Jacobi method applied to (3.14) using Fourier mode as a start iterate with k=3, and N=128

Noting that the matrix $\mathbb{S}_J$ has the eigenvectors of the matrix A.

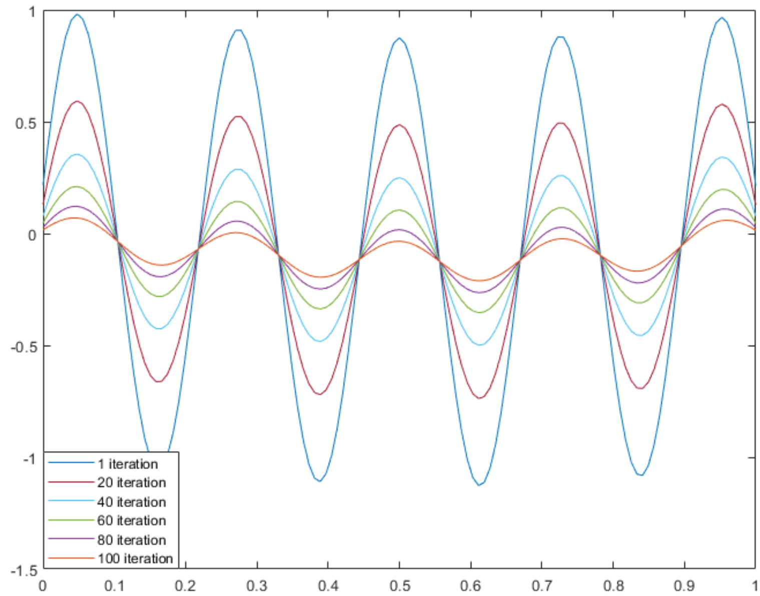

Figure 3.4: Smoothing effect of Jacobi method applied to (3.14) using Fourier mode as a start iterate with k=9, and N=128

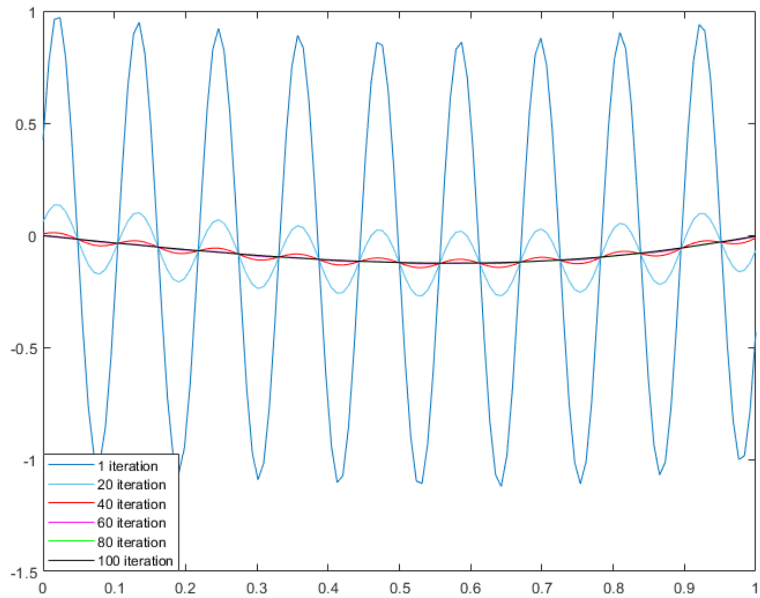

Figure 3.5: Smoothing effect of Jacobi method applied to (3.14) using Fourier mode as a start iterate with k=18, and N=128

Let $\mathbb{S}_J$ has N-1 u_k distinct eigenvectors, so they form a basis:

$$\forall u \in \mathbb{R}^{N-1}, \quad \exists c_k \text{ such that: } u = \sum_{k=1}^{N-1} c_k u_k. \tag{3.18}$$

Let the eigenvalues of $\mathbb{S}_J$ defined as

$$\lambda_k = \cos\frac{k\pi}{N}, \qquad 1 \le k \le N-1, \tag{3.19}$$

and the u_k eigenvectors of $\mathbb{S}_J$ are defined as (3.15).

Expand the start error in 3.18

$$e^0 = \sum_{k=1}^{N-1} c_k u_k, \tag{3.20}$$

then after n iterations we get:

$$e^n = \mathbb{S}_J^n e^0 = \mathbb{S}_J^n \sum_{k=1}^{N-1} c_k u_k = \sum_{k=1}^{N-1} c_k \lambda_k^n u_k. \tag{3.21}$$

Consequently: $\lim_{n\to\infty} e^n \longrightarrow 0$ *if* $|\lambda_k| < 1, \quad 1 \le k \le N-1.$
Which mean that this iterative method converges if the spectral radius is smaller than 1.
Observation (3.6):

since $|\lambda_k| < 1$, then this method converges. However, for some eigenvectors the spectral radius is close to 1, then the convergence is slow. Figure (3.6) shows that the error is

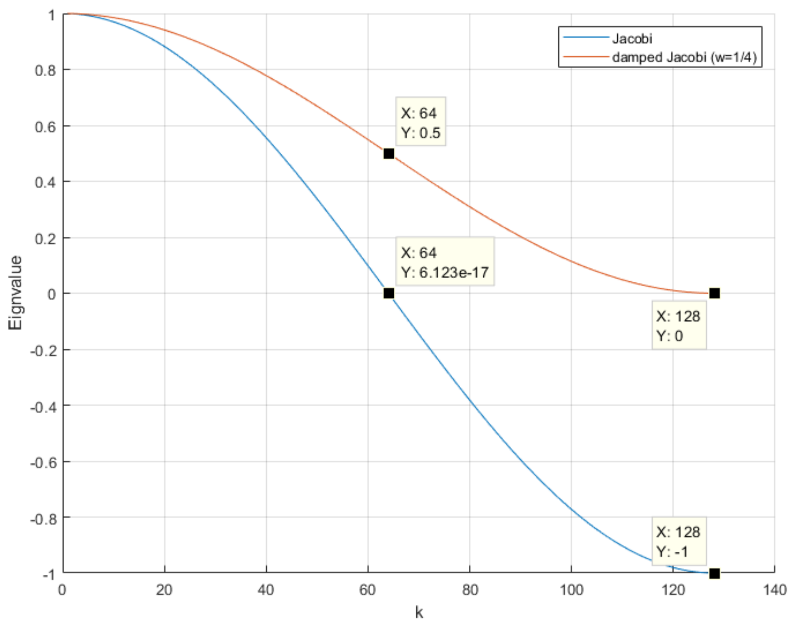

Figure 3.6: Eigenvalues of Jacobi iteration matrix and damped Jacobi iteration matrix as function of the modes m

reduced at least by a factor $\frac{1}{2}$ per iteration for $m \ge \frac{N}{2}$. Therefore, the convergence rate $\rho = \frac{1}{2}$ of those iterations is restricted for $\frac{N}{2} \le m \le N$.

Definition 24 *Eigenvectors are said to be low-frequency if* $1 \leq k < \frac{N}{2}$ *and are called high-frequency if* $\frac{N}{2} \leq k < N$.

As we have seen above, there are many relaxation methods that have the smoothing property to eliminate the oscillatory error modes effectively, while smooth error modes are eliminated very slowly. In the next section we will see how to eliminate effectively the low-frequency error by using the coarse grid correction (Two-grid idea).

3.2 Multigrid methods for solving linear equations

Description of the two-grid method

We consider the following linear elliptic partial differential equations

$$Au = f, \quad in\ \Omega. \tag{3.22}$$

To apply two-grid method for Solving a linear system (3.22), it is necessary to construct a hierarchy

$$h_l < h_{l-1} < ... < h_1$$

of mesh sizes that correspond to the discretized domains Ω_l, $\Omega_{l-1}, ...,$ Ω_1. h_l *denote the mesh size of the finest level.*

The discretization by finite difference or by conforming finite elements of the partial differential equation (3.22) leads to the solution of the linear system

$$A_{h_k} u_{h_k} = f_{h_k}, \quad in\ \Omega_{h_k}. \tag{3.23}$$

Let the fine grid h_k *and the coarse grid* h_{k-1} *are a grids for two discretization levels for the same discrete problem. The idea of two-grid method is to apply a few iterations of relaxation method which have the smoothing property, i.e., serve well to reduce a high-frequency components of the error, while the low frequencies error can be approximated successfully on a coarser grid and reduced there.*
To simplify the notation, we put

$$\Omega_k = \Omega_{h_k},\ f_k = f_{h_k},\ A_k = A_{h_k}.$$

Pre-smoothing

let $\bar{u}_k$ *be the result of* α *iterations of an iterative method applied to a given initial vector* u_k^0, *denoted by*

$$\bar{u}_k = M_{RELAX}(A_k; f_k; u_k^0). \tag{3.24}$$

for the relaxation method to solve (3.23).

Coarse-grid correction

Setting the error $e_k = u_k^* - \bar{u}_k$ *and the residual* $d_k = f_k - A_k\bar{u}_k$ *we have*

$$A_k e_k = f_k - A_k\bar{u}_k = d_k. \tag{3.25}$$

Therefore, to determine e_k *entirely, we require to compute* e_{k-1} *at level* $(k-1)$ *as the solution of the coarse grid system*

$$A_{k-1}e_{k-1} = d_{k-1}. \tag{3.26}$$

Where

$$e_{k-1} = R_k e_k, \qquad A_{k-1} = R_k A_k P_k, \qquad d_{k-1} = R_k d_k,$$

Hence, we need to define the intergrid transfer operators R_k *and* P_k.

Restriction

Denote by $\mathcal{H}_k$ *for the space vector of* u_k *and* f_k*. The restriction* R_k *is a linear transfer from the space of vectors* $\mathcal{H}_k$ *to to the space* $\mathcal{H}_{k-1}$.

$$R_k : \mathcal{H}_k \longrightarrow \mathcal{H}_{k-1}.$$

A simple injection can be used for the restriction, but a full weighted transfer is preferable. In our numerical example we have choose the the restriction R_k *as follow:*

$$R_k = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & & & & & \emptyset \\ & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & & & \\ & & & & & . & & & & \\ & & & & & & . & & & \\ \emptyset & & & & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}_{N_{k-1}\times N_k}$$

Prolongation

The prolongation P_k *is a linear transfer from the space of vectors* $\mathcal{H}_{k-1}$ *to the space* $\mathcal{H}_k$.

$$P_k : \mathcal{H}_{k-1} \longrightarrow \mathcal{H}_k.$$

Usually, linear interpolation is an effective and simple method for prolongation. In our numerical example we have choose the prolongation P_k *as follow:*

$$P_k = \begin{bmatrix} \frac{1}{2} & & & \varnothing \\ 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ & 1 & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2} \\ & & & 1 \\ \varnothing & & & \frac{1}{2} \end{bmatrix}_{N_k \times N_{k-1}}$$

Add the correction

We determine an improved iterate for solving (3.23) at the level k by

$$u_k = \bar{u}_k + P_k e_{k-1}. \tag{3.27}$$

Post-smoothing

Apply a few steps of the relaxation method (3.24) on $\mathcal{H}_k$, with the improved estimate u_k. The two-grid method can be presented in a more compact formulation as the following algorithm 1.

Algorithm 1 Two-grid algorithm

$u \leftarrow MGM(A_k, f_k, u_0)$
$u_0 := smoother(A_k, f_k, u_0, \alpha_1)$; % α_1 is a number of iterations performed (presmoothing step)
$d_k := f_k - A_k u_0$; % residual computation
$A_{k-1} := R_k A_k P_k$; % compute the restriction of A_k
$d_{k-1} := R_k d_k$; % compute the restriction of d_k
$e_{k-1} := (A_{k-1})^{-1} d_{k-1}$; % Direct solving on the coarsest grid
$e_k := P e_{k-1}$; % compute the prolongation of e_{k-1}
$u_k := u_0 + e_k$; % add correction to the solution
$u_k := smoother(A_k, f_k, u_k, \alpha_2)$; % α_2 is a number of iterations performed (postsmoothing step)
return u_k

3.3 The convergence of the two-grid method

Before one studying the convergence of the two-grid method, it is necessary to determine its iteration matrix. Denoted by u_k^n for the approximate solution before the presmoothing, *and by u_k^{n+1} for the updated solution. After α iterations of the presmoothing, we get*

$$e^{\alpha} = M_{RELAX}^{\alpha} e^{0}. \tag{3.28}$$

Where

$$e^0 = u - u_k^n, \quad e^{\alpha} = u - \bar{u}_k^n, \text{ and } M_{RELAX} \text{ is the smoothing iteration matrix.}$$

It is well known from the two-grid algorithm that

$$\bar{u}_k^n = u - M_{RELAX}^{\alpha}(u - u_k^n).$$

Then

$$d_k = f_k - A_k \bar{u}_k^n = f_k - A_k(u - M_{RELAX}^{\alpha}(u - u_k^n)) = A_k M_{RELAX}^{\alpha}(u - u_k^n). \tag{3.29}$$

Applying (3.29) in 3.27, we get:

$$\begin{aligned}
u_k^{n+1} &= \bar{u}_k^n + P_k e_{k-1}. \\
&= u - M_{RELAX}^{\alpha}(u - u_k^n) + P_k (A_{k-1})^{-1} R_k d_k \\
&= M_{RELAX}^{\alpha}(u_k^n) + (I - M_{RELAX}^{\alpha})(A_k)^{-1} f_k \\
&+ P_k (A_{k-1})^{-1} R_k A_k M_{RELAX}^{\alpha}((A_k)^{-1} f_k - u_k^n) \\
&= (I - P_k (A_{k-1})^{-1} R_k A_k) M_{RELAX}^{\alpha}(u_k^n) \\
&+ ((I - M_{RELAX}^{\alpha}) + P_k (A_{k-1})^{-1} R_k A_k M_{RELAX}^{\alpha})(A_k)^{-1} f_k.
\end{aligned}$$

Thus, the two-grid iteration matrix is as follows:

$$TG_k = (I - P_k (A_{k-1})^{-1} R_k A_k) M_{RELAX}^{\alpha}. \tag{3.30}$$

Using the splitting of the two-grid iteration matrix developed by Hackbusch ([28]), then (3.30) can be written as

$$TG_k = ((A_k)^{-1} - P_k (A_{k-1})^{-1} R_k) A_k M_{RELAX}^{\alpha}. \tag{3.31}$$

The two-grid convergence is based on its contraction number $\| TG_k \|$, so that

$$\| TG_k \| \leq \| (A_k)^{-1} - P_k (A_{k-1})^{-1} R_k \| \| A_k M_{RELAX}^{\alpha} \| . \tag{3.32}$$

The two essential components of the two-grid iteration matrix can be analyzed separately, the first one $\| (A_k)^{-1} - P_k (A_{k-1})^{-1} R_k \|$ describes the approximation property, while the second $\| A_k M_{RELAX}^{\alpha} \|$ describes the smoothing property.

Definition 25 (smoothing property) *One can said that the matrix M^{α}_{RELAX} has the smoothing property, if there exist a number $\beta > 0$ and functions $\xi(\alpha)$ and $\bar{\alpha}$ independent h_k such that*

$$\| A_k M^{\alpha}_{RELAX} \| \leq \xi(\alpha) h^{-\beta}, \quad for\ all \quad 1 \leq \alpha \leq \bar{\alpha}, \tag{3.33}$$

$$with \quad \xi(\alpha) \longrightarrow 0,\ when\ \alpha \longrightarrow \infty,$$
$$and\ \bar{\alpha} = \infty\ or\ \bar{\alpha} \longrightarrow \ \infty\ when\ h_k \longrightarrow 0$$

Remark 26 *It is not necessary to use a convergent smoothing iteration in order to fulfil the smoothing property. In fact, there are a divergent iterative methods which are good smoothers.*

Definition 27 (approximation property) *One can be said that the approximation property is satisfied, if there exists a constant C independent of h_k such that*

$$\| (A_k)^{-1} - P_k (A_{k-1})^{-1} R_k \| \leq C h_k^{\beta}, \tag{3.34}$$

is fulfiled.

If these two properties are fulfilled, the two-grid method's convergence rate follows easily.

Theorem 28 ([28]) [Convergence of Two-grid method]
Suppose that the approximation property and the smoothing property are satisfied, and Let a fixed number ρ be such that $\rho > 0$. If $\bar{\alpha}(h_k) = \infty$ for all h_k, then there exists $\underline{\alpha}$ such that

$$\| TG_k \| \leq C \xi(\alpha) \leq \rho, \tag{3.35}$$

$$as\ long\ as\ \alpha \geq \underline{\alpha}$$

.

Multigrid algorithm

Notting that the coarse grid system (3.26) is of the same form as the system (3.23). Therefore, we can apply the two-grid Scheme (1) recursively to solve (3.26) approximately. Thus the multigrid method for solving (3.23) is as the following algorithm 2.

Remark 29 *The multigrid cycle is called V-cycle, if $\tau = 1$, and W-cycle, if $\tau = 2$ see the figure (3.7).*

Algorithm 2 Muligrid methods

$u \leftarrow MGM(A_k, f_k, u_0)$
$u_0 := smoother(A_k, f_k, u_0, \alpha_1)$; % α_1 is a number of iterations performed (presmoothing step)
$d_k := f_k - A_k u_0$; % residual computation
$A_{k-1} := R_k A_k P_k$; % compute the restriction of A_k
$d_{k-1} := R_k d_k$; % compute the restriction of d_k
$e_{k-1} := d_{k-1} \cdot 0$; % start value for coarse grid iteration
if $size(A_k) \leq \mu$; **then**
  $e_{k-1} := (A_{k-1})^{-1} d_{k-1}$; % Direct solving on the coarsest grid Ω_μ
else
  $e_{k-1} := MGM(A_{k-1}, d_{k-1}, e_{k-1})$;
end if
$e_k := Pe_{k-1}$; % prolongation of e_{k-1}
$u_k := u_0 + e_k$; % add correction to the solution
$u_k := smoother(A_k, f_k, u_k, \alpha_2)$; % α_2 is a number of iterations performed (postsmoothing step)
return u_k

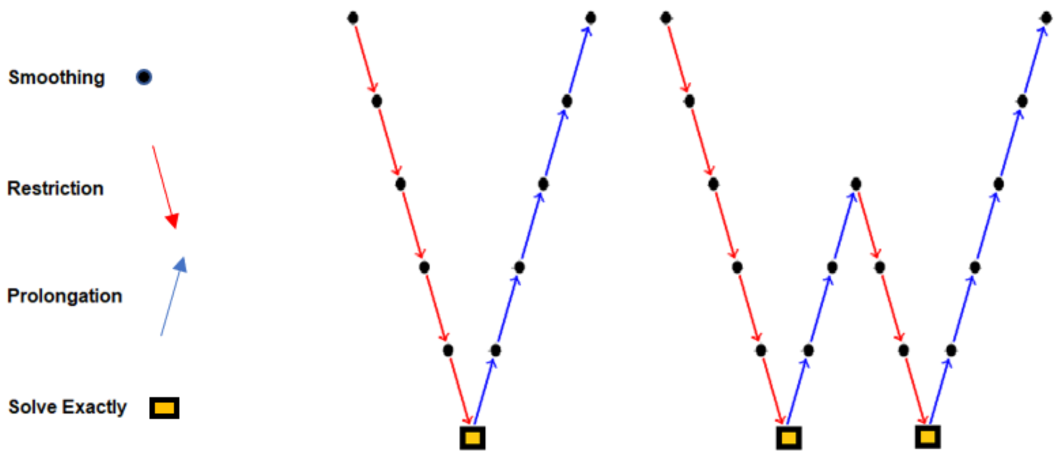

Figure 3.7: Multigrid V- and W-cycle

3.4 The convergence of Multigrid method

The iteration matrix of the multigrid method is defined as:

$$MG_k = \begin{cases} TG_k & if\, k = 1, \\ TG_k + M^{\alpha}_{RELAX} P_k MG_{k-1} (A_{k-1})^{-1} R_k (A_k) M^{\alpha}_{RELAX}, & if\ \ k = 1, 2, ... \end{cases} \tag{3.36}$$

The convergences in L^2 or H^2 norms have been proved for W-cycles only. The uniform convergence has been proved for the 2-grid method [48, 49] for elliptical PDEs and by Breamble [13] in a case of VI. where is assumed that the approximation and smoothing properties hold. We will recall a simplified version of the convergence results demonstrated *by Hackbusch in [28]. For more general case, the reader is refered to [28].*

Theorem 30 ([54]) *Suppose that the approximation and the smoothing properties are satisfied. Let P_k be such that*

$$\| P_k u_{k-1} \| \geq c_p^{-1} \| u_{k-1} \|, \quad c_p^{-1} \text{ is independent of } k \tag{3.37}$$

$$\| P_k u_{k-1} \| \leq C_p \| u_{k-1} \|, \quad C_p \text{ is independent of } k \tag{3.38}$$

Then there exists a number $\bar{\alpha}$ independent of K such that

$$\| MG_k(\alpha, 0) \| \leq \zeta < 1, \quad \zeta \in (0,1) \tag{3.39}$$

whenever $\alpha \geq \bar{\alpha}$.

3.5 Multigrid methods for solving non linear equations

In the previous section, it has been shown that multigrid methods can be successfully applied to a large class of problems since the residual equation obtained (in the coarse-grid correction step) is linear. But, this is not true when dealing with nonlinear problems, and two basic approaches are required in this case. In this section, we describe how to apply multigrid methods for solving nonlinear problems, there are two multigrid approaches to doing this. The first is to apply a standard linearization, like by Newton's method, and use multigrid for solving the Jacobian system in each iteration (has been described briefly in chapter 5). The second approach is to apply multigrid directly to the given nonlinear problem by the so-called Full Approximation Scheme (FAS).

Full Approximation Scheme

Consider a system of nonlinear algebraic equations on the fine grid

$$A_k(u_k) = f_k, \quad in\ \Omega_k. \tag{3.40}$$

Suppose $\bar{u}_k$ is an approximate solution to the exact solution u_k on the fine grid. It is possible to define the residual and the error for the system (3.40), as we did previously: Once again define the error as $e_k = u_k - \bar{u}_k$ while the residual is $d_k = f_k - A_k \bar{u}_k$. If we subtract the original equation (3.40), from the residual, we get

$$A_k(u_k) - A_k(\bar{u}_k) = d_k. \tag{3.41}$$

Since the operator A is non-linear, we cannot conclude the linear residual equation(3.25), as in the standard multigrid. Instead, we can rewrite (3.41) as the non-linear residual *equation:*

$$A_k(\bar{u}_k + e_k) - A_k(\bar{u}_k) = d_k. \tag{3.42}$$

The coarse-grid version of the residual equation appears as

$$A_{k-1}(\bar{u}_{k-1} + e_{k-1}) - A_{k-1}(\bar{u}_{k-1}) = d_{k-1}. \tag{3.43}$$

Here, we have chosen the coarse-grid residual to be the restriction of the residual on the fine-grid

$$d_{k-1} = R_k d_k = R_k(f_k - A_k\bar{u}_k).$$

In the same way, we have restricted the fine-grid approximation using the same interpolation operator, $\bar{u}_{k-1} = R_k\bar{u}_k$*. Substituting these into the coarse-grid residual equation, (3.44) can be written as*

$$A_{k-1}\underbrace{(R_k\bar{u}_k + e_{k-1})}_{u_{k-1}} = \underbrace{A_{k-1}(R_k\bar{u}_k) + R_k(f_k - A_k\bar{u}_k)}_{f_{k-1}}. \tag{3.44}$$

The right-hand side of this nonlinear equation is known and is of the same form as the fine-grid system (3.40). If we assume that we can find a solution u_{k-1} *to this equation, we can then extract the coarse grid error from the solution by* $e_{k-1} = u_{k-1} - R_k\bar{u}_k$*, and can then be interpolated back to the fine grid and used to improve the fine-grid approximation* $\bar{u}_k$*:*

$$\bar{u}_k \leftarrow \bar{u}_k + P_k e_{k-1}.$$

This approximation scheme is called the full approximation scheme (FAS), because the coarsegrid system is solved for the full approximation rather than only the error e_{k-1}*. One can easily see that if the operator A is linear, then the FAS is equivalent to the linear multigrid scheme, this mean that the FAS is a generalization of coarse-grid correction to the nonlinear problems. The FAS can be viewed as an enhancement of the coarse-grid equations. This can be seen if the coarse-grid equation (3.44) is written as*

$$A_{k-1}u_{k-1} = f_{k-1} + \tau_{k-1}. \tag{3.45}$$

Where the tau correction τ_{k-1} *is given by*

$$\tau_{k-1} = A_{k-1}(R_k\bar{u}_k) - R_k(A_k\bar{u}_k). \tag{3.46}$$

Since $\tau_{k-1} \neq 0$ *in general, the coarse grid FAS equation's solution* u_{k-1} *is not the same as the original coarsened solution*

$$A_{k-1}(u_{k-1}) = R_k f_k.$$

A solution u_{k-1} is actually convergent to an accuracy similar to the solution of the fine grid, albeit at the resolution of the coarse grid $k-1$.
A complete FAS multigrid cycle is described in the following algorithm (3):

Algorithm 3 Full Approximation Scheme Multigrid Cycle

$u \leftarrow FAS(A_k, f_k, u_k, \mu)$
$\bar{u}_k \leftarrow smoother(A_k, f_k, \bar{u}_k, \alpha_1)$;
$d_{k-1} \leftarrow R_k(f_k - A_k\bar{u}_k)$;
$\bar{u}_{k-1} \leftarrow R_k\bar{u}_k$;
$f_{k-1} \leftarrow d_{k-1} - A_{k-1}\bar{u}_k)$;
if $size(f_k) \leq \mu$ % coarsest grid Ω_μ **then**
  $e_{k-1} \leftarrow Solve(A_{k-1}, d_{k-1}, e_{k-1})$
else
  $e_{k-1} \leftarrow FAS(A_{k-1}, d_{k-1}, e_{k-1}, \mu)$;
end if
$u_k \leftarrow \bar{u}_k + P_k e_{k-1}$;
$u_k \leftarrow smoother(A_k, f_k, u_k, \alpha_2)$;
return u_k

3.6 Full multigrid methods (Nested iteration)

When the initial guess is unfortunate, the algorithm may even diverge for the nonlinear problem. If there is no a priori information about the solution to assist choosing the initial guess on the fine grid, starting the computation on the fine grid is obviously wasteful, thus it is better to use the coarse grids to provide an improved initial guess. In the strategy of nested iteration (also called full multigrid) we use the systems on coarse grids to obtain an improved starting vector for the multigrid method with relatively low computational costs. In the full multigrid we use the multigrid solver on all levels we get the following algorithm 4.

Algorithm 4 Full Muligrid

$u \leftarrow MGM(A_k, f_k, u_0)$
$u_0^* := A_0^{-1} f_0; \quad u_0^\mu := u_0^*$;
for $k := 1$ *to* $\bar{k}$ **do**
  $u_k^0 := P_k u_{k-1}^\mu$
end for
for $i = 1$ *to* τ **do**
  $u_k^i := MGM(A_k, d_k, u_k^{i-1})$;
end for
return the improved u_k

Nested iteration is based on the fact that $P_k u_{k-1}^*$ provides a reasonable approximation to u_k^*, because the system on the coarse grid and the system on the fine grid are discretiza-

tions of the same continuous problem, the following figure (3.8) shows the structure of nested iteration for 5 grids.

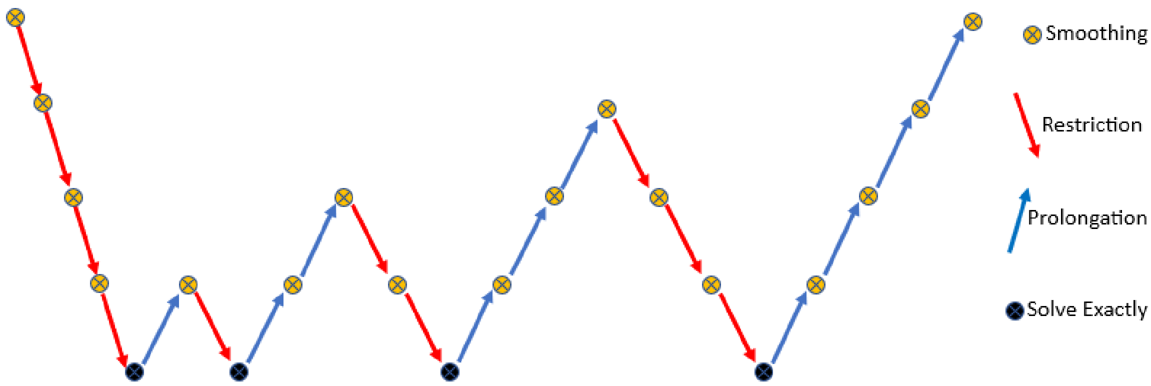

Figure 3.8: Nested iteration for 5 levels.

3.7 Numerical example for linear equations

In this section, the results of a linear model problem are given to test the efficiency of a standard multigrid method. We construct a linear problem like a Poisson's equation where we know the exact solution (we want the picture (3.9) to be an exact solution of the Poisson's equation ($\Delta u = f$)), then compute the right hand side by the method of manufactrued solution [53](see: A). Then we see how fast Multigrid method whould converge for that, to see it, we are going to compare it with Jacobi and Gaus-Seidel methods.

Figure 3.9: Exact solution of the Poisson's equation

Figure 3.10: Solution of the Poisson's equation by one iteration of Multigrid method

Figure 3.11: Solution of the Poisson's equation after 1000 iterations of Gauss-seidel method

Figure 3.12: Solution of the Poisson's equation after 1000 iterations of Jacobi method

Comparison

It turns out that Gauss-Seidel and Jacobi methods are still not good even after a large number of iterations (1000 iterations in this example!). Unlike the multigrid method, which converges in few iterations.

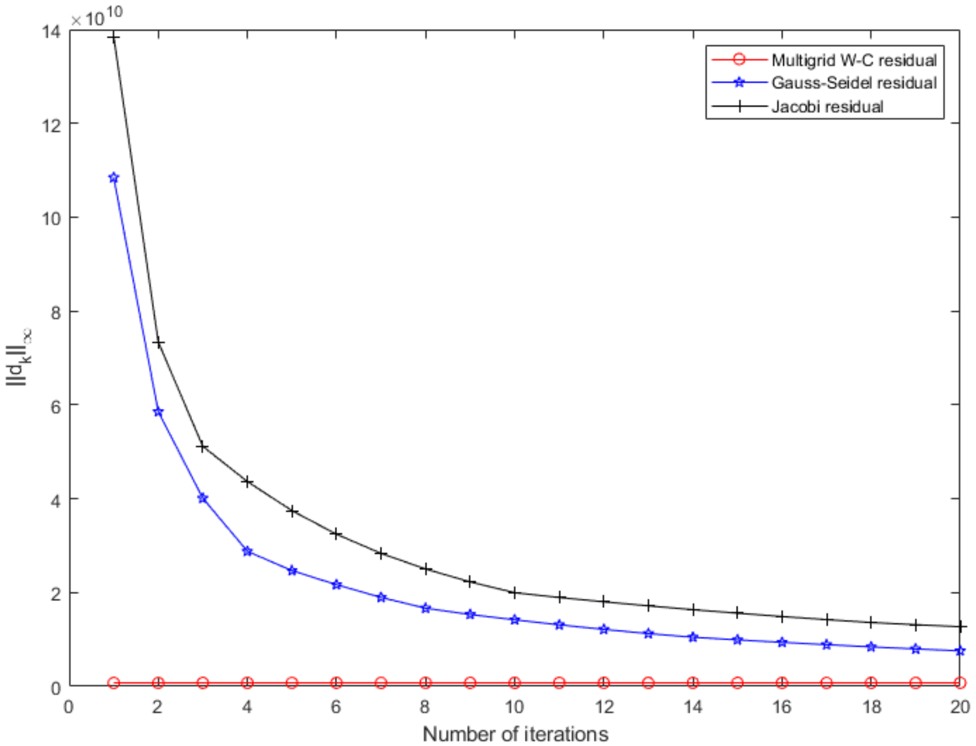

Figure 3.13: Comparison between the convergence behavior of Multigrid method, Gauss-Seidel and Jacobi methods (using 20 iterations)

4

The L^∞-Norm Convergence of Multigrid Method for Solving Elliptic Quasi-Variational Inequalities

In this chapter, algebraic multigrid methods on adaptive finite element discretisation are applied to solve elliptic quasi-variational inequality. The uniform convergence of the multigrid scheme has been established which proves that the multigrid method has a contraction number with respect to the L^∞-norm. Numerical results which demonstrate the high efficiency of these methods are given for a quasi-variational inequality arising from impulse control problem on a domain with nonpolygonal boundaries.

We briefly outline the remainder of this chapter. In §2, we introduce a continuous problem and we present some regularity results. In §3 we apply linear finite element discretizations to get a system of equations. After the HJB reformulation, in §4 we describe a multigrid method for the solution of the algebraic system equations obtained. In §5 we prove the uniform convergence of multigrid method. The results of numerical experiments are given in §6 for a QVI leading to a stochastic inventory problem with impulse control.

4.1 Continuous problem

Let Ω be an open in $\mathbb{R}^N$, with sufficiently smooth boundary $\partial\Omega$ for $u, w \in V$ $\left(V = H_0^1(\Omega)\right)$, consider

$$\vartheta(u,w)=\int_{\Omega}[\sum_{1\le i,j\le N}\vartheta_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial w}{\partial x_j}+\sum_{1\le i\le N}\vartheta_i(x)\frac{\partial u}{\partial x_i}+\vartheta_0(x)u\cdot w]dx, \tag{4.1}$$

a bilinear form related to the linear operator A defined by

$$A=\sum_{1\le i,j\le N}\vartheta_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{1\le i\le N}\vartheta_i\frac{\partial}{\partial x_i}+\vartheta_0,$$

where $\vartheta_{ij}(x),\vartheta_i(x),\vartheta_0(x), x\in\overline{\Omega}, 1\le i,j\le n$ *are sufficiently regular and satisfy the following conditions:*

$$\sum_{1\le i,j\le n}\vartheta_{ij}\xi_i\xi_j\ge\varrho|\xi|^2,\ \xi\in\mathbb{R}^N,\varrho>0,$$
$$\vartheta_0(x)\ge\beta>0,\ \beta \textit{ is a constant.}$$

Consider the following obstacle problem: Finding $u\in K_g(u)$ *the solution of*

$$\begin{cases}\vartheta(u,w-u)\ge\langle f,w-u\rangle & w\in K_g(u),\\ u\le Mu & Mu\ge 0,\\ u=g & on\ \ \partial\Omega.\end{cases} \tag{4.2}$$

Where the operator M is the operator defined in 2.18, and $K_g(u)$ *is an implicit convex and non empty set defined in 2.17. Under Theorem 19, the problem (4.2) admits a unique solution which satisfies the regularity property 2.20.*

4.2 Discrete problem

To apply a multigrid cycle, we propose a decreasing sequence (mesh size parameter)

$$\{h_k\}_{k=0}^{l},\quad h_{k+1}<h_k, 0\le k\le m-1.$$

We introduce a quasi-uniform family of nested triangulations of $\Omega_k=\bigcup_{k\in N}T_k$. *Therefore*

$$\Omega\supset\Omega_{k+1}\supset\Omega_k.$$
$$distance(\partial\Omega_k,\partial\Omega)\le c_0h_k^2.$$
$$h_{k+1}\le\tfrac{c_1}{h_k}.$$

On each level k, we choose V_k *a space of piecewise linear finite elements*

$$V_k=\left\{w_k\in C(\Omega)\cap H^1(\Omega)|\ w_{k|\Omega_k}\in P_1\right\}, \tag{4.3}$$

and to each h_k *an analogous discretization of the Q.V.I (4.2) by FEM is associated. To* *facilitate the notations, we put*

$$A_k = A_{h_k},\ V_k = V_{h_k},\ \Omega_k = \Omega_{h_k}.$$

The standar basis functions $\varpi_k^j, j \in (1,...,z(h_k))$ *are defined by*

$$\varpi_k^j\left(x_k^i\right) = \delta_{ji}.$$

Where x_k^j *is a vertex of* T_k. *Let* $U_k = \mathbb{R}^{z_k}$, *to define the finite element restriction operator, we use the usual operator defined from* U_k *into* V_k *as*

$$r_k v(x) = \sum_{j=1}^{z(h_k)} v\left(x_k^j\right) \varpi_k^j(x). \tag{4.4}$$

On U_k *we use the scaled euclidean scalar product*

$$\langle u, v\rangle_k = h_k^2 \sum_{i=1}^{z_k} u_i v_i, \text{ and a corresponding norm } \|u\|_k = \langle u, u\rangle_k^{1/2},$$

moreover, the adjoint operator $r_k^* : V_k \longrightarrow U_k$ *satisfies*

$$\langle r_k u, v\rangle_{L^2} = \left\langle u, r_k^* v\right\rangle,\ \forall u \in U_k,\ v \in V_k.$$

***Lemma 31** **([48])** For the restriction operator* r_k *defined by (4.4), there exist constants* C_1 *and* C_2 *independent of k such that*

$$\begin{aligned} \|r_k(u)\|_{L^\infty} &= \|u\|_\infty,\ \forall u \in U_k, \\ C_1 \|v\|_{L^\infty} &\leq \left\|r_k^*(v)\right\|_\infty \leq C_2 \|v\|_{L^\infty},\ \forall v \in V_k. \end{aligned}$$

The norm $\|.\|_\infty$ *(on* U_k*) and the norm*$\|.\|_{L^\infty}$ *(on* V_k*) are equivalent, which are denoted by* $\|.\|_\infty$.

We naturally introduce the discretization matrices A_k *by the generic coefficient matrices*

$$\vartheta(\varphi_k^l(x), \varphi_k^s(x)),\ \ s \in (1, ..., m(h_k)).$$

The application of FEM to discretise the QVI (4.2) leads to the solution of the discrete QVI (4.5) in finite dimension. Find $u_k \in K_{g,k}$ *such that*

$$\begin{cases} \langle A_k u_k, v_k - u_k\rangle \geq \langle f, v_k - u_k\rangle, & \forall v_k \in K_{g,k}, \\ u_k \leq M_k u_k, \quad v_k \leq M_k u_k, \end{cases} \tag{4.5}$$

where f, M_k, *and* $K_{g,k}$ *are defined as in 2.22.*

Denoted by π_k the interpolation operator on $\partial\Omega$. Assume that the discrete maximum principle holds, then under Theorem 21 the problem 4.5 admits a unique discrete solution which satisfies the regularity result 2.26.

4.3 Multigrid method for solving Q.V.Is

The HJB-formulation of the discret problem (4.5)

Usually, we can write the QVI (4.5) as the HJB equation (4.6). Let u_k^ν be a unique solution of the following HJB equation

$$\max_{1\le i\le N}\left(A_{k,i}u_{k,i}^{\nu}-f_{k,i}\,,u_{k,i}^{\nu}-M_k u_{k,i}^{\nu-1}\right)=0. \tag{4.6}$$

Choose a start vector $u_k^0 \in U_k$. Starting from the iterate $u_k^\nu \in U_k$, $\nu \ge 0$, using the set

$$J_k=\{1,2,\dots,m_k\}\ ,\ \ J_k=\bigcup_{p=1}^{3} J_k^p\left(u_k^\nu\right)$$

which may be splitted as

$$\begin{aligned} J_k^1\left(u_k^\nu\right)&=\left\{i\in J_k \mid \left(A_k u_k^\nu-f_k\right)_i > u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right\},\\ J_k^2\left(u_k^\nu\right)&=\left\{i\in J_k \mid \left(A_k u_k^\nu-f_k\right)_i < u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right\},\\ J_k^3\left(u_k^\nu\right)&=\left\{i\in J_k \mid \left(A_k u_k^\nu-f_k\right)_i = u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right\}. \end{aligned} \tag{4.7}$$

And compute $u_k^{\nu+1} \in U_k$ as the solution of the system

$$A_k^\nu u_k^{\nu+1}=f_k^\nu, \tag{4.8}$$

where

$$A_k^\nu=\begin{cases} A_{k,i} & \text{if } i\in J_k^1\left(u_k^\nu\right),\\ I_{k,i} & \text{if } i\in J_k^2\left(u_k^\nu\right)\cup J_k^3\left(u_k^\nu\right). \end{cases} \tag{4.9}$$

$$f_k^\nu=\begin{cases} f_{k,i} & \text{if } i\in J_k^1\left(u_k^\nu\right),\\ M_k u_{k,i}^{\nu-1} & \text{if } i\in J_k^2\left(u_k^\nu\right)\cup J_k^3\left(u_k^\nu\right), \end{cases} \tag{4.10}$$

with $A_{k,i}$ (resp. $f_{k,i}$)is the i^{th} row of the discretization matrix A_k (resp. the i^{th} element of the right-hand side f_k), and $I_{k,i}$ is the i^{th} unit vector.

Assume that A_k is continuously differentiable, we can prove the monotone convergence of the iterates,

***Theorem 32** ([33]) Let $\left(u_k^\nu\right)$ be the iterate obtained by the preceding iterative scheme so it satisfies the H.J.B equation (4.6), furthermore suppose that A_k is continuously differ-*

entiable, then the sequence $\left(u_k^\nu\right)_{\nu\geq 0}$ *converges monotonously decreasingly to the unique solution* u_k^* *of (4.5).*

Proof. *We will prove that the iterates* $u_k^\nu, \nu \geq 0$, *monotonely decrease towards* u_k^* *as follows*

$$u_k^* \leq u_k^{\nu+1} \leq u_k^\nu. \tag{4.11}$$

For this purpose we strate with u_k^0 *satisfying*

$$\max\left(A_k u_k^0 - f_k\, , u_k^0 - M_k u_k^0\right) \geq 0, \tag{4.12}$$

with using of this and

$$\max\left(A_k u_k^{\nu+1} - f_k\, , u_k^{\nu+1} - M_k u_k^\nu\right) \geq A_k^\nu u_k^{\nu+1} - f_k^\nu \geq 0,\ \nu \geq 0. \tag{4.13}$$

There holds

$$\max\left(A_k u_k^\nu - f_k\, , u_k^\nu - M_k u_k^{\nu-1}\right) \geq 0, \text{ for all } \nu \geq 0. \tag{4.14}$$

In view of (4.9) and (4.14) we get

$$\max\left(A_k u_k^\nu - f_k\, , u_k^\nu - M_k u_k^{\nu-1}\right) \geq A_k^\nu u_k^\nu - f_k^\nu \geq 0. \tag{4.15}$$

Which by (4.8) it follows that

$$A_k^\nu u_k^{\nu+1} \leq A_k^\nu u_k^\nu. \tag{4.16}$$

In particular, with using of (4.9) we get $u_{k,i}^{\nu+1} \leq u_{k,i}^\nu, i \in I_k^2\left(u_k^\nu\right) \cup I_k^3\left(u_k^\nu\right)$. *Consequently,* $\left(u_{k,i}^{\nu+1} - u_{k,i}^\nu\right)^+ = 0, i \in I_k^2\left(u_k^\nu\right) \cup I_k^3\left(u_k^\nu\right)$. *By using of (4.9) and (4.16) we get*

$$\left(A_k u_k^{\nu+1} - A_k u_k^\nu, \left(u_{k,i}^{\nu-1} - u_{k,i}^\nu\right)^+\right) = \left(A_k^\nu u_k^{\nu+1} - A_k^\nu u_k^\nu, \left(u_{k,i}^{\nu+1} - u_{k,i}^\nu\right)^+\right) \leq 0. \tag{4.17}$$

Because A_k *are strictly T-monotone (4.17) implies* $\left(u_{k,i}^{\nu+1} - u_{k,i}^\nu\right)^+ = 0$, *so*

$$u_k^{\nu+1} \leq u_k^\nu.$$

On the other hand, we have

$$0 = \max\left(A_k u_k^* - f_k\, , u_k^* - M_k u_k^*\right) \geq A_k^\nu u_k^* - f_k^\nu. \tag{4.18}$$

Whence, due to (4.8) gives

$$A_k^\nu u_k^* \leq A_k^\nu u_k^{\nu+1}. \tag{4.19}$$

As the same previous arguments we get

$$u_k^* \le u_k^{\nu+1}.$$

Now by (4.11) there exists an $u_k^{**} \in \mathbb{R}^{n_k}$ *such that* $u_k^\nu \longrightarrow u_k^{**}$ *for* $\nu \longrightarrow \infty$. *Then*

$$\begin{aligned}\max\left(A_k u_k^\nu - f_k, u_k^\nu - M_k u_k^{\nu-1}\right) &= A_k^\nu u_k^\nu - A_k^\nu u_k^{\nu+1} \\ &= \int_0^1 \nabla A_k^\nu \left(u_k^{\nu+1} + \lambda\left(u_k^\nu - u_k^{\nu+1}\right)\right) d\lambda \left(u_k^\nu - u_k^{\nu+1}\right).\end{aligned}$$

Since A_k *is continuously differentiable, so* A_k^ν *is and hence, for* $\nu \longrightarrow \infty$ *we get*

$$\max\left(A_k u_k^{**} - f_k, u_k^{**} - M_k u_k^{**}\right) = 0.$$

yielding $u_k^{**} = u_k^*$, *because (4.6) has a unique solution.* ■

The HJB-formulation for restating the discret QVI (4.5) as an HJB-equation (4.8) may be described as the algorithm 5.

Algorithm 5 HJB-formulation

$[A, b, u_k^\nu] \leftarrow HJB(A, F, M)$
Select a start guess u_k^0
$u_k^\nu \leftarrow smoother(A, F, u_0, \nu)$; %compute u_k^ν
for $k := 1$ *to size*$(A, 1)$ **do**
  if $\max(A_k u_k^\nu - F_k, u_k^\nu - M u_k^\nu) = 0$ **then**
    $A_{[k,:]} \leftarrow Id_{[k,:]}$; % Id(k) is the identity matrix
    $F_k \leftarrow M u_k^\nu$;
  end if
end for
return $[A, b, u_k^\nu]$

Multigrid Algorithm

In the multigrid method, choose an iterate $u_k^\nu, \nu > 0$, *we get* $\bar{u}_k^\nu$ *by* α *applications of a smoothing iterative method for solving (4.8), which defined as*

$$\bar{u}_k^\nu = S_k^\alpha\left(u_k^\nu\right). \tag{4.20}$$

S_k *is the iteration matrix of the smoothing iterative method, and* α *is a number of iterations performed.*

Denoted by u_k^* *for the solution of (4.8). Setting the error* $e_k^\nu = \bar{u}_k^\nu - u_k^*$, *and the residual* $d_k^{(\nu)} = f_k^\nu - A_k^\nu \bar{u}_k^\nu$, *we can write the equation (4.8) as*

$$A_k^\nu\left(\bar{u}_k^\nu + e_k^\nu\right) = f_k^\nu.$$

Which results in the residual equation

$$A_k^\nu e_k^\nu = f_k^\nu - A_k^\nu \bar{u}_k^\nu = d_k^{(\nu)}.$$

On the fine grid, after the relaxation on $A_k^\nu \bar{u}_k^\nu = f_k^\nu$ the error will be smooth, while on the coarse grid this error seems to be more oscillatory, and the relaxation will be so efficient. Therefore, to determine e_k^ν completely, we require to compute e_{k-1}^ν at level $(k-1)$ as the solution of on the coarse grid

$$A_{k-1}^\nu e_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{4.21}$$

We can interpret e_{k-1}^ν $\left(resp\ A_{k-1}^\nu, d_{k-1}^{(\nu)}\right)$as an approach at a level $k-1$ of $e_k^\nu\left(resp\ A_k^\nu, d_k^{(\nu)}\right)$.

$$e_{k-1}^\nu = R_k e_k^\nu, \qquad A_{k-1}^\nu = R_k A_k^\nu P_k, \qquad d_{k-1}^{(\nu)} = R_k d_k^{(\nu)}.$$

Consequently, the improved iterate at the level k can be defined as

$$u_k^{\nu+1} = \bar{u}_k^\nu + P_k\left(e_{k-1}^\nu\right). \tag{4.22}$$

Due to the nestedness we use the well-defined identity operator

$$\begin{aligned} \Upsilon \quad &: \quad V_{k-1} \longrightarrow V_k \\ \Upsilon v \quad &= \quad v \end{aligned}$$

to construct the prolongation operator and the restriction operator, i.e.,

$$R_k = P_k^t, \qquad P_k = r_k^{-1} r_{k-1}. \tag{4.23}$$

Remark 33 *The preceding algorithm represents one cycle of a two-grid iteration for the solution of (4.8) at two hierarchies of grids Ω_k and Ω_{k-1}. It is clear that the coarse grid system (4.21) has the same form as the system (4.8) Therefore, the system (4.21) can be solved approximately by performing the two-grid iteration recursively to all hierarchies of grids $\{\Omega_k,\ k = 0, ..., m_k\}$.*

The multigrid iteration for solving (4.8) may be described as the algorithm 2.

4.4 Multigrid convergence on the maximum norm

In this section we present the uniform convergence analysis for the multigrid algorithm described in the previous section.

4.4.1 The iteration matrix of the multigrid methods

The iteration matrix of Two-grid scheme with α_2 postsmoothing iterations and α_1 presmoothing iterations on level k is defined as

$$TG_k(\alpha_1,\alpha_2) = S_k^{\alpha_2}\left(\left(A_k^{\nu}\right)^{-1} - P_k\left(A_{k-1}^{\nu}\right)^{-1}R_k\right)\left(A_k^{\nu}\right)S_k^{\alpha_1}. \tag{4.24}$$

It is easy to verify that the multigrid methods is a linear iterative method.

Theorem 34 ([49]) *Multigrid methods can be defined as a linear iterative methods associated with an iteration matrix MG_k defined as*

$$\begin{aligned} MG_0 &= 0, && (4.25a)\\ MG_k &= S_k^{\alpha_2}\left(I_k - P_k(I_k - MG_{k-1})\left(A_{k-1}^{\nu}\right)^{-1}R_k\right)\left(A_k^{\nu}\right)S_k^{\alpha_1}, && (4.25b)\\ &= TG_k + S_k^{\alpha_2}P_k MG_{k-1}\left(A_{k-1}^{\nu}\right)^{-1}R_k\left(A_k^{\nu}\right)S_k^{\alpha_1},\ k=1,2,\ldots && (4.25c) \end{aligned}$$

4.4.2 Approximation property

The proof of the approximation property introduced by Arnold in [48] yields the approximation property presented here.

Theorem 35 ([29]) *Under the preceding hypotheses, the matrix*

$$\chi = \left[\left(A_k^{\nu}\right)^{-1} - P_k\left(A_{k-1}^{\nu}\right)^{-1}R_k\right],$$

fulfils the approximation property:

$$\|\chi\|_\infty \le Ch_k^2\left|\log h_k\right|^2. \tag{4.26}$$

Proof. *Theorem (22) states*

$$\|u-u_k\|_{L^\infty} \le Ch_k^2\left|\log h_k\right|^2\|f_k\|_\infty, \qquad u \in W^{2,p}.$$

Then

$$\left\|u-u_k^*\right\|_{L^\infty} \le Ch_k^2\left|\log h_k\right|^2\|f_k\|_{L^\infty}.$$

Therefore, we obtain

$$\begin{aligned} \left\|u_k^*-u_{k-1}^*\right\|_{L^\infty} &\le \left\|u_k^*-u\right\|_{L^\infty} + \left\|u_{k-1}^*-u\right\|_{L^\infty} && (4.27)\\ &\le C_1h_k^2\left|\log h_k\right|^2\|f_k\|_{L^\infty} + C_2h_k^2\left|\log h_k\right|^2\|f_k\|_{L^\infty} \\ &\le Ch_k^2\left|\log h_k\right|^2\|f_k\|_{L^\infty}. \end{aligned}$$

Furthermore, there exists $g \in U_k$ a right-hand side, such that

$$\begin{aligned}\vartheta\left(r_k\left(A_k^\nu\right)^{-1} g, v_k\right) &= \left\langle\left(r_k^*\right)^{-1} g, v_k\right\rangle_{L^2}, \ \forall v \in K_{g,k},\\ \vartheta\left(r_{k-1}^{-1}\left(A_{k-1}^\nu\right)^{-1} R_k g, v_{k-1}\right) &= \left\langle\left(r_k^*\right)^{-1} g, v_{k-1}\right\rangle_{L^2}, \ \forall v \in K_{g,k-1}.\end{aligned}$$

Using (4.27) and Lemma 31 we get

$$\left\|r_k^{-1}\left(A_k^\nu\right)^{-1} g - r_{k-1}^{-1}\left(A_{k-1}^\nu\right)^{-1} R_k g\right\|_\infty \leq C h_k^2 \left|\log h_k\right|^2 \|g\|_\infty .$$

Then

$$\left\|\left(A_k^\nu\right)^{-1} - r_k^{-1} r_{k-1}\left(A_{k-1}^\nu\right)^{-1} R_k\right\|_\infty \leq C h_k^2 \left|\log h_k\right|^2 .$$

As a result, the proof is completed

$$\left\|\left(A_k^\nu\right)^{-1} - P_k\left(A_{k-1}^\nu\right)^{-1} R_k\right\|_\infty \leq C h_k^2 \left|\log h_k\right|^2 .$$

■

4.4.3 Smoothing prorperty

In order to prove a smoothing prorperty we decompose the matrix $A_k^\nu = E_k - N_k$, and use the following hypotheses:

$$E_k \text{ is regular and } \left\|E_k^{-1} N_k\right\|_\infty \leq 1, \text{ for all } k. \tag{4.28}$$

$$\|E_k\|_\infty \leq C h_k^{-2}, \text{ with } C \text{ independent of } k. \tag{4.29}$$

As a smoother we use the relaxation iterative method with the iteration matrix

$$S_k = I_k - \omega E_k^{-1} A_k^\nu, \qquad \omega \in (0,1). \tag{4.30}$$

A smoothing property can be proved by using the following basic lemma.

Lemma 36 ([48]) *Let A_k^ν is a square $(n \times n)$-matrix such that $\left\|A_k^\nu\right\|_\infty \leq 1$. Then we have the following inequality:*

$$\left\|(I_k - A_k^\nu)(I + A_k^\nu)^\alpha\right\|_\infty \leq 2^{\alpha+1}\sqrt{\frac{2}{\pi\alpha}}, \qquad (\alpha = 1, 2, ...) \tag{4.31}$$

Theorem 37 ([48]) *Assume that the preceding hypotheses and notations are satisfied, therefore there is a constant C independent of k and α for which the following smoothing property holds:*

$$\left\|\left(A_k^\nu\right) S_k^\alpha\right\|_\infty \leq C \frac{1}{\sqrt{\alpha}} h_k^{-2}. \tag{4.32}$$

Proof. *Let $\mathbb{A}_k := I_k - 2\omega E_k^{-1} A_k^\nu = 2\omega E_k^{-1} N_k + (1-2\omega)I$. Then due 4.29 we get*

$$\|\mathbb{A}_k\|_\infty \leq 1.$$

Applying Lemma36, we obtain

$$\begin{aligned}
\left\|\left(A_k^\nu\right) S_k^\alpha\right\|_\infty &= \left\|\left(A_k^\nu\right)(I_k - \omega E_k^{-1} A_k^\nu)^\alpha\right\|_\infty \\
&= \left\|\frac{1}{2\omega} E_k (I_k - \mathbb{A}_k)(\frac{1}{2})^\alpha (I_k + \mathbb{A}_k)^\alpha\right\|_\infty \\
&\leq \frac{1}{2\omega} \|E_k\|_\infty (\frac{1}{2})^\alpha 2^{\alpha+1} \sqrt{\frac{2}{\pi\alpha}} \\
&\leq \frac{1}{\omega}\sqrt{\frac{2}{\pi}} C \frac{1}{\sqrt{\alpha}} h_k^{-1} \\
&\leq C \frac{1}{\sqrt{\alpha}} h_k^{-2}.
\end{aligned}$$

■

4.4.4 Convergence result of the multigrid methods

In this part we prove the uniform convergence of a multigrid algorithm. Besides the approximation and smoothing property, we have to prove the following stability bound:

$$\text{for all } k \text{ and } \alpha, \text{ there exists } C_s \text{ such that } : \left\|S_k^\alpha\right\|_\infty \leq C_s. \tag{4.33}$$

The convergence analysis of the multigrid methods is based on the splitting of the iteration matrix of the two-grid iteration:

$$\begin{aligned}
\|TG_k(\alpha_1, 0)\|_\infty &= \left\|\left(\left(A_k^\nu\right)^{-1} - P_k\left(A_{k-1}^\nu\right)^{-1} R_k\right)\left(A_k^\nu\right) S_k^{\alpha_1}\right\|_\infty, \\
&\leq \left\|\left(A_k^\nu\right)^{-1} - P_k\left(A_{k-1}^\nu\right)^{-1} R_k\right\|_\infty \left\|\left(A_k^\nu\right) S_k^{\alpha_1}\right\|_\infty.
\end{aligned}$$

The following theorem represents the main result of our work.

Theorem 38 *Under the preceding hypotheses, there is a constant C independont of α and k, such that the iterate $u_k^\nu, \nu \geq 0$ for the two grids k and $k-1$ fulfils:*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_\infty \leq \left(\frac{C}{\sqrt{\alpha}} \left|\log h_k\right|^2\right) \left\|u_k^\nu - u_k^*\right\|_\infty. \tag{4.34}$$

Proof. *We have:*

$$\begin{aligned}
\left\|u_k^{\nu+1}-u_k^*\right\|_\infty &= \left\|\left(\left(I_k-P_k\left(I_k-MG_{k-1}\right)\left(A_{k-1}^{\nu}\right)^{-1}R_k\right)\left(A_k^{\nu}\right)S_k^{\alpha_1}\right)\left(u_k^{\nu}-u_k^*\right)\right\|_\infty \\
&\le \left\|I_k-P_k\left(I_k-MG_{k-1}\right)\left(A_{k-1}^{\nu}\right)^{-1}R_k\right\|_\infty \left\|\left(A_k^{\nu}\right)S_k^{\alpha_1}\right\|_\infty \left\|\left(u_k^{\nu}-u_k^*\right)\right\|_\infty \\
&\le \left(C_2\frac{1}{\sqrt{\alpha}}h_k^{-2}\right)\left(C_1h_k^2\left|\log h_k\right|^2\right)\left\|u_k^{\nu}-u_k^*\right\|_\infty \\
&\le \frac{C_1C_2}{\sqrt{\alpha}}\left|\log h_k\right|^2\left\|u_k^{\nu}-u_k^*\right\|_\infty .
\end{aligned}$$

■

generally, we will choose more than two grids hierarchies. In this case the iteration matrix (4.25) has been defined by recurrence using the iteration matrix of the two-grids scheme (4.24) at all levels. Therefore, if we suppose that (4.33) holds, then the L^∞-convergence result can be easily deduced from the preceding result.

Theorem 39 ([49]) *Let the multigrid methods with iteration matrix defined in (4.25). So under the preceding hypotheses and for parameter values $\alpha_2 = 0$, $\alpha_1 = \alpha > 0$, $\tau = 2$.*

$\forall \zeta \in (0,1)$, $\exists \alpha^*$ *such that* $\alpha \ge \alpha^*$

$$\|MG_k\|_\infty \le \zeta, \qquad k = 0,1,... \tag{4.35}$$

holds.

Proof. *After the combination of the approximation and smoothing properties with (4.33), then we can apply the same arguments as in [49, Theorem 7.20].* ■

4.5 Numerical experiments

In this section, we introduce a numerical example of a quasi-variational inequality arising from a stochastic inventory problem with impulse control [33].

In order to apply this model to our example, we suppose that the data of our problem to be sufficiently smooth and apply the Bellman's principle of dynamic programming, we find that the minimal cost function u which is the solution of impulse control problem related to the QVI (5.1), Then we solve (5.1) as we discussed before with the following data:

$$\begin{cases} Au \leq f, \qquad in\ \Omega = \left\{(x_1, x_2)\,|x_1^2 + x_2^2 \leq 1\right\}, \\ \left\langle Au - f\ , u^{\nu} - k - u^{\nu-1}\right\rangle = 0, \\ u^{\nu} \leq k + u^{\nu-1}, \qquad k \geq 0, \\ u = 0, \qquad in\ \ \partial\Omega. \end{cases} \tag{4.36}$$

Where

$$\begin{aligned} Au &= -\Delta u, \\ f(x) &= x_1 + x_2, \\ k &= 0.01. \end{aligned}$$

We are confine ourselves to the FEM discretization with a uniform triangulation and P1 nested finite element function spaces. For the discretization of the Domain, we have used the PDE toolbox in MATLAB ($R2018a$) to generate the mesh and then the multigrid FEM can be used to efficiently solve (4.36) as mentioned above.

This numerical example is reported to verify the high efficiency of the multigrid method. As a pre/post-smoothing we have chosen Gauss-Seidel method in the multigrid code. Regarding the recursion in multigrid method, we stop the recursion of the multigrid algorithm when the DOFs (the number of interior grid points) smaller than 5.

Figure 4.1 illustrates the convergence behavior of the multigrid solver (red and black curves for the maximum norm of the residual of multigrid (V and W_cycle)) with respect to the number of iterations performed. For comparison, the convergence behavior of Gauss-Seidel and damped Gauss-Seidel solvers(ω=1.5) (green and blue curves) are included. The multigrid V-cycle is carried out on the finest grid with 1045 nodes and 4 nodes on the coarsest one, and then we have applied Matlab-Operator solver, Gauss-Seidel, and damped Gauss-Seidel on this finest grid to get the solutions in figures 4.2, 4.3, 4.4 and 4.5. We have used the following table (4.1) to get more information about the convergence behavior of multigrid and the other methods.

Notting that, if we perform more than 24 *iterations, the multigrid solver is better than*

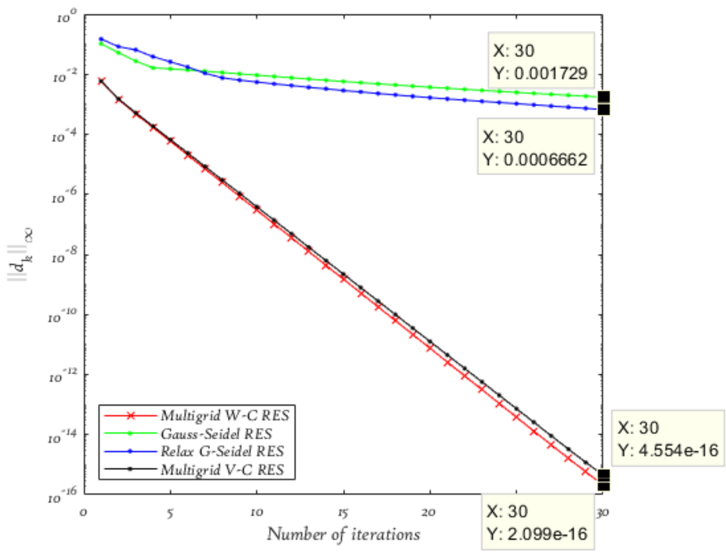

Figure 4.1: Comparison between the convergence behavior of Multigrid method, Gauss-Seidel and weighted Gauss-Seidel methods

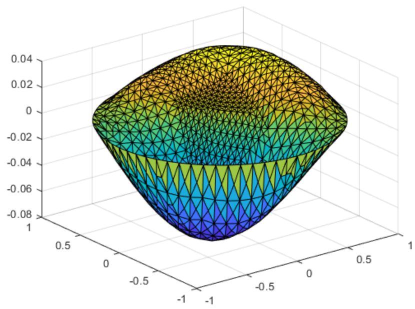

Figure 4.2: Solution of the problem (4.36) on a fine grid with 1045 DOFs after 30 iterations of multigrid method

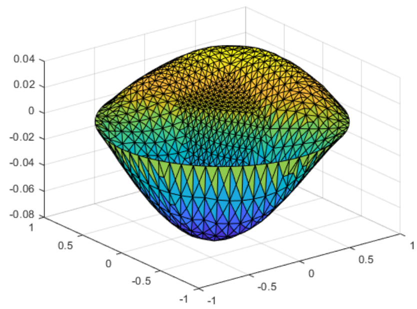

Figure 4.3: Solution of the problem (4.36) on a fine grid with 1045 DOFs using Matlab-Operator solver

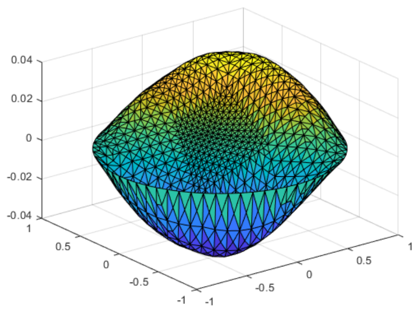

Figure 4.4: Solution of the problem (4.36) on a fine grid with 1045 DOFs after 30 iterations of a Gauss-seidel method

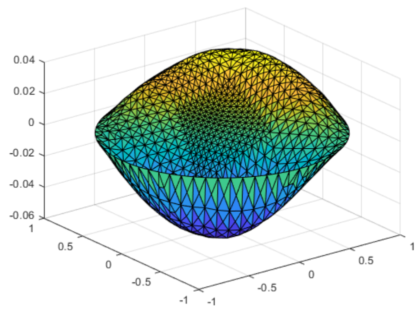

Figure 4.5: Solution of the problem (4.36) on a fine grid with 1045 DOFs after 30 iterations of a damped Gauss-seidel method

Table 4.1: **Convergence behavior of the multigrid and the other methods**

number of iterations	1	5	10	15	20
$\|\|d_k\|\|_\infty$ *GAUS_SEIDEL residual*	3.3390e-02	1.4805e-02	9.4917e-03	6.1365e-03	4.0521e-03
$\|\|d_k\|\|_\infty$ *RELAX−G_S residual*	4.8369e-02	1.1943e-02	5.9492e-03	3.1745e-03	1.7460e-03
$\|\|d_k\|\|_\infty$ *MULTIGRID V_C residual*	5.7658e-03	2.5791e-05	3.2102e-08	3.9659e-11	4.8950e-14
$\|\|d_k\|\|_\infty$ *MULTIGRID W_C residual*	5.6163e-03	2.2277e-05	2.3693e-08	2.4975e-11	2.6286e-14

Matlab-Operator solver.

$$\begin{cases} \text{Norm of residual obtained by Matlab-Operator solver :=1.2663e-16} \\ \text{After 25 iterations:Norm of residual obtained by multigrid solver :=8.9338e-17} \end{cases}$$

Discussion

In the following we have discussed the multigrid methods for solving the problem (4.36) in the finest grid with 225 interior grid points. We have chosen a startiterate $u_k^0 = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \in \mathbb{R}^{225}$, *given* $u_k^\nu \in \mathbb{R}^{225}$ *(we have chosen* u_k^ν *locally for which the maximum of the equation (5.3) is attained), compute* $u_k^{\nu+1} \in \mathbb{R}^{225}$ *as the solution of the equation*

$$A_k^\nu u_k^{\nu+1} = f_k^\nu, \tag{4.37}$$

by multigrid method. Where A_k^ν *and* f_k^ν *are defined in (5.5) and (5.6) with the data of the problem (4.36). The solution of (4.37) satisfies*

$$\max_{1\le i\le N}\left(A_{k,i}u_{k,i}^{\nu+1} - f_{k,i}, u_{k,i}^{\nu+1} - k - u_{k,i}^\nu\right) = \begin{cases} A_{k,i}u_{k,i}^{\nu+1} - f_{k,i}, & k \ge 0.02, \\ \max\left(A_{k,i}u_{k,i}^{\nu+1} - f_{k,i}, u_{k,i}^{\nu+1} - k - u_{k,i}^\nu\right)_{0\le k\le 0.02}. \end{cases} \tag{4.38}$$

On the other hand, if we choose $u_k^{\nu-1} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \in \mathbb{R}^{225}$, *and* $0 < k < 0.02$, *the solution (4.8) of the problem (4.36) is the solution of the following variational Inequalitie:*

$$\begin{cases} Au \le f, & \text{in } \Omega = \left\{(x_1, x_2) \mid x_1^2 + x_2^2 \le 1\right\}, \\ \langle Au - f, u^\nu - k\rangle = 0, & \\ u^\nu \le k, & k = 0.001, \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{4.39}$$

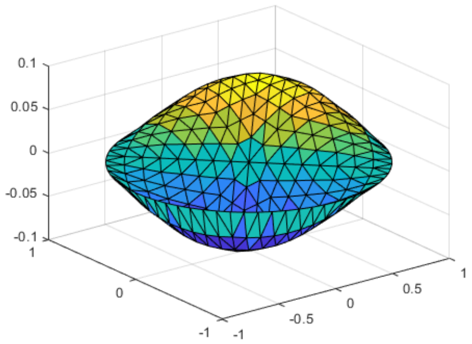

Figure 4.6: Solution of the equation (4.38) by 25 iterations of multigrid method with $k = 1$

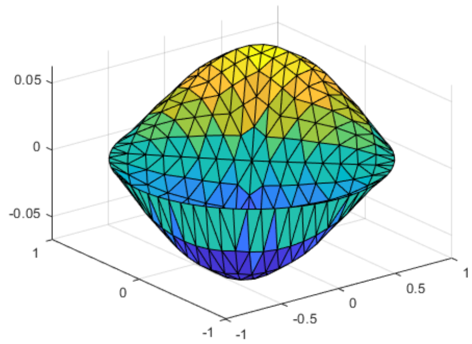

Figure 4.7: Solution of the equation (4.38) by 25 iterations of multigrid method with $k = 0.001$

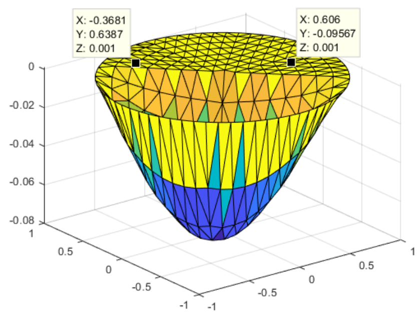

Figure 4.8: Solution of the VI (4.39) by 25 iterations of MGM with k=0.001

All codes were implemented in MATLAB and executed on a computer with Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz 1.80 GHz and 8 GB memory.

5

The L^∞-Norm Convergence of Non linear Multigrid Method for Solving Elliptic Quasi-Variational Inequalities with Nonlinear Source Terms

In this chapter, Newton-multigrid method on adaptive finite element discretisation is employed for the solution of the nonlinear Q.V.Is. Newton's method has been used as an outer iteration for the standard linearization, and the standard multigrid is used as the inner iteration for solving the Jacobian system obtained at each step. The L^∞-Norm convergence of Newton-multigrid method has been shown in the sense that the multigrid scheme has a contraction number with respect to the L^∞-norm.

The outline remainder of this chapter is as follows. In §2, we state some hypotheses and we present our continuous problem. In §3 the linear finite element discretizations has been applied to get a system of equations. After the HJB formulation, in §4 Newton-multigrid method has been described for solving the algebraic system obtained. In §5 the L^∞-norm convergence results of this multigrid method is proved.

5.1 Continuous problem

Assume that $\Omega \in \mathbb{R}^N$ is a bounded open set, and we denote by $\partial\Omega$ for its sufficiently smooth boundary, for $u, w \in V$ $\left(V = H_0^1(\Omega)\right)$, Let $\vartheta(u, v)$ is a variational form associated with the continuous surjective M-function and non-linear operator A, furthermore, assume that A is a strictly T-monotone and satisfies the coerciveness assumption. Also, Let $f(u)$ represent a nonlinear right-hand side, such that:

$$f(u) \in L^\infty \cap C^1(\bar{\Omega}).$$

Consider the nonlinear obstacle problem: Find $u \in K_g(u)$ solution of

$$\begin{cases} \vartheta(u, w-u) \geq \langle f(u), v-u\rangle & v \in K_g(u), \\ u \leq Mu \qquad Mu \geq 0, & \\ u = g \qquad on \;\; \partial\Omega. & \end{cases} \tag{5.1}$$

Where the operator M and the set $K_g(u)$ are given in 2.18 and 2.17 respectively. According to the preceding assumptions, the problem (5.1) has a unique solution [22].

5.2 Discrete problem

Similarly, as in the linear case, we present a nested quasi-uniform triangulations, then we associate on each mesh size parameter h_k an analogous discretization of the Q.V.I (5.1) by FEM.

The application of FEM to discretise the QVI (5.1) leads to the solution of the discrete QVI (5.2) in finite dimension. Find $u_k \in K_{g,k}$ such that

$$\begin{cases} \langle A_k u_k, v_k - u_k\rangle \geq \langle f(u_k), v_k - u_k\rangle, & \forall v_k \in K_{g,k}, \\ u_k \leq M_k u_k, \quad v_k \leq M_k u_k, & \end{cases} \tag{5.2}$$

where

$$\begin{aligned} f(u_k) &\in L^\infty(\Omega), \\ M_k u_k &= K + \inf_{\epsilon \geq 0,\ (x+\epsilon)\in\bar{\Omega}} u_k(x+\epsilon),\ k \text{ is a positive constant}, \\ K_{g,k} &= \{v \in V_k : v = \pi_k g \text{ on } \partial\Omega, v \leq M_k u_k \text{ in } \Omega\}. \end{aligned}$$

The regularity results presented by Ph. Cortey-Dumont in [22] are satisfied for our problem. Assume that the assumptions seted in [22] for the non-lineair case are fulfilled, then the uniqueness and existence of the solution of the discrete problem (5.2) are well-known.

Moreover, we get the regularity 2.26.

5.3 Newton-Multigrid method for solving Q.V.Is

The HJB-formulation of the discret problem (5.2)

Usually, we can write the QVI (5.2) as the HJB equation (5.3). Let u_k^ν be a unique solution of the following HJB equation

$$\max_{1\le i\le N}\left(A_{k,i}[u_k^\nu]u_{k,i}^\nu - f[u_k^\nu]_{k,i}\,, u_{k,i}^\nu - M_k u_{k,i}^{\nu-1}\right) = 0. \tag{5.3}$$

Choose a start vector $u_k^0 \in U_k$. Starting from the iterate $u_k^\nu \in U_k$, $\nu \ge 0$, and compute $u_k^{\nu+1} \in U_k$ the solution of the non-linear system

$$A_k^\nu[u_k^\nu]u_k^{\nu+1} = f_k^\nu[u_k^\nu], \tag{5.4}$$

where

$$A_k^\nu[u_k^\nu] = \begin{cases} A_{k,i}, & \text{if } i \in J_k^1\left(u_k^\nu\right), \\ I_{k,i}, & \text{if } i \in J_k^2\left(u_k^\nu\right) \cup J_k^3\left(u_k^\nu\right). \end{cases} \tag{5.5}$$

$$f_k^\nu[u_k^\nu] = \begin{cases} f_{k,i}, & \text{if } i \in J_k^1\left(u_k^\nu\right), \\ M_k u_{k,i}^{\nu-1}, & \text{if } i \in J_k^2\left(u_k^\nu\right) \cup J_k^3\left(u_k^\nu\right). \end{cases} \tag{5.6}$$

with $A_{k,i}$ (resp. $f_{k,i}$)is the i^{th} row of the discretization matrix $A_k[u_k^\nu]$ (resp. the i^{th} element of the right-hand side $f_k[u_k^\nu]$), and the splitting sets J_k are defined in 4.7 (with non-linear data).

Similarly, as in the linear case (32), the monotone convergence of the iterates can be proved in the nonlinear case.

Newton-Multigrid Algorithm

The multigrid method is also used for solving nonlinear partial differential equations. In this part, we do this for the solution of the nonlinear system, we need to apply non linear multigrid methods in which Newton's method is used as a global linearization is used. Then the multigrid methods can be applied for the solution of the Jacobian system of equations in each linearization step.

Assume that v_k^ν is an approach to the exact solution u_k^ν of the system (5.4), set the error $e_k = u_k^\nu - v_k^\nu$.

Select $\mathcal{R}_k = f_k^\nu[u_k^\nu] - A_k^\nu[u_k^\nu]v_k^\nu$ to be residual. Subtracting (5.4) from the residual equation, we get

$$A_k^\nu[u_k^\nu]u_k^\nu - A_k^\nu[u_k^\nu]v_k^\nu = \mathcal{R}_k. \tag{5.7}$$

furthermore, because $A_k^\nu[u_k^\nu]$ is a non-linear operator, so $A_k^\nu[u_k^\nu](e_k) \neq \mathcal{R}_k$, this means that for the nonlinear problem, we can not find the error by solving the linear system on the coarsest grid, as in the standard multigrids. but, we have to use (5.7) as residual equation. Consequently, By the application of Newton's method to (5.4), (5.7) can be used as a basis for the standard multigrid: To simplify the idea, we use $\mathcal{F}_k$ as the non-linear operator, and we define the system (5.4) on fine grid as follow:

$$\mathcal{F}_k(u_k^\nu) = A_k^\nu[u_k^\nu]u_k^\nu - f_k^\nu[u_k^\nu] = 0. \tag{5.8}$$

The residual is rewritten on the fine grid as:

$$\mathcal{R}_k(u_k^\nu) = -\mathcal{F}_k(u_k^\nu).$$

Then the Newton iteration to solve (5.8), is represent as

$$u_k^{\nu+1} = u_k^\nu + (\mathbb{J}_k(u_k^\nu))^{-1}\mathcal{R}_k(u_k^\nu), \quad k = 0, 1, 2, 3, \ldots$$

With $\mathbb{J}_k$ denote the Jacobian matrix of $\mathcal{F}_k$.
The multigrid methods for solving the nonlinear system (5.8) can be obtained by the solution of the following Jacobian system

$$\mathbb{J}_k(u_k^\nu)e_k^\nu = \mathcal{R}_k(u_k^\nu). \tag{5.9}$$

In the linear multigrid, we choose an iterate $e_k^\nu, \nu > 0$, we get $\bar{e}_k^\nu$ by α applications of a smoothing iterative method for solving (5.9), which defined as in 4.20.

Denoted by e_k^* for the solution of the system (5.9). Setting the error as $\mathcal{E}_k^\nu = \bar{e}_k^\nu - e_k^*$, and the residual $d_k^{(\nu)} = \mathcal{R}_k(u_k^\nu) - \mathbb{J}_k(u_k^\nu)\bar{e}_k^\nu$, therefore, the system (5.9) can be represented as

$$\mathbb{J}_k(u_k^\nu)\left(\bar{e}_k^\nu + \mathcal{E}_k^\nu\right) = \mathcal{R}_k(u_k^\nu).$$

Then, we get a residual equation

$$\mathbb{J}_k(u_k^\nu)\mathcal{E}_k^\nu = \mathcal{R}_k(u_k^\nu) - \mathbb{J}_k(u_k^\nu)\bar{e}_k^\nu = d_k^{(\nu)}.$$

Therefore, to determine $\mathcal{E}_k^\nu$ completely, we require to compute $\mathcal{E}_{k-1}^\nu$ at level $(k-1)$ as the solution of the coarse grid system

$$\mathbb{J}_{k-1}(e_{k-1}^\nu)\mathcal{E}_{k-1}^\nu = d_{k-1}^{(\nu)}. \tag{5.10}$$

Were $\mathcal{E}^{\nu}_{k-1}\left(resp\ \mathbb{J}^{\nu}_{k-1}(e^{\nu}_{k-1}), d^{(\nu)}_{k-1}\right)$ *represent an approach at the level* $k-1$ *of* $\mathcal{E}^{\nu}_{k}\left(resp\ \mathbb{J}^{\nu}_{k}(e^{\nu}_{k}), d^{(\nu)}_{k}\right)$.

$$\mathcal{E}^{\nu}_{k-1} = R_k\mathcal{E}^{\nu}_{k}, \qquad \mathbb{J}^{\nu}_{k-1}(e^{\nu}_{k-1}) = R_k\mathbb{J}^{\nu}_{k}(e^{\nu}_{k})P_k, \qquad d^{(\nu)}_{k-1} = R_k d^{(\nu)}_{k},$$

Where the restriction operator R_k *and the prolongation* P_k *operator are defined in 4.23. Consequently, the improved iterate at the level k can be defined as*

$$e^{\nu+1}_{k} = \bar{e}^{\nu}_{k} + P_k\left(\mathcal{E}^{\nu}_{k-1}\right). \tag{5.11}$$

Newton-multigrid algorithm may be described as 6.

Algorithm 6 Newton-Muligrid

Select a start guess u^0_k and a required tolerance η.
while $(\mathcal{R}_k < \eta)$ **do**
 Calculate $\mathbb{J}_k$ a Jacobian matrix of the nonlinear system and compute the residual: $\mathcal{R}_k$
 The muligrid solution of the Jacobian system: $e^{\nu}_{k} \leftarrow MGM(\mathbb{J}_k, \mathcal{R}_k, e^{\nu}_{k})$
 Setting the improved iterate: $u^{\nu}_{k} \leftarrow u^{\nu}_{k} + e^{\nu}_{k}$;
 Compute the residual: $\mathcal{R}_k \leftarrow f^{\nu}_{k}[u^{\nu}_{k}] - A^{\nu}_{k}[u^{\nu}_{k}]u^{\nu}_{k}$;
end while

5.4 Multigrid convergence on the maximum norm

In this part, the convergence analysis of Multigrid methods and Newton's method is described under hypotheses that are similar to those that have been imposed for the nonlinear systems.

We now sitting the main assumptions:

1. $\exists\, u^*_k \in K_{g,k}$ *for which* $\mathcal{F}_k(u^*_k) = 0$.

2. *For all* u_k *in the neighborhood of* u^*_k *there is a linear function* $\mathcal{F}'_k(u_k)$ *such that:*

$$\forall\, \varepsilon > 0,\ \exists\, \eta > 0,\ \|\mathcal{F}_k(u_k) - \mathcal{F}_k(u^*_k) - \mathcal{F}'_k(u^*_k)(u_k - u^*_k)\| \le \varepsilon\|u_k - u^*_k\|,$$

whenever $\|u_k - u^*_k\| < \eta$.

3. *The derivative of* $\mathcal{F}_k(u_k)$ *is invertible and its inverse represent a bounded linear operator, for all* u_k *in the neighborhood of* u^*_k*, and*

$$\|(\mathcal{F}'_k(u_k))^{-1}\| \le \kappa,$$

with a constant κ*. In addition, assume that* $(\mathcal{F}'_k(u_k))^{-1}$ *represent continuous mapping* *in u_k. For any $\varepsilon > 0$ there is an $\eta > 0$ such that*

$$\|I - \mathcal{F}_k'(u_k^*)(\mathcal{F}_k'(u_k))^{-1}\| \leq \varepsilon,$$

and

$$\|I - (\mathcal{F}_k'(u_k^*))^{-1}\mathcal{F}_k'(u_k)\| \leq \varepsilon,$$

holds, whenever $\|u_k - u_k^*\| < \eta$.

5.4.1 The iteration matrix of the multigrid methods

The iteration matrix of Two-grid scheme with α_2 postsmoothing iterations and α_1 presmoothing iterations on level k is defined as in (4.24)

$$TG_k(\alpha_1, \alpha_2) = \mathcal{S}_k^{\alpha_2}\left(\mathbb{J}_k^{-1} - P_k\mathbb{J}_{k-1}^{-1}R_k\right)\mathbb{J}_k\mathcal{S}_k^{\alpha_1}. \tag{5.12}$$

One can easily prove that Multigrid methods represent a linear iterative method associated with the iteration matrix defined as in 4.25.

5.4.2 Approximation property

The demonstration of the following theorem is based on the combination of Theorem 22 and Lemma 31.

Theorem 40 ([29]) *Under the preceding hypotheses, the matrix*

$$\chi = \left[\mathbb{J}_k^{-1} - P_k\mathbb{J}_{k-1}^{-1}R_k\right],$$

fulfils the approximation property:

$$\|\chi\|_\infty \leq Ch_k^2\left|\log h_k\right|^2. \tag{5.13}$$

Proof. *Theorem (22) states*

$$\|u - u_k\|_{L^\infty} \leq Ch_k^2\left|\log h_k\right|^2 \|\nabla(f)\|_{L^\infty}, \qquad u \in W^{2,p}.$$

Then

$$\left\|u - u_k^*\right\|_{L^\infty} \leq Ch_k^2\left|\log h_k\right|^2 \|\nabla(f)\|_{L^\infty}.$$

Therefore, we obtain

$$\begin{aligned} \left\|u_k^* - u_{k-1}^*\right\|_{L^\infty} &\leq \left\|u_k^* - u\right\|_{L^\infty} + \left\|u_{k-1}^* - u\right\|_{L^\infty} \\ &\leq Ch_k^2 \left|\log h_k\right|^2 \|\nabla(f)\|_{L^\infty} + Ch_{k-1}^2 \left|\log h_k\right|^2 \|\nabla(f)\|_{L^\infty} \\ &\leq Ch_k^2 \left|\log h_k\right|^2 \|\nabla(f)\|_{L^\infty}. \end{aligned} \tag{5.14}$$

Galerkin discretization leads to:

$$\vartheta(r_k u, r_k v) = \langle A_k u, v\rangle_{L^2}, \quad \forall u, v \in U_k$$

Then

$$(\vartheta(r_k u, r_k v))' = \langle(\mathbb{J}_k)u, v\rangle_{L^2}, \quad \forall u, v \in U_k$$

Furthermore, we have

$$(\vartheta\left(r_k^{-1}(A_k)^{-1}(f_k), v\right))' = \left\langle \left(r_k^*\right)^{-1} \nabla(f), v\right\rangle_{L^2}, \ \forall v \in K_{g,k},$$

Assume that $u_k \in K_{g,k}$ and $u_{k-1} \in K_{g,k-1}$, be such that

$$\vartheta(u_k, v) = \left\langle \left(r_k^*\right)^{-1} f, v\right\rangle_{L^2},$$

$$\vartheta(u_{k-1}, v) = \left\langle \left(r_{k-1}^*\right)^{-1} f, v\right\rangle_{L^2},$$

therefore $u_k^* = r_k^{-1}\mathbb{J}_k^{-1}\nabla(f)$ and $u_{k-1}^* = r_{k-1}^{-1}\mathbb{J}_{k-1}^{-1}R_k\nabla(f)$.
The combination of (5.14) and Lemma 31 gives

$$\left\|r_k^{-1}\mathbb{J}_k^{-1}\nabla(f) - r_{k-1}^{-1}\mathbb{J}_{k-1}^{-1}R_k\nabla(f)\right\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2 \|\nabla(f)\|_{L^\infty}.$$

Then

$$\left\|\mathbb{J}_k^{-1} - r_k^{-1}r_{k-1}\mathbb{J}_{k-1}^{-1}R_k\right\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2.$$

As a result, the proof is completed

$$\left\|\mathbb{J}_k^{-1} - P_k(\mathbb{J}_{k-1})^{-1}R_k\right\|_\infty \leq Ch_k^2 \left|\log h_k\right|^2.$$

■

5.4.3 Smoothing prorperty

In order to prove a smoothing prorperty we decompose the matrix $\mathbb{J}_k = E_k - N_k$, and use the assumptions 4.28 and 4.29. For the presmoothing and the post-smoothing, we use a relaxation iterative method defined in 4.30.

Theorem 41 ([48]) *Assume that the preceding hypotheses are satisfied, therefore there is a constant C independent of k and α for which the following smoothing property holds:*

$$\left\|(\mathbb{J}_k)\,\mathcal{S}_k^{\alpha}\right\|_{\infty} \le C\frac{1}{\sqrt{\alpha}}h_k^{-2}, \tag{5.15}$$

5.4.4 Convergence result of multigrid methods

Theorem 42 *According to the preceding hypotheses, there is a constant C independont of k and α, such that the iterate $u_k^{\nu}, \nu \ge 0$ for the two grids k and $k-1$ fulfils:*

$$\left\|u_k^{\nu+1} - u_k^*\right\|_{\infty} \le \left(\frac{C}{\sqrt{\alpha}}\left|\log h_k\right|^2\right)\left\|u_k^{\nu} - u_k^*\right\|_{\infty}. \tag{5.16}$$

Usually, we will choose more than two grids hierarchies. The iteration matrix of the multigrid methods in this case has been defined by recurrence using the iteration matrix of the two-grids scheme (5.12) at all levels. Therefore, as in the linear case the L^{∞}-convergence result (39) can be easily deduced from the preceding result if we suppose that (4.33) holds.

For the outer iteration, once the approximate solution is close to the actual solution of the non-linear system, Newton's method converges quadratically [47].

$$\|u_k^{\nu+1} - u_k^*\| \le C\|u_k^{\nu} - u_k^*\|^2. \tag{5.17}$$

6

Conclusions and Future Works

In this dissertation, we have used a multigrid method as an efficient iterative solution to solve a certain classes of discretized elliptic quasi-variational inequalities. In the first part, we have chosen the variational form as a bilinear form associated with the linear operator A and chosen the right-hand side f as a linear form. For the discretization, adaptive finite element approximation has been applied on a non-polygonal domain. After the discretization, an iterative process proposed by Hoppe in [33] is used to construct an algebraic linear system which can be solved by multigrid, then we have explained how to apply a multigrid method for the solution of this discrete algebraic linear system. We have established a uniform convergence of our problem and proved that a multigrid method has a contraction number with respect to the L^∞-norm. In the numerical experiments, we have presented an example of a quasi-variational inequality arising from a stochastic inventory problem with impulse control. From these numerical results, we show that Gauss-Seidel and relaxed Gauss-Seidel are still not good even after a large number of iterations. Unlike the Multigrid method, which has a debugging feature (the high-frequency error is reduced by relaxation, while low-frequency error is mapped to coarse grids and reduced there), it converges in a few iterations.

In the second part of this dissertation, we have chosen the variational form as a form associated with a non-linear operator A and chosen the right-hand side $f(u)$ as a non-linear form. Newton-Multigrid methods have been applied for solving a nonlinear quasi-variational inequality related to HJB equation. Then, three numerical methods have been proposed. For the discretization, FEM is used to drive the discrete system of equations, *and the two approaches of Newton-Multigrid method: For the outer iteration, Newton's method has been used as a global linearization and use a multigrid scheme for the solution of the Jacobian system. The L^{∞}-norm convergence of this nonlinear multigrids has been proved successfully.*

In the future work, we are interested to apply the parallel full multigrid methods for the solution of parabolic quasi-variational inequalities associated with stochastic control problems. For more complicated problems, we are interested in addressing this method for solving three-dimensional problems. Additionally, it is well known that the iteration of any non-linear method become much faster if the start value is more accurate, therefore, it is also interesting to apply nested iteration, which provides a good initial guess by first solve the problem on the coarsest grid, and doing that at all levels, which is Known as FMG approach.

Appendices

A

MATLAB Codes

Code A.1: **Matlab Programme of The Method of Manufactrued Solution & Multigrid Algorithm**

```
A = imread('BOYYA2','jpg');
BOYYA2 = A;
ue=BOYYA2(:,:,1);
ue=double(ue);
b=zeros(797);
u_expanded=zeros(797);
u_expanded(2:end-1,2:end-1)=ue;
dx=1/(795);
for i=2:796
for j=2:796
    b(i,j)=(u_expanded(i+1,j)+u_expanded(i-1,j)+...
        u_expanded(i,j+1)+...
        u_expanded(i,j-1)-4*u_expanded(i,j))/dx^2;
end
end
b = b(2:end-1,2:end-1);
u0=zeros(795);
um=u0;
um=Multigrid(um,b);
figure; imshow(um/255); title('Multigrid solution after one
```

```
    iteration');
figure; imshow(ue/255); title('Exact solution');
ug=u0;
for i=1:1000
ug=Gouss(ug,b);
end
figure; imshow(ug/255); title('Gaus seidel solution after 1000
    iterations');
uj=u0;
for i=1:1000
uj=Jacoby(uj,b);
end
figure; imshow(uj/255); title('Jacobi solution after 1000
    iterations');
ujj=u0;

%%
```

Code A.2: **Matlab Function of Multigrid Method**

```
function  [u1]= Multigrid(u0,b)
  n_inter=size(b,1)-1;
dx=1/n_inter;
for i=1:5
    u0=Gouss(u0,b);
end
r=zeros(size(b));
for i=2:n_inter
    for j=2:n_inter
           r(i,j)=b(i,j)-(u0(i+1,j)+...
        u0(i-1,j)+...
        u0(i,j+1)+...
        u0(i,j-1)-4*u0(i,j))/dx^2;
    end
end
   rc=zeros(n_inter/2+1);
   rc(2:end-1,2:end-1)=r(3:2:end-2,3:2:end-2);
   du=zeros(size(rc));
   if size(rc,1)<4
   for i=1:5
       du=Gouss(du,rc);
   end
   else
       for gamma=1:2
      du=Multigrid(du,rc);
       end
   end
   duf=zeros(size(b));
   duf(3:2:end-2,3:2:end-2)=du(2:end-1,2:end-1);
   duf(2:2:end-1,3:2:end-2)=0.5*(duf(1:2:end-2,3:2:end-2)...
                              +duf(3:2:end,3:2:end-2));
   duf(:,2:2:end-1)=0.5*(duf(:,1:2:end-2)...
                              +duf(:,3:2:end));
    u1=u0+duf;
    for i=1:5
       u1=Gouss(u1,b);
   end
end
```

List of Publications and Presentations

Refereed Journals/Manuscripts Under Preparation

1. *M.E.Belouafi and M.Beggas. Maximum Norm Convergence of Newton-Multigrid Methods for Elliptic Quasi-Variational Inequalities with Nonlinear Source Terms, Advances in Mathematics: Scientific Journal,* ***11(10)*** *(2022), 969–983.*

2. *M.E.Belouafi and M.Beggas. Uniform Convergence of Multigrid Methods for Elliptic Quasi-Variational Inequalities and its Implementation, Communications in Mathematics and Applications, 2023.(accepted)*

Conference Abstracts/Posters/Presentations

1. *M.E.Belouafi and M.Beggas. Multigrid methods for Elliptical Quasi-Variational Inequalities, International Pluridisciplinary PhD Meeting (IPPM'20), University of El Oued, Algeria, February 23-26, 2020.*

2. *M.E.Belouafi and M.Beggas. Uniform Convergence Of Multigrid Methods For Variational Inequalities, 1st International Conference on Pure and Applied Mathematics (IC-PAM'21), University of Kasdi Merbah Ouargla, Algeria, May 26-27, 2021.*

3. *M.E.Belouafi and M.Beggas. Uniform Convergence of Multigrid Methods for Elliptic Quasi-Variational Inequalities, National Conference Applied Mathematics and Mathematical Modeling(A3M'2021), University of El Oued, Algeria, March 30-31, 2021.*

4. *M.E.Belouafi and M.Beggas. Maximum Norm Convergence of Newton-Multigrid Methods for Elliptic Quasi-Variational Inequalities with Nonlinear Source Terms, National Conference on Mathematics and Applications (NCMA2022), Abdelhafid Boussouf University Center of Mila, Algeria, November 29-30, 2022.*

5. *M.E.Belouafi and M.Beggas. Implementation of Algebraic Multigrid Method for Variational Inequality Related to HJB-Equation, International Conference in Op-*

erator Theory, EDP and Applications (CITO'22), University of El Oued, Algeria, December 07-08, 2022.

6. *M.E.Belouafi and M.Beggas. The Uniform Convergence of Non-linear Multigrid Methods for Solving a Nonlinear Obstacle Problems, 2nd National Conference on Pure and Applied Mathematics (NCPAM'2022), Amar Telidji University – Laghouat, December 18 - 19, 2022.*

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