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Domaine : Mathématiques et Informatique

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Les Théorèmes de Point Fixe dans Espace Uniforme : principe de contraction de Banach dans des structures non métrique

Présentée Par : OUANI Afaf
Soutenu le : 08/06/2023

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Presented by : OUANI Afaf
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In front of the jury composed of :

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مــلـخص

الهدف من أطروحتنا هو تمديد بعض المفاهيم و النتائج في التحليل الدالي, و بعبارة أخرى البنى التي لا تملك مسافة. لهذا نقدم بنية غير مترية أكثر ثراء في الخصائص من البنية الطبولوجية, على سبيل المثال: الطبولوجيا المنتظمة, و خاصة الفضاءات المزودة بعائلة من أشباه النضيمات. في وقتنا الحالي, كتطبيقات لنظرية النقطة الصامدة, فإن الأكثر اعتبارا هو مبدأ التقلص لبناخ و نظريات النقطة الصامدة للدوال الرتيبة في فضاءات بناخ المرتبة.

الكلمات المفتاحية : المعادلات التفاضلية الكسرية, الشروط النبضية , نظريات النقطة الثابتة

Abstract

The aim of our thesis is the extension of certain notions and results in functional analysis to non-metric topological spaces, in other words structures without distance. For this, we introduce a non-metric structure richer than the topological structure, for example : the uniform topology, and especially the spaces endowed with a family of semi-norms. At present, as applications of the fixed point theory, the most considered are the fixed point theorems of monotonic function in ordered Banach spaces.

Keywords : Fractional differential equations, impulsive conditions, fixed point theorems.

Résumé

Le but de notre thèse est l'extension de certaines notions et résultats en analyse fonctionnelle à des espaces topologiques non métriques, autrement dit des structures sans distance. Pour cela, nous introduisons une structure non métrique plus riche que la structure topologique, par exemple : la topologie uniforme, et spécialement les espaces muni d'une famille de semi-normes. A présent, comme applications de la théorie du point fixe, les plus considérés sont les théorèmes du point fixe de la fonction monotone dans les espaces de Banach ordonnés.

Mots clés : differentiales fractionaires , conditions impulsives, thèorèmes de point fixe.

Thanks

First of all thanks to **ALLAH** Almighty for the opportunity Who asked me to do this job.
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Dedication

Thanks to my parents for their constant support, encouragement and trust.
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Contents

General Introduction

In this thesis we study the extension of certain fixed point theorems to topological vector spaces not necessarily endowed with a Banach structure. Especially the principle of Banach contraction, and theorems in ordered Banach spaces.

The fixed point theorems are the heart of non-linear analysis, because it is an important tools for proving the existence of solutions of many problems modelled various phenomena in: physics, chemistry, biology,...

Let $\mathbb{X}$ be a set ,and T be a map such that:

$$\begin{array}{cccc} T & : & \mathbb{X} & \longrightarrow & \mathbb{X} \\ & & x & \longmapsto & T(x), \end{array}$$

the solution of equation $T(x) = x$ is called a fixed point of T. The original of fixed point an important branch of non-linear functional analysis, witch dates back to the latter of the 19^{th} century. The rest in use successive approximations of the existence, uniqueness of the solutions, particular to differential equations.

The development of this theories is the interesting branch of most research works, it rich the advancement of non-linear analysis. For that, it appeared two type of fixed point theory.

- **Topological-type**: gives only the existence of fixed point, for example: Brower (1912), schauder (1934), schauefer (1955),...

- **Metric-type**: The famous of them is the principle of Banach (1922), also called theorem of Picard. This theory provides an approximation fixed point-algorithm by a limit of iterative sequence introduce by Liouville (1837), and it developed by Picard (1920). It prove all contraction map define in complete metric-space admits a unique fixed point.

In other hand, the last conditions on the map and space don't verified in the most of problems. Hence, the necessity of extension of Banach contraction principle to broader its scope. That leads to the natural question:

By what means can we generalise this principle?

The answer is our theme.

Our these continued three chapter. In the first chapter, we present many notions and mathematical definitions; Especially the notion of uniformed space, this structure of uniformity consists a necessary based for our study. In the second chapter, we recall the theory of fixed point of Banach. After that, we generalise the last theorem in uniform structure. And we extend many notions such that: the boundedness, the contraction and the set of indices of semi-norm's family. Also, we applied that for intial value problem of non-linear fractional differential equation with impulsive conditions and maxima defined as follow:

$$ {}^{C}D^{\alpha}u\left(t\right)=f\left(t,\max_{\sigma\in[a(t),b(t)]}u\left(\sigma\right),u\left(t-\tau_{1}\left(t\right)\right),...,u\left(t-\tau_{N}\left(t\right)\right)\right),\qquad t>0, \tag{1} $$

$$ \Delta u\mid_{t=t_k}=I_k\left(u\left(t_k^- -\rho(t_k^-)\right)\right), \tag{2} $$

with the initial condition function

$$ u\left(t\right)=\phi\left(t\right),\quad t\leq 0, \tag{3} $$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative operator of order $\alpha\in\,]0,1[$, N is a positive integer, a, b and τ_i (with $0\leq i\leq N$) are real continuous functions defined on $\mathbb{R}_+=[0,+\infty[$ subject to conditions which will be specified later, $\phi:]-\infty,0]\longrightarrow\mathbb{R}$ is a continuous function such that $\phi\left(0\right)=\phi_0>0$, and $f:\mathbb{R}_+\times\mathbb{R}^{1+N}\longrightarrow\mathbb{R}$ is a non-linear continuous function.
such that $I_k:\mathbb{R}\longrightarrow\mathbb{R}$, and $\Delta u\mid_{t=t_k}=u\left(t_k^+\right)-u\left(t_k^-\right)$, where $u\left(t_k^+\right)=\lim_{h\longrightarrow 0^+}u\left(t_k+h\right)$ and $u\left(t_k^-\right)=\lim_{h\longrightarrow 0^-}u\left(t_k+h\right)$ represent the right and left limits of $u(t)$ at $t=t_k$.
Where the impulsive differential equations (for $\alpha\in\mathbb{N}$) was very useful in the physical and social science problems.
We denote that the existence and uniqueness of solution for problem (2.8-2.10) has been tried in [43]; where the very famous theories of fixed point are used. In [43] the authors obtained the result by using three approaches: the first based on Banach contraction principle, the second based on fixed point theorem of Scheafer, and the third based on the nonlinear alternative of Leray-Schauder.
In our work, we proved our result by the new generalization of Banach contraction principle, where we broader the space to complete semi-normed space provide with a family of semi norms.

In the third chapter, we extend the fixed point theory of monotone functions in an ordered Banach space to uniform space. We generalise some properties related to the notion of order, and we weaken some conditions on the functions, in particular the compactness. Finally, we give an example of application of our results.

General Notations

Let $\mathbb{X}, \mathbb{Y}$ be two set, and $\mathbb{A}, \mathbb{B}$ are a sub-set of $\mathbb{X}$.

Notation	:	**Definition**
$\mathbb{A} \subset \mathbb{X}$	:	$\forall x \in A \Rightarrow x \in X$
$\emptyset$	:	the empty set
$\mathbb{C}$	:	the set of complex numbers
$\mathbb{R}$	:	the set of real numbers
$\mathbb{N}$	:	the set of natural numbers
$C_X\mathbb{A}$	:	the complement of $\mathbb{A}$ in $\mathbb{X}$, such that $\mathbb{A} \subset \mathbb{X}$
$\mathbb{A} \cap \mathbb{B}$	$\doteqdot$	$\{x \in \mathbb{A} \text{ and } x \in \mathbb{B}\}$
$\mathbb{A} \cup \mathbb{B}$	$\doteqdot$	$\{x \in \mathbb{A} \text{ or } x \in \mathbb{B}\}$
$\mathbb{X} + \mathbb{Y}$	$\doteqdot$	$\{x + y; x \in \mathbb{X}, y \in \mathbb{Y}\}$
$\mathbb{X} \times \mathbb{Y}$	$\doteqdot$	$\{(x, y); x \in \mathbb{X}, y \in \mathbb{Y}\}$
$C(\mathbb{X}, \mathbb{Y})$	$\doteqdot$	$\{f : \mathbb{X} \longrightarrow \mathbb{Y} \text{ s.t.: } f \text{ is continuous}\}$
$C(\mathbb{X})$	$\doteqdot$	$\{f : \mathbb{X} \longrightarrow \mathbb{X} \text{ s.t.: } f \text{ is continuous}\}$
$B(\mathbb{X}, \mathbb{K})$	$\doteqdot$	$\{f : \mathbb{X} \longrightarrow \mathbb{K} \text{ s.t. : } f \text{ is bounded } \}$ such as $\mathbb{K} = \mathbb{R}$ ou $\mathbb{C}$
$\partial\mathbb{X}$	:	the boundary $\mathbb{X}$
i.e.	:	it means
In ordered space $x \ll y$	:	x greater than y
In ordered space $x \gg y$	:	x less than y
hlctvs	:	Hausdroff locally convex topological vector space

Chapter 1

Uniform Space

In the first chapter, we recall a many mathematical notions. we begin by the principle axioms of the topology. Then, we define the metric space, the normed space. After that, we present the uniform space witch is the important tool in our study, and the relation between it and the semi-normed space. In the final of this chapter, we give many definitions, witch are more generalise for example: the notion of contraction, fixed point, bounded set in the new structure. The main works used in this part of our work are [9], [39] and [31].

1.1 Topological Space

THE Topology is mathematical formalisation of intuitive geometric concepts such as proximity, distance, neighbourhood, limits,ect, in Euclidean space of 2 or 3 dimensions.
This formalisation is expressed in a very general axiomatic language. The generality introduced into the language has advantage of being applicable to very different situations, such as the study of "functional spaces".
The downside is that in the particular case we stared with the lose of a certain number of intuitively obvious properties; for example, the possibility of separating two points by disjoint neighbourhoods. In fact, in many cases, people try to restrict the generality by attaching axioms to approach the original intuition; the study of separation spaces, metric spaces, ect.
Recall that if $\mathbb{X}$ is a set, we denote the set of all parts of $\mathbb{X}$ by $\mathcal{P}(X)$.

Definition 1.1 (see [39])**.** *Let $\mathcal{T}$ a set of parts of $\mathbb{X}$(i.e. $\mathcal{T} \subset \mathcal{P}(X)$).*
We say that $\mathcal{T}$ be a topology on $\mathbb{X}$, if it verified the following properties (axioms of open sets):

(O1) The empty set $\varnothing$, and the set $\mathbb{X}$ include in $\mathcal{T}$.

(O2) If $u, v \in \mathcal{T}$,so $u \cap v \in \mathcal{T}$.

(O3) If $(u_i)_{i\in I}$ be a family of parts of $\mathbb{X}$ include in $\mathcal{T}$, then $\cup_{i\in I}u_i \in \mathcal{T}$ (where I is any set of indices).

The set $\mathbb{X}$ provided with the topology $\mathcal{T}$ called topological spaces. We denote the elements of $\mathcal{T}$ open set, or open of $\mathbb{X}$.Properties (O1),(O2) and (O3) can be formulated in the following way:

- The empty set $\emptyset$ and the set $\mathbb{X}$ are open parts.
- The intersection of two open parts is open part.
- The union of all family of open parts is open part.

Construction of topology from base of open sets:

Definition 1.2 (see [39])**.** *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B}$ a part of $\mathcal{T}$. We said that $\mathcal{B}$ a basic of the topology $\mathcal{T}$. If it has the following properties:*

(B1) For all $x \in X$, there exists $U \in \mathcal{B}$ such as $x \in U$.

(B2) For all $U_1, U_2 \in \mathcal{B}$, and all $x \in U_1 \cap U_2$, there exists $U \in \mathcal{B}$ such as $x \in U \subset U_1 \cap U_2$.

Reciprocally; let $\mathbb{X}$ be a set and if $\mathcal{B}$ include in $\mathcal{P}(X)$ verified (B1)and (B2), then there exists a unique topology $\mathcal{T}$ on $\mathbb{X}$, for that $\mathcal{B}$ be a basic.

Indeed, we choose:

$$\mathcal{T} = \{\emptyset\} \cup \{U \subset X; U \textit{ is any union of sets blonging to } \mathcal{B}\}.$$

Namely, U be a subset of $\mathbb{X}$ is open on $\mathcal{T}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \in U$.

Certain topological structures possessing characteristics such as to produce properties in these spaces, which make it rich in important results and very useful in functional analysis. They are for example: separation, completeness, uniformity, ect

· To define and manipulate the previous notions, we must use a central tool called: **neighbourhood**

Definition 1.3 (see [39])**.** *We call for all part of $\mathbb{X}$ containing at least open containing it self x, a neighbourhood of x. We will denote by $\mathcal{B}(x)$ the set of neighbourhood of x. Which verified the following properties, besides the property of containing all x :*

1. *All part that contains a neighbourhood of x be a neighbourhood of x.*

2. *All intersection of a finite family of neighbourhood of x is a neighbourhood of x.*

Construction of topology from basis of neighbourhood:

Neighbourhood system

Definition 1.4 (see [39]). *Let X be a non empty set. suppose that for all $x \in X$ there exists a non empty set $\mathcal{V}(x) \subseteq \mathcal{P}(X)$ that satisfies:*

(T1) If $V \in \mathcal{V}(x)$, so $x \in V$;

(T2) If $V \in \mathcal{V}(x)$ and $V \subseteq W$, so $W \in \mathcal{V}(x)$;

(T3) If $V_1, V_2 \in \mathcal{V}(x)$, so $V_1 \cap V_2 \in \mathcal{V}(x)$;

(T4) If $x \in V$ and $V \in \mathcal{V}(x)$, so there exists $W \in \mathcal{V}(x)$ such that $V \in \mathcal{V}(y)$ for all $y \in W$.

*That entity $\mathcal{V}(x)$ is called **a neighbourhood system** of $x \in X$. The element $V \in \mathcal{V}(x)$ is called **a neighbourhood** of $x \in X$.*

Thus, we can define a topology on X from a base of neighbourhood as defined below. A sub-set $U \subseteq X$ is called **open** if $U \in \mathcal{V}(x)$ for all $x \in X$. The set $\mathcal{T}$ of all open sets is called a **topology** on X and the pair $(X, \mathcal{T})$ is called **topological space**.

Base of neighbourhood

Definition 1.5 (see [39]). *Let $(X, \mathcal{T})$ be a topological space, $x \in X$. We call base of neighbourhood of $\mathbb{X}$, all family $\mathcal{B}(x)$ of neighbourhoods of $\mathbb{X}$ such that for all neighbourhood V of $\mathbb{X}$, there exists $W \in \mathcal{B}(x)$ such that $W \subset V$. Note that if $\mathcal{B}(x)$ be a base of neighbourhood of $\mathbb{X}$, so we have :*

$$\mathcal{V}(x) = \{V \subset X;\ \textit{il existe } W \in \mathcal{B}(x) \textit{ with } W \subset V\}.$$

Namely, $\mathcal{V}(x)$ be the set of parts of X containing a element of $\mathcal{B}(x)$. So, we get $\mathcal{V}(x)$ from $\mathcal{B}(x)$.

The base of open sets in X, and neighbourhood systems of point de X have comparable roles, nevertheless this last notion is more appropriate for the local study and analysis in these structures.

Continuity

Definition 1.6 (see [39])**.** *Let X and Y two non empty sets. If for all $x \in X$ we can associated a unique element $y \in Y$, so this association is called map on X into Y.*

Definition 1.7 (see [39])**.** *Let T be a map of topological space E in topological space F.*
We say that T be continuous at a point a in E if, for all V be a neighbourhood of $T(a)$, there exists U a neighbourhood of a, such that: $T(U) \subset V$.
If the inverse image by T of all neighbourhood of $T(a)$ be a neighbourhood of a.
A map on X into Y is continuous, if it is continuous at all point of $\mathbb{X}$.

Theorem 1.1 (see [39])**.** *Let T be a map of topological space X in topological space Y, we say that it is continuous, if and only if the inverse image by T of all open in Y be open in X.*

Theorem 1.2 (see [39])**.** *For all map T in topological space X into topological space Y be continuous, if and only if the inverse image by T of all closed in Y is a closed in X.*

Convergent sequence

Definition 1.8 (see [39])**.** *Let $(x_n)_{n\geq 0}$ be a sequence in topological space $\mathbb{X}$.*

1. *We say that a point l in $\mathbb{X}$ is* ***limit of the sequence*** *$(x_n)_{n\geq 0}$, or this sequence* ***converge to*** *l, or x_n tends to l when n tends to $+\infty$ if, for all neighbourhood V of l in $\mathbb{X}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in V$. We notice $\lim\limits_{n\longrightarrow+\infty} x_n = l$*
2. *the sequence $(x_n)_{n\geq 0}$ is called* ***convergent*** *if it a limit, else it called* ***divergent****.*

Construction of a topology from the axioms of the closed:

Definition 1.9 (see [39])**.** *Let A be a sub-set of topological space $(X, \mathcal{T})$. We call that A is a closed or closed part of X if the complement of A in X is open.*
In other words, if we have $C_X A \in \mathcal{T}$.

All family of closed $\mathcal{F}$ in topological space $(X, \mathcal{T})$ verified the following properties, call axioms of the closed.

(F1) $\varnothing, E \in \mathcal{F}$, i.e. the empty part $\varnothing$ and the set $\mathbb{X}$ are closed set.

(F2) Si $A, B \in \mathcal{F}$, so $A \cup B \in \mathcal{F}$, i.e. the union of two closed sets is closed set.

(F3) If $(A_i)_{i\in I}$ be a family of parts of X belonging to $\mathcal{F}$, so $\cap_{i\in I} A_i \in \mathcal{F}$, i.e. the intersection of all family of closed sets is a closed set.

Reciprocally, we can define a topology from closed sets. More precise, if X is a set and if $\mathcal{F} \subset \mathcal{P}(X)$ verifying (F1),(F2) and (F3), then $\mathcal{T} = \{C_X A, A \in \mathcal{F}\}$ is a topology on X whose set of closed is $\mathcal{F}$.

Hausdorff space

Definition 1.10 (see [39]). *A topological space is said to be separate or Hausdorff if and only if for all two distinct points $x_1, x_2 \in X$ there are two disjoint open $U_1, U_2 \in \mathcal{T}$ such that $x_1 \in U_1$ and $x_2 \in U_2$. Which is equivalent to: the intersection of all closed neighbourhoods of $x \in X$ contains only x.*

1.2 Metric Space

Definition 1.11 (see [39]). *A metric space is a set E with a distance function, that is a map d of $E \times E$ in $\mathbb{R}_+$ ($\mathbb{R}_+ = \{x; x \in \mathbb{R}, x \geq 0)$.*
which, to the pair $(x; y)$ of $E \times E$, corresponds to a number $d(x; y) \geq 0$, called the distance of $\mathbb{X}$ and of y.
This distance must have the following three properties:

1. *Symmetric: $d(x, y) = d(y, x)$;*

2. *Positivity: $d(x, y) \geq 0$ if $x \neq y$, and $d(x, x) = 0$;*

3. *Triangular inequality: $d(x, z) \leq d(x, y) + d(y, z)$*

(each side of a triangle is at most equal to the sum of the other two).

The inequality is immediately deduced:

$$d(x, z) \geq |d(x, y) - d(y, z)|$$

(each side of a triangle is at least equal to the difference of the other two).

This also implies that if $x_1.x_2, ..., x_n$ are n arbitrary points of E we have:

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{n-1}, x_n).$$

Definition 1.12 (see [39]). *Let (E, d) be a metric space, $a \in E$, $r > 0$.*

1. *We call* ***open ball*** *of center a and radius r the whole:*

$$B(a, r) = x \in E; d(a, x) < r.$$

2. *We call* ***closed ball*** *of center a and radius r the whole:*

$$B'(a, r) = x \in E; d(a, x) \leq r.$$

3. *We call* ***sphere*** *of center a and radius r the set:*

$$S(a, r) = x \in E; d(a, x) = r.$$

Topology of metric spaces:

Proposition 1.1 (see [31]). *Let (E, d) be a metric space.*

1. *The set E is the meeting of open balls.*

2. *Let $a \in E$ and $r > 0$. Let $x \in B(a, r)$ and $\rho = r - d(a, x)$, then $\rho > 0$ and we have $B(x, \rho) \subset B(a, r)$.*

3. *Let $a, b \in E$ and $r_1, r_2 > 0$. For any $x \in B(a, r_1) \cap B(b, r_2)$, there exists $\rho > 0$ such as $B(x, \rho) \subset B(a, r_1) \cap B(b, r_2)$. So the intersection of two open balls is union of open balls.*

Proof. 1. Let $a \in E$ and $r > 0$, then we have $a \in B(a, r)$, so E is the union of open balls.

2. Let $y \in B(x, \rho)$, then we have $d(x, y) < \rho = r - d(a, x)$, hence $d(a, x) + d(x, y) < r$. According to the triangular inequality, we have $d(a, y) \leq d(a, x) + d(x, y)$. Therefore, we have $d(a, y) < r$. That is, we have $y in B(x, r)$, so $B(x, \rho) \subset B(x, r)$.

3. Let $\rho = \inf\{r_1 - d(a, x), r_2 - d(b, x)\}$, then $\rho > 0$ and it follows from 2 that we have $B(x, \rho) \subset B(a, r_1) \cap B(b, r_2)$.

$\square$

Let (E, d) be a metric space. It is deduced from the previous proposal that open balls of (E, d) verify the properties 1.1. Therefore, there is a single topology $\mathcal{T}_d$ on E for which the open balls form a basis of opens. The topology $\mathcal{T}_d$ is called **topology associated with distance** d. A metric space (E, d) will always be considered a topological space, with topology $\mathcal{T}_d$.

What to remember about the definition of the $\mathcal{T}_d$ topology is:

Let U be a subset of E, then U is an open for topology $\mathcal{T}_d$ if and only if for all $x \in U$, there exists $r > 0$ such that $B(x,r) \subset U$.

Proposition 1.2 (see [31]). *Let (E,d) be a metric space. Then it is separated.*

Proof. Let be $x, y \in E$ such that $x \neq y$, then we have $d(x,y) > 0$. Let $r = \dfrac{d(x,y)}{2}$, then we have $B(x,r) \cap B(y,r) = \emptyset$.
Therefore, (E,d) is separated. □

Continuity

Proposition 1.3 (see [31]). *Let (E,d_E) and (F,d_F) two metric spaces, we say that a function $T : E \longrightarrow F$ is continuous, if for all $\varepsilon > 0$, there exists $\eta > 0$ such that $d_E(a,x) \leq \eta$ Involves $d_F(T(a),T(x)) \leq \varepsilon$; we have here a generalization of continuity as it was known for $E = F = \mathbb{R}$.*

If, for all ball with center $T(a)$, there exists a ball of center a whose image by T is in the previous one.

Proof. This results from defining the continuity 3.3, and the topology associated with a distance. □

Uniform continuity

Definition 1.13 (see [31]). *Let (X,d_X) and (Y,d_Y) two metric spaces, and $T : X \longrightarrow Y$. f is said to be uniformly continuous on $\mathbb{X}$ if*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, x_0 \in X; d_X(x,x_0) < \delta \Rightarrow d_Y(T(x),T(x_0)) < \varepsilon.$$

With respect to the definition of continuity, we replaced "$\forall x \in X, \exists \delta > 0$" by "$\exists \delta > 0, \forall x \in X$" : δ always depends on ε, but no longer depends on $\mathbb{X}$. Every uniformly continuous function is continuous, but the reverse is false.

The Lipschitzian function

Definition 1.14 (see [31]). *A mapping $T : M \subset E \to E$ is said to be Lipschitzian if, $\exists k > 0$ s.t.: $\forall x, y$ in M, we have*

$$d(T(x),T(y)) \leq k.d(x,y)$$

($k \doteqdot$ Lipschitz coefficient).

The Contracting function

Definition 1.15 (see [31]). *A mapping $T : M \subset E \to E$ is said to be contracting, if Lipschitzian with Lipschitz coefficient $k in]0.1[$.*

Remark 1.1 (Non expansive function). *If $k = 1$, we call that T is non expansive.*

One can define on metric spaces a wide structure that calls **the norm**.

Normed space

Definition 1.16 (see [39]). *a* ***norm*** *on a $\mathbb{K}$- vector space E is a map:*

$$\begin{array}{rcccl} \|\| & : & E & \to & \mathbb{R}_+ \\ & & x & \to & \| x \| \end{array}$$

admits the following properties:

(N1) For all $x \in E$ non null, we have $\| x \| \neq 0$;

(N2) For all $x \in E$ and all $\lambda \in \mathbb{K}$, we have $\| \lambda x \| = |\lambda| \, \| x \|$;

(N3) For all $x, y \in E$, we have $\| x + y \| \leq \| x \| + \| y \|$ ***(convexity inequality)****.*

The space E, provided with a norm $\|\|$, is called ***normed space*** *or* ***normed vector space*** *(or $\mathbb{K}$-****normed vector space****, if we want to specify the (corps) $\mathbb{K}$). We often denote such a space $(E, \|\|)$. We deduce from the property (N2) that we have $\| 0 \| = 0$.*

Proposition 1.4 (see [39]). *Let $(E, \|\|)$ be a normed space.*

1. *The above property (N2) is equivalent to the following weakened form:*
   (N2') For all $x \in E$ and all $\lambda \in \mathbb{K}$, we have $\| \lambda x \| \leq |\lambda| \, \| x \|$.

2. *We can replace the property (N3) above by the following property :*
   (N3') For all $x, y \in E$ and all $t \in [0, 1]$, we have:

$$\| tx + (1 - t)y \| \leq t \| x \| + (1 - t) \| y \| .$$

3. *For all $x, y \in E$, we have the inequality $|\| x \| - \| y \|| \leq \| x - y \|$.*

4. *The inequality (N3) generalizes to n points. For all $x_1, x_2, ..., x_n \in E$, we have:*

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} \| x_i \|$$

Proof. 1. It's clear that (N2)$\Longrightarrow$ (N2').Reciprocally, we suppose (N2') verified. Let $x \in E$ and $\lambda \in \mathbb{K}$. If $\lambda = 0$, so $0 \leq\| \lambda x \|\leq |\lambda| \| x \|= 0$. If $\lambda \neq 0$, so we have $|\lambda| \| x \|= |\lambda| \| \lambda^{-1}(\lambda x) \|=\| \lambda x \|\leq |\lambda| \| x \|$. We deduce (N2).

2. We suppose (N3) verified. So for all $x, y \in E$ and all $t \in [0, 1]$, we have:

$$\| tx + (1-t)y \|\leq\| tx \| + \| (1-t)y \|= t \| x \| +(1-t) \| y \| .$$

Reciprocally, we suppose (N3') verified. Let $x, y \in E$, taking $t = \frac{1}{2}$, we get:

$$\| x + y \|=\| \frac{1}{2}(2x) + \frac{1}{2}(2y) \|\leq \frac{1}{2} \| 2x \| +\frac{1}{2} \| 2y \|= \frac{1}{2}2 \| x \| +\frac{1}{2}2 \| y \|=\| x \| + \| y \| .$$

3. We have $\| x \|=\| y+(x-y) \|\leq\| y \| + \| x-y \|$, so $\| x \| - \| y \|\leq\| x-y \|$. Also, we have $\| y \| - \| x \|\leq\| y - x \|=\| x - y \|$. Consequently, we have $| \| x \| - \| y \| | \leq\| x - y \|$. the last propriety is trivial. We verify that by induction on n.

$\square$

Distance associated by a norm

Let $(E, \|\|)$ be a normed space. For all $x, y \in E$, we put $d(x, y) =\| x - y \|$. It is easy to verify that d is a distance on E, called **the distance associated by a norm** . The corresponding topology will be called **normal topology** . Except mention of contrary, we provide all normed spaces to the normed topology. The distance d norm associated by the norm has the following properties:

(a) $d(x + z, y + z) = d(x, y)$ for each $x, y, z \in E$.

(b) $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x \in E$, and all $\lambda \in \mathbb{K}$.
Conversely, if E be a $\mathbb{K}$-vector space provided with a distance d having the properties (a) and (b) above, so d is defined from a norm; It suffices to put $\| x \|= d(x, 0)$, for all $x \in E$.

Proposition 1.5 (see [39])**.** *Let $(E, \|\|)$ be a normed space. Then we have :*

1. *The function:*

$$\begin{array}{ccc} E \times E & \longrightarrow & E \\ (x, y) & \longmapsto & x + y \end{array}$$

is lipschitzien, so uniformly continuous.

2. *The following function is continuous.*

$$\begin{array}{ccc} \mathbb{K}\times E & \longrightarrow & E \\ (\lambda, x) & \longmapsto & \lambda x \end{array}$$

3. *The following function is lipschitzien, so uniformly continuous.*

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \parallel x \parallel \end{array}$$

4. *The translation, and the homotheties of non-null ratio are homeomorphisms of E. In the other words, for all $a \in E$ and all non-null $\lambda \in \mathbb{K}$, the following function are homeomorphisms of E.*

$$\begin{array}{ccc} E & \longrightarrow & E \\ x & \longmapsto & x + a \end{array}$$

and

$$\begin{array}{ccc} E & \longrightarrow & E \\ x & \longmapsto & \lambda x \end{array}$$

Corollary 1.1 (see [31]). *Let $(E, \|\|)$ a $\mathbb{K}$-normed vector space, $(x_n)_{n\geq 0}$,$(y_n)_{n\geq 0}$ two converging sequences in E and $(\lambda)_{n\geq 0}$ be converging sequence in $\mathbb{K}$. Then $(x_n + y_n)_{n\geq 0}$, $(\lambda_n x_n)_{n\geq 0}$ and $(\parallel x_n \parallel)_{n\geq 0}$ are convergent in their respective spaces, and we have $\lim_{n\longrightarrow+\infty} x_n + y_n = \lim_{n\longrightarrow+\infty} x_n + \lim_{n\longrightarrow+\infty} y_n$, $\lim_{n\longrightarrow+\infty} \lambda_n x_n = (\lim_{n\longrightarrow+\infty} \lambda_n)(\lim_{n\longrightarrow+\infty} x_n)$ et $\lim_{n\longrightarrow+\infty} \parallel x_n \parallel = \parallel \lim_{n\longrightarrow+\infty} x_n \parallel$.*

Definition 1.17 (see [31]). *Tow norms $\|\|$ and $\|\|'$ on a vector space E are called* ***equivalents*** *if there exists $\alpha, \beta \in]0, +\infty[$ such that for all $x \in E$, we have $\alpha \parallel x \parallel \leq \parallel x \parallel' \leq \beta \parallel x \parallel$.*

We will see that tow norms on E are equivalents if and only if they defined the same topology on E.

Remark 1.2. *Let E be vector space and $\|\|$,$\|\|'$ two norms on E.If there exists a sequence $(x_n)_{n\geq 0}$ in E such that the sequence $(\parallel x_n \parallel)_{n\geq 0}$ is bounded and the sequence $(\parallel x_n \parallel')_{n\geq 0}$ isn't bounded so two norms $\|\|$ and $\|\|'$ aren't equivalents.*

Converging sequence

Proposition 1.6 (see [31]). *Let $(x_n)_{n\geq 0}$ be a sequence in topological space (E, d).*
We call that the sequence $(x_n)_{n\geq 0}$ is ***convergent*** *if it admits a limit.*

In other word, $(x_n)_{n\geq 0}$ *is converging sequence if*

$$\lim_{n\longrightarrow+\infty} x_n = l \Leftrightarrow \forall\varepsilon > 0, \exists N \in \mathbb{N},\ \textit{tel que}\ \forall n \geq N, d(x_n, l) \leq \varepsilon.$$

Cauchy sequence

Definition 1.18 (see [31])**.** *Let* (E, d) *be a metric space.*
The sequence $(x_n)_{n\geq 0}$ *in* (E, d) *is Cauchy sequence if for all* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0, m \geq n_0$*, we have* $d(x_m, x_n) < \varepsilon$*.*

It amounts to the same thing to say that for all $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and for all* $p \in \mathbb{N}$*, we have* $d(x_{n+p}, x_n) < \varepsilon$*.*

Proposition 1.7 (see [31])**.** *Let* (E, d) *be a metric space.*

1. *All converging sequence in* E *is Cauchy.*
2. *All Cauchy sequence in* E *is bounded.*
3. *All sub-sequence of Cauchy sequence is Cauchy.*

Proposition 1.8. *Let* (E, d) *be a metric space and* $(x_n)_{n\geq 0}$ *be Cauchy sequence in* E*.*

The sequence $(x_n)_{n\geq 0}$ *is convergent if and only if it has a convergent subsequence. In other words, a Cauchy sequence in* E *with an* **adherence value** *is convergent. The adherence value is then unique. This is the limit of the sequel.*

Complete space

Definition 1.19 (see [31])**.** *Let* (E, d) *be a metric space.*

1. *We say that* (E, d) *is* ***complete*** *if every Cauchy sequence in* (E, d) *is convergent.*
2. *A subset* A *of* (E, d) *is said to be complete if* A *with the induced distance is a complete metric space.*

The fundamental importance of complete metric spaces is in the fact that, to demonstrate that a sequence is convergent in an abstract space, it isn't necessary to know in advance the value of the limit.

Example 1.1. *Let* (E, d) *be a discrete metric space, so* (E, d) *is complete because we have* $d(x, y) = 0$ *if* $x \neq y$*, so all Cauchy sequence in* E *is stationary, then convergent.*

Banach space

Definition 1.20 (see [31]). *A **Banach** space be a normed space $(E, \|\|)$, witch is complete for the distance associated to the norm.*

Remark 1.3 (see [31]). *Let E be a vector space and $\|\|,\|\|'$ two equivalent norms on E. So $(E, \|\|)$ is a Banach space if and only if $(E, \|\|')$ be a Banach space.*

As (basic) topologies are poor in fundamental properties in functional analysis such as: uniform continuity, contraction, ect. We are forced to define new structures to generalise these notions to topological spaces. These are the so-called **uniform structures**

1.3 Uniform structure

Uniformity

Let X be a non-empty set. We consider a collection *mathcalU* of subset E of $X \times X \doteqdot \{(x_1, x_2) : x_1, x_2 \in X\}$. The set $\Delta \doteqdot \{(x, x) : x \in X\}$ is called the **diagonal**. For $E \subseteq X \times X$ we note $E^{-1} \doteqdot \{(x_2, x_1) : (x_1, x_2) \in E\}$ and $E \diamond E \doteqdot \{(x_1, x_2) : \exists x \in X : (x_1, x), (x, x_2) \in E\}$.

Definition 1.21 (see [9]). *Let X be a non-empty set. a set $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ is called a **uniformity** on X if*

(U1) $E \in \mathcal{U}$, $E \subseteq E'$ *implies* $E' \in \mathcal{U}$ *and* $E_1, E_2 \in \mathcal{U}$ *implies* $E_1 \cap E_2 \in \mathcal{U}$*;*

(U2) If $E \in \mathcal{U}$*, then* $\Delta \subseteq E$*;*

(U3) If $E \in \mathcal{U}$*, so there exists* $E' \in \mathcal{U}$ *such that* $E' \subseteq E^{-1}$*;*

(U4) If $E \in \mathcal{U}$*, so there exists* $E' \in \mathcal{U}$ *such that* $E' \diamond E' \subseteq E$.
*the couple $(X,\mathcal{U})$ be a uniformed space. the elements of $\mathcal{U}$ are called **surroundings**. the Uniformity $\mathcal{U}$ is called separate if*

(U5) $\cap_{E \in \mathcal{U}} E = \Delta$.
*a uniformity $\mathcal{U}$ on X is a filter by (U1), (U4). The filter base $\mathcal{U}_B$ for filter $\mathcal{U}$ is called **the basis of uniformity** $\mathcal{U}$.*

The set $\mathcal{U}_B \subseteq \mathcal{P}(X \times X)$ be a basis of uniformity $\mathcal{U}$ if it verified (U2), (U3), (U4) with $\mathcal{U}$ replace by $\mathcal{U}_B$,and also

(U1') Si $E_1, E_2 \in \mathcal{U}_B$, there exists $E \in \mathcal{U}_B$ with $E \subseteq E_1 \cap E_2$.

In this case, the uniformity is obtained by the subset of $\mathcal{P}(X \times X)$ of elements of $\mathcal{U}_B$.
Let $(X,\mathcal{U})$ be uniformed space. The family of sets $U(x) \doteqdot \{U_E(x) : E \in \mathcal{U}\}$ where $U_E(x) \doteqdot \{x' \in X : (x,x') \in E\}$ is a local system of neighbourhoods of $x \in X$, i.e. it satisfies (T1) - (T4) of the definition 1.4 . Similarly, the basis of the uniformity generate the basis of neighbourhoods for all $x \in X$. In this way, the unique topology defined is called the **uniform topology** on X associated to the uniformity $\mathcal{U}$. If the uniformity is separate, so the corresponding uniform topology is also a separate topology.

Definition 1.22. *Let X be a separate topological space.*

1. *We call that X is a* ***regular space*** *or T_3-space if for all closed part F of X and for all point x of X such that $x \notin F$, there exists two disjoint opens U et V in X such that $x \in U$ and $F \subset V$.*
2. *We call that X is a* ***complete regular space*** *or space of* ***Tychonoff*** *if for all closed part F of X and for all point x of X such that $x \notin F$, there exists a continuous mapping $f : X \longrightarrow [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in F$.*
3. *We call that X is a* ***normal space*** *or T_4-space if for all two disjoint closed sets A and B in X, there exists two disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.*

All the above spaces are uniformed spaces.

1.4 Semi-Normed Space

Firstly, we introduce topological spaces, after that the metric spaces. All metric space are topological, but all its richness comes from its metric structure and don't from its topology. Indeed, it permit to consider other purely metric aspects (Cauchy sequence, complete spaces, bounded sets, uniformly continuous functions, ect).
On other hand, there exists topology non-**metric**. For that, we can can generalise the notion of metric structure.

Definition 1.23 (see [31])**.** *Let $\mathbb{X}$ a $\mathbb{K}-$vector space.*
We denote a function $p : E \longrightarrow \mathbb{R}^+$ semi-norm on E if:

1) $\forall x \in X, \forall \lambda \in \mathbb{K}; \quad p(\lambda x) = |\lambda| p(x)$

2) $\forall x, y \in X; \quad p(x+y) \leq p(x) + p(y)$

Or

1') $\forall x, y \in X, \forall \alpha, \beta \in \mathbb{K}; \quad p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y)$

However, $\forall x, y \in X; \quad p(x, y) = 0$ *don't imply* $x = y$.

It would hardly be interesting to consider a set provided with a single semi-distance; he would don't be a separated topological space.

Definition 1.24. *We call semi-metric space all set E provided with a family $(p_i)_{i \in I}$ of semi-distances (where the set of indices is of arbitrary power) satisfying the following condition: The family $(p_i)_{i \in I}$ est "filtering", in other words, for any finite part J of I, there exists $k \in I$ such that $d_k \geq d_j$ for all $j \in J$.*

We call open semi-ball $B_{i.0}(a, R)$ (resp. closed $B_i(a, R)$) with center $a \in E$, and radius $R > 0$, of index $i \in I$, the set of $\mathbb{X}$ in E such that $d_i(a, x) < R$ "(resp. $\leq R$).

A semi-metric space is then provided with a topology (non necessarily separated), defined as follows: a sub-set ν of E is open if, it contains a semi-ball, with its center in it, in other words if:

$$\forall a \in \nu, \exists i \in I \quad \text{and} \quad \varepsilon > 0 \quad \text{such that} \quad B_i(a, \varepsilon) \subset \nu.$$

We check that the axioms 1.1 ;of open sets are satisfied. The only one who isn't completely trivial is axiom 1.1 of finite intersection: all finite intersection of opens is open. Let $O_1, O_2, ..., O_n$ be open sets, and O their intersection. Let $a \in O$. there are so the indices $i_1, i_2, ..., i_n$, and numbers $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n > 0$, such that for each semi-ball $B_{i_j}, (a, \varepsilon_\xi)$, a sub-set of O_i, $j = 1, 2, ..., n$. If k is an index such that p_k bound $p_{i_1}, p_{i_2}, ..., p_{i_n}$ (axiom of filtration) and if $\varepsilon = \min(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, the semi-ball $B_k(a, \varepsilon)$ is in O, which is therefore open. We can also say that a set is open if and only if it is a union of semi-open balls. The semi-balls $B_i(a, \varepsilon)$ of center a (when E and i vary) form a fundamental system of neighbourhoods of a.

This topology is separated (axiom p of Hausdorff) if and only if: For every $\mathbb{X}$ and y distinct in E, there don't exist a $i \in I$ such that $p_i(x, y) > 0$ Or again: " $p_i(x, y) = 0$ for all i " results $x = y$.

The semi-metric space is said to be separated if its topology is separated.

Any separated semi-metric is regular, because the semi-balls $B_i(a, R)$ are closed.

the semi-distance functions d_i are trivially continuous from $E \times E$ to $\mathbb{R}_+$. The previous topology $\mathcal{T}_0$ is moreover the least fine of the topologies $\mathcal{T}$ on E, and it defined the continuous maps p_i from $E \times E$ into $\mathbb{R}_+$. Indeed, for such a topology $\mathcal{T}$, any "semi-open ball"

$$B_{i,0}(a, R) = \{x \in E; d(a, x) < R\}$$

is necessarily open; Then all open from $\mathcal{T}_0$ be a union of semi- open balls is open for $\mathcal{T}$, and $\mathcal{T}_0$ is least fine of $\mathcal{T}$.

In fact, it may happen that all the semi-distances p_i are distance. The space is still called semi-metric and non-metric, since there is a different distance in the family.

We will say that a topological space is semi-metric if its topology can be defined by a family of semi-distances.

Theorem 1.3 (see [31])**.** *For a topological space E be a semi-metric, it is necessary and sufficient that it has the following property:*

For each $a \in E$ and the open neighbourhood Ω of a, there exists a continuous function γ of E with values $[0, 1]$, equal to 1 in a and null in $C_E\Omega$.

Remark 1.4 (see [31])**.** *In fact, this theorem doesn't give such a practical criterion for recognizing arise that a space is semi- metric. Anyway, if it is true that many spaces useful in analysis aren't metric, most of them are semi-metric; we demonstrate that any compact or locally compact space is semi-metric.*

The relation into the uniformity and semi-metric space

The following proposition gives the relation into the uniformity and semi-metric space. This result is fundamental for the uniform spaces.

Proposition 1.9. *(i) Let $(X, \mathcal{U})$ be a uniformed space (separate). then there exists a family*

$\{p_i\}_{i\in I}$ *of semi-distances on* X *such that the sets*

$$B_{i,r} \doteqdot \{(x_1, x_2) \in X \times X : p_i(x_1, x_2) < r\}, \quad i \in I, r > 0, \tag{1.1}$$

form a basis of uniformity $\mathcal{U}$.

(ii) Let X *be a non empty set and* $\{p_i\}_{i\in I}$ *be a family (separated) of semi-distances on* X*. So the sets given by 1.1 is a based of uniformity* $\mathcal{U}$ *(separate) on* $\mathbb{X}$.

Corollary 1.2 (see [37])**.** *A topological space* $(X, \mathcal{T})$ *is a uniformed space (separate) if and only if its topology* $\mathcal{T}$ *can be generated by a family (separate) of semi-distances.*

Topological vector space defined by a semi-metric

Topological vector space:

We call topological vector space a set E, equipped on the one part with vector space structure on the corps K of reals or complexes, on other hand of a topology, such that the addition $(x, y) \longrightarrow x + y$ in $E \times E$ into E and for the multiplication by the scalars $(\lambda, x) \longrightarrow \lambda x$ in $K \times E$ into E are continuous, or K is endowed with its natural topology. We say also that the topological structure and the vector structure are compatible. If x_n converges to x, y_n to y, and λ_n to λ in K, $x_n + y_n$ tends to $x + y$, and $\lambda_n x_n$ to λx.

Generality on the vector topological spaces:

We have seen the definition of vector topological spaces. it obviously results from this definition that, in such a space E, a similarity $x \longrightarrow \lambda x + b$ is a continuous function; it is a homeomorphism if $\lambda \neq 0$, because it is so bijective and its reciprocal map $x \longrightarrow \dfrac{x-b}{\lambda}$ is also continuous.

If F be a sub-vector space of E, it is obviously a vector topological space when it provided with the induce topology.

Now let $(E_i)_{i\in I}$ be a family of vector spaces. Recall that vector space product est the set product $E = \prod\limits_{i\in I} E_i$ equipped with addition law
$((x_i)_{i\in I}, (y_i)_{i\in I}) \longrightarrow (x_i + y_i)_{i\in I}$ and of the multiplication law by the scalars $(\lambda, (x_i)_{i\in I}) \longrightarrow (\lambda x_i)_{i\in I}$.
So:

Theorem 1.4. *Let $(E_i)_{i\in I}$ be a family of vector topological spaces. If we provide the product $E = \prod_{i\in I} E_i$ of the product vector structure and the product topology, it is a vector topological space.*

Theorem 1.5. *Let E be a vector topological space, F be a sub-vector space of E. If we provide the vector space quotient E/F of the topology quotient, it is a vector topological space, and the canonic map in E on E/F is open. the neighbourhoods of 0 in E/F are exactly the canonic ranges of neighbourhoods of 0 in E.*

Lemma 1.1. *Don't confuse $U + V$ with the union $U \cup V$; in a vector space vector$E$, $A + B$ is the set of $x + y, x \in A, y \in B$.*
As for λA, it is the set of $\lambda x, x \in A$. Note that, under this conditions, $A + A$ contains $2A$ but generally different. Finally $A - B$ is the set of $x - y, x \in A, y \in B$, which forbids using this notation for designate $A \cap C_E B$.

Properties of neighbourhoods of 0 in a vector topological spaces

Definition 1.25.

1. *A part A of a vector space E is called balanced set, if, for each $\lambda \in K$ with $|\lambda| \leq 1$, and $x \in A$, we have $\lambda x \in A$; or also if, for all $\lambda \in K$ such that $|\lambda| \leq 1$, we have $\lambda A \subset A$.*
2. *A part A of E is called absorbant set if, for all x of E, there exists $\alpha > 0$, such that $|\lambda| \leq \alpha$ Causes $\lambda x \in A$; in other words, if, for all x of E, « the* **homothetic** *of x fairly small ratio » are in A.*

Theorem 1.6. *Let E be a vector topological space and $\mathfrak{F}(a)$ the set of neighbourhoods of $a \in E$. So,*

1. *$\mathfrak{F}(a) = a + \mathfrak{F}(0)$.*
2. *$V \in \mathfrak{F}(0)$ and $V_1 \supset V$ result in $V_1 \in \mathfrak{F}(0)$.*
3. *$V_1, V_2 ... V_n \in \mathfrak{F}(0)$ result in $\cap_{i=1}^{n} V_i \in \mathfrak{F}(0)$.*
4. *If $V \in \mathfrak{F}(0)$, there exists $W \in \mathfrak{F}(0)$.Such that $W + W \subset V$.*
5. *$\forall V \in \mathfrak{F}(0)$ and $\lambda \neq 0$, we have $\lambda V \in \mathfrak{F}(0)$.*
6. *$\forall V \in \mathfrak{F}(0)$, V is absorbent set.*
7. *There exists a fundamental system of neighbourhoods of 0 is balanced set.*

Conversely, it is proved that, if, in a vector space E on K, at any point $a \in E$ is associated a set $\mathfrak{F}(a)$ of parts of E so as to verify the seven properties of the theorem 1.6, there exists a topology and only one on E, which makes E a topological vector space admitting for all a of E, $\mathfrak{F}(a)$ as a set of neighbourhoods of a. (See, N. BOURBAKI, Esp. vect. top., Chap. 1, parag. 1, $n^{circ}3$, prop. 5).

So these seven properties are characteristic properties of the neighbourhood sets of points of a topological vector space.

Corollary 1.3. *If a linear map u of a topological vector space E in another F is continuous at the origin, it is continuous everywhere.*

This partially extends the If a linear map u of a topological vector space E in another F is continuous at the origin, it is continuous everywhere.

This partially extends the theorem 1.4 relating to normed vector spaces; What relates to uniform continuity or the Lipschitzian character can't be extended at present.

Corollary 1.4. *If a multi-linear map u of a product of topological spaces in a topological vector space is continuous at the origin, it is continuous everywhere.*

Theorem 1.7. *In a topological vector space, the origin admits a fundamental system of open balanced neighbourhoods , and a fundamental system of closed balanced neighbourhoods.*

Corollary 1.5. *A separated topological vector space is regular.*

Corollary 1.6. *In a topological vector space E in separated if and only if the intersection of neighbourhood of* 0 *reduces to* 0*, or if and only if* 0 *is closed.*

Corollary 1.7. *Let E be a topological vector space, and F a vector sub-space. For that the quotient topological vector space E/F is separated, it is necessary and sufficient that F is closed in E.*

Indeed, E/F is separated if and only if the set reduced to its origin is closed, this which by virtue of the definition of the closed of the quotient topology, amounts to saying that F is closed in E.

□

Remark 1.5. *We see again here what we had already noted (2, VII, 4): the quotient of a separated space isn't necessarily separated. But it can also happen that E isn't separated and that E/F is; it suffices for this that F is closed in E (example: $F = E, E/F = 0$).*

Locally convex separated topological vector space defined by semi-norms

Definition 1.26. *We call a locally convex separated topological vector space, all* $\mathbb{X}$ *vector space with a topology* $\mathcal{T} \subset P(X)$ *such that:*

* $\mathcal{T}$ *is compatible with the structure of v.s.:*

a. $X \times X \longrightarrow X, (x, y) \longrightarrow x + y$ *is continuous.*

b. $\mathbb{R} \times X \longrightarrow X, (\lambda, x) \longrightarrow \lambda x$ *is continuous.*

* $\mathcal{T}$ *is locally convex, i.e. has a based of convex neighbourhoods:*

c. $\forall x \in X$, $\forall U \in \mathcal{T}$, $x \in U$, *there exists* $V \in \mathcal{T}$ *convex (i.e.*$\forall \alpha, \beta \in V : t\alpha + (1-t)\beta \in V$*) such that* $x \in V \subset U$.

* $\mathcal{T}$ *is separated:*

d. For all $(x, y) \in X^2$ *such that* $x \neq y$, *there exists* $U, V \in \mathcal{T}$ *such that* $x \in U$, $y \in V$ *and* $U \cap V = \emptyset$.

It is also possible to define the notion of topological vector space (these are those verifying axioms (a) and (b)). In practice, the spaces encountered in analysis have a much richer structure (at least that of hlctvs).

Proposition 1.10 (Second definition of a hlctvs)**.** *Let* $\mathbb{X}$ *vector space and* $\mathcal{T}$ *a topology on* $\mathbb{X}$*. So* $(X, \mathcal{T})$ *be a hlctvs if, and only if,* $\mathcal{T}$ *admits neighbourhood basis sets*

$$\{x + tB; x \in X, t > 0, B \in \mathfrak{B}\}$$

where the "basis of open balls centred at the origin"$\mathfrak{B} \subset \mathcal{P}(X)$ *is verified*

e. All $B \in \mathfrak{B}$ *is convex, balanced* $(-B = B)$*, absorbent* $(\forall x \in X, \exists t > 0,$ *tel que* $tx \in B)$*;*

f. $\mathfrak{B}$ *is separated* $(\forall x \in X, \exists B \in \mathfrak{B}, \exists t > 0,$ *tel que* $tx \notin B)$

In other words, in this second characterization it is assumed that $\mathbb{R}_+^*$. $\mathfrak{B}$ *is a basis of neighbourhoods of 0 and a basis of neighbourhoods of* X *is obtained by translation of it. It is then obvious that*

$$\textit{pour tout } O \textit{ ouvert de } X, x \in X, \lambda \in \mathbb{R}_+^*, x + O \textit{ et } \lambda O \textit{ sont des ouverts .} \tag{1.2}$$

The notion of "basis of open balls centred at the origin" may not be standard. Its interest is to be able to characterize with as few open as possible a topology of hlctvs. In the following we will use only this second definition of an hlctvs. The proof of the proposal refp can therefore be omitted at first reading.

Remark 1.6. *For all $J \subset I$ finite, the ball $B_J(0,1)$ is a convex, balanced, absorbing set and $(B_J(0,1))_{J \text{ fini}}$, $J \subset I$ is separating.*

Theorem 1.8. *Let $(X, \mathcal{P})$ be a vector space with a family of semi-norms that separates the points. It is associated with the $\mathcal{T}_{\mathcal{P}}$ the topology whose neighbourhood basis consists of the open balls defined from $\mathcal{P} = (p_i)_{i \in I}$. So $(X, \mathcal{T}_{\mathcal{P}})$ is a hlctvs.*

The canonical topology of a finite-dimensional vector space

Theorem 1.9. *Let E be a vector space finite dimension on $K = \mathbb{R}$ or $\mathbb{C}$. Then there exists a unique topology, that make E a separate topological vector space; it is called the canonical topology of E. It is the finest topology that is compatible with the vector structure.*

Next, we define a important structure in a Hausdorff complete vector semi-normed space Y called "**order**".

1.5 Order

The relation

1. **Binary relations**

   Definition 1.27 (see [1]). *Let $\mathbb{X}$. We know that $\mathbb{X} \times \mathbb{X}$ is the set of all couples $(x; y)$ s.t. $x, y \in \mathbb{X}$, i.e.:*

   $$\mathbb{X} \times \mathbb{X} = \{(x; y)/x, y \in \mathbb{X}\}$$

   We call a binary relation on $\mathbb{X}$ (denote $\mathcal{R}$) the data of a part Υ of the product $\mathbb{X} \times \mathbb{X}$, we write so, for $x, y \in \mathbb{X}$:

   $$x\mathcal{R}y \text{ if } (x; y) \in \Upsilon$$

   Definition 1.28 (see [1]). *A binary relation $\mathcal{R}$ on the set $\mathbb{X}$ be an association between the elements of $\mathbb{X}$ en couples elements relate by a properties well define. We note $x\mathcal{R}y$ where the property is true for the couple $(x, y) \in \mathbb{X} \times \mathbb{X}$.*

Definition 1.29. *Let $\mathcal{R}$ be a relation on the set $\mathbb{X}$.*

- *$\mathcal{R}$ is reflexive if for all $x \in \mathbb{X}$, on a $x\mathcal{R}x$;*
- *$\mathcal{R}$ is symmetric if for all $x, y \in \mathbb{X}$, we have $x\mathcal{R}y \Rightarrow y\mathcal{R}x$;*
- *$\mathcal{R}$ is anti-symmetric if for all $x, y \in \mathbb{X}$, $(x\mathcal{R}y$ et $y\mathcal{R}x) \Rightarrow x = y$;*
- *$\mathcal{R}$ is transitive if for all $x, y, z \in \mathbb{X}$, $(x\mathcal{R}y$ et $y\mathcal{R}z) \Rightarrow x\mathcal{R}z$.*

2. **Equivalent relations**

Definition 1.30. *A binary relation is equivalence if and only if it is reflexive, symmetric and transitive.*

Total order

Definition 1.31 (see [48])**.** *a relation of order $\mathcal{R}$ on the set $\mathbb{X}$ be a relation of total order if for all $x, y \in \mathbb{X}$ we have a relation between them. with can traducing:*

$$\forall x, y \in \mathbb{X}, x\mathcal{R}y \vee y\mathcal{R}x$$

Definition 1.32 (Partial order)**.** *We call that a relation $\mathcal{R}$ on the set $\mathbb{X}$ is a relation of partial order if there exists at least two element of $\mathbb{X}$ don't have a relation by $\mathcal{R}$.*
In other word,

$$\exists x, y \in \mathbb{X}, x \not\mathcal{R} y \wedge y \not\mathcal{R} x$$

Definition 1.33. *We call that $(\mathbb{X}, \mathcal{R})$ is ordered set.*

Definition 1.34. *Let $\mathcal{R}$ be an equivalent relations on $\mathbb{X}$, $x \in \mathbb{X}$ an element . We call equivalent class of x the set $\mathcal{C}(x) = \{y \in X, y\mathcal{R}x\}$.*

Definition 1.35. *Let $(Y, \{\rho_\alpha\}_{\alpha \in I})$ be a hlctvs defined by semi-norms(hlctvs-ns) and $S \subset Y$ be a non-empty set closed. We call that S a cone, if it verified the following conditions:*

1. *$\forall \alpha, \beta \in \mathbb{R}$; $\forall m, n \in S$ we have $\alpha m + \beta n \in S$.*
2. *$\forall m \in S$ and $-m \in S$ so $m = 0$.*

The order in X ($\leqslant$) satisfies:
$\forall a, b \in X : a \leqslant b \Longleftrightarrow b - a \in S.$

Proposition 1.11. *All order verified :*

1) $\forall a, b$ *and* $c \in Y$*:*

$a \leqslant b \Longleftrightarrow b - a \in S \Longleftrightarrow (a + c) - (b - c) \in S \Longleftrightarrow a + c \leqslant b + c.$

2) $\forall a, b$ *in* Y*:*

$a \leqslant b \Longleftrightarrow b - a \in S \Longleftrightarrow (-a) - (-b) \in S \Longleftrightarrow -b \leqslant -a.$

Definition 1.36 (Ordered space)**.** *We said that* $(Y, \{\rho_\alpha\}_{\alpha \in I}, \leqslant)$ *is ordered space, when it be hlctvs-ns provided with the order(*$\leqslant$*), witch defined on* Y*.*

In the following, we propose the important type of order called "Normal order". witch help us to generalise the most of notions and theorems, for example: fixed point theory, ...

Normal order

Definition 1.37. *see [16] Let* $(Y, \{\rho_\alpha\}_{\alpha \in I}, \leqslant)$ *be hlctvs-ns. We say that* $(\leqslant)$ *be normal order on* Y*, when for every* $\alpha \in I$*,* $\exists M_\alpha > 0$ *we have:*

$$\forall\ a,\ b \in X\ :\ \mathbf{o} \leqslant a \leqslant b \Longrightarrow \rho_\alpha(a) \leqslant M_\alpha \rho_\alpha(b).$$

Example 1.2. *Let* $Y = C[0; 1]$ *with the family of semi norms* $\{\rho_\alpha\}_{\alpha \in I}$ *where* I *be the set of all the compacts is* $\mathbb{R}$ *and* $\rho_\alpha(g) = \{e^{-\iota t} g(t)\}, \iota \in \mathbb{R}_+$*; the order* $\mathfrak{P} = \{f \in Y : f \geq 0\}$ *be normal with* $M_\alpha = 1, \forall \alpha$

Proposition 1.12. *Let* $(Y, \{\rho_\alpha\}_{\alpha \in I}, \leqslant)$ *provided with a normal order. The interval* $[a,\ b] \subset Y$ *is a bounded set.*

Proof. Let $\alpha \in I$, and $c \in [a,\ b]$. Since, $a \leqslant c \leqslant b$ then $\mathbf{o} \leqslant c - a \leqslant b - a$, and it follows that for every $\alpha \in I$,

$$\rho_\alpha(c) \leqslant \rho_\alpha(c - a) + \rho_\alpha(a) \leqslant M_\alpha \rho_\alpha(b - a) + \rho_\alpha(a) = C,$$

where the constant C is independent of c. Thus, the interval $[a,\ b]$ is bounded. $\square$

Monotonic operators

Definition 1.38. *Let* $(Y, \{\rho_\alpha\}_{\alpha \in I}, \leqslant)$ *be an ordered semi-normed space, and* $G : D(G) \subset Y \longrightarrow Y$ *be a map on* Y*.*

1- The map G *is increasing, if* $\forall a, b \in D(G)$ *and* $a \leqslant b \implies G(a) \leqslant G(b)$*.*

2- The map G is decreasing, if $\forall a, b \in D(G)$ and $a \leqslant b \implies G(b) \leqslant G(a)$.

Definition 1.39. *Let G be a map. If for every $a \in D(F)$ such that $\mathbf{o} \leqslant a \implies \mathbf{o} \leqslant G(a)$, then it is positive.*

1.6 Fixed point

Definition 1.40. *see [39] Let Y be a topological space and $G : Y \to Y$ be a map. The point $a \in Y$ is the fixed point of G if $G(a) = a$.*

Chapter 2

Fixed point in uniformed space

In the second chapter, we recall Banach fixed point theory in metric space. After that, we define the generalisation of last theorem in uniformed space. Then, we extend it. Finally, we applied our results in initial value problem for nonlinear fractional differential equation with maxima.

2.1 Banach theory (metric type)

For more details on the notion of fixed point presented in this paragraph, see [8].

Introduction

The fixed point theory is one of the most important in functional analysis, witch is the study tools of the existence and uniqueness of solutions for the problems; Where they are modelling of phenomena in physic, chemistry, ...
The most famous of them is Banach fixed point theory, also called fixed point theorem of Picard; It used in 1922, for the resolution of an integral equation. This theory is an abstract of the classical method of successive approximations introduce by Liouville in 1837, and it developed by Picard in 1890. The principle of Banach is generalised on topological space, for example: Schauder, Schaefer,...
There is another type of extension in non-metric topological space, especially on topological vector semi-normed spaces.

Theorem 2.1 (Principe de contraction de Banach, 1922)**.** *Let (E, d) be a complete metric space and $T : E \to E$ be a contractive map with contraction constant l. So :*

a T has a unique fixed point $x_l \in E$.

b For all $x \in E$; $x_l = \lim_{n \longrightarrow +\infty} T^n(x)$ where $T^0(x) = x$ and $T^n(x) = T(T^{n-1}(x))$.

c The convergence speed can be estimated by :

$$d(T^n(x); x_l) \leq \frac{l^n}{1-l} d(x; T(x)).$$

We will present two demonstration of theorem 3.7. The first one is constructive, this is the original demonstration of Banach, witch give the approximation of fixed point; the second is non-constructive, and it proved only the existence and uniqueness of fixed point. It rests in the principle of Cantor's empoite sets.

Proof. First method :

1. The uniqueness of fixed point:
   Let x_0, y_0 two fixed points of T, so

   $$d(x_0; y_0) = d\left(T(x_0); T(y_0)\right) \leq k.d(x_0; y_0)$$

   it is verified if $d(x_0; y_0) = 0$; i.e. $x_0 = y_0$.

2. The existence of fixed point:
   We proved if $x \in E$, the sequence $(T^n(x))_n \subset E$ is of Cauchy. For all $n \in \mathbb{N}$, we have

   $$d(T^{n+1}(x); T^n(x)) \leq k.d(T^n(x); T^{n-1}(x)) \leq ... \leq k^n d(T(x); x).$$

   Then, for all $n; p \in \mathbb{N}$ we have :

   $$\begin{aligned} d(T^{n+p}(x); T^n(x)) &\leq d(T^{n+p}(x); T^{n+p-1}(x)) + ... + d(T^{n+1}(x); T^n(x)) \\ &\leq (k^n + k^{n+1} + k^{n+2} + ... + k^{n+p-1} d(T(x); x) \\ &\leq k^n(1 + k + k^2 + ... + k^{p-1}) d(T(x); x) \\ &\leq k^n . \frac{1-k^p}{1-k} d(T(x); x), \end{aligned} \tag{2.1}$$

   where the last inequality is obtained by calculating the sum of geometric sequence of ratio k. We see that if $n \longrightarrow +\infty$, $d(T^{n+p}(x); T^n(x)) \longrightarrow 0$ and so the sequence $(T^n(x))_n$ be of Cauchy in E. Consequently, as E is complete, this sequence converge to the limit that we will call.

Now, we proved that the limit point is fixed point of T. Firstly, we remark

$$\lim_{n \longrightarrow \infty} (T^{n+1}(x)) = \lim_{n \longrightarrow \infty} (T^n(x)).$$

After that, the function T is continuous because it is contractive, so

$$x_0 = \lim_{n \longrightarrow \infty} (T^{n+1}(x)) = \lim_{n \longrightarrow \infty} (T(T^n(x))) = T(\lim_{n \longrightarrow \infty} (T^n(x))) = T(x_0)$$

Finally, in the inequality 2.1, if we fixed n and in fact tend p to infinity, we will have

$$d(x_0; T^n(x)) \leq \frac{k^n}{1-k} d(T(x); x)$$

Second method :

Let $a = \inf\{d(x; T(x)) : x \in E\} \geq 0$, we supposed that $x_0 > 0$. So for $\varepsilon > 0$, there exists $x \in E$ such as $d(x; T(x)) < a + \varepsilon$, then we will have:

$$a \leq d(T(x); T^2(x)) \leq k.d(x; T(x)) < k.(a + \varepsilon),$$

which brings us back to a contradiction when ε tend to 0, so $x_0 = 0$.

For $n \in \mathbb{N}^*$, we consider the sets

$$D_n = \left\{ x \in E : d(x; T(x)) \leq \frac{1}{n} \right\}.$$

- D_n is non empty, because $a = 0$.
- D_n is closed, because all sequence of D_n is converge in E has a limit in D_n.
- Also, the sets D_n form a increasing sequence when $n \longrightarrow \infty$ with $\lim_{n \longrightarrow \infty} diam(D_n) = 0$. In effect, $\forall x; y \in D_n$

$$d(x; y) \leq d(x; T(x)) + d(T(x); T(y)) + d(T(y); y) \leq \frac{2}{n} + k.d(x; y).$$

This give $d(x; y) \leq \dfrac{2}{n(1-k)} \longrightarrow 0$ when $n \longrightarrow \infty$.

Then, the theory of Cantor's empoite sets ensures that the intersection of D_n, $n \in \mathbb{N}^*$ is reduce a point which is the unique fixed point of T.

□

2.2 Generalisation of contraction principle for non-metric structures

In this part, we present a generalisation in a relatively general scope (see [57]). Then, we propose a generalisation in a particular scope but nevertheless very useful in applications of functional analysis. Let's first fix some definitions in non-metric topological space (but semi-metric space).

Let E be a topological vector space, A be a set of indices such that for $S \subseteq A$ we define a map $i : S \to A$, and $\mathfrak{p} = \{p_\alpha\}_{\alpha \in A}$ a family of semi-norms defined on E.

The function T is said to be contractive if ($T : X \to E$) verifies

$$p_\alpha(Tx - Ty) \leq K_\alpha . p_{i(\alpha)}(x - y) \qquad (0 < K_\alpha \leq 1)$$

for all $x, y \in X$ and $\alpha \in A$. We note ∂X the sphere of X.

Definition 2.1 (i-bounded set)**.** *Let E be a Hausdorff space, and $i : S \to A$ self mapping in indices set. We call that E is i-bounded if for all $x, y \in E, \alpha \in A$*

$$\sup\{p_{i^n(\alpha)}(x - y) : n = 0, 1, 2, ...\} < \infty,$$

such that $i^n(\alpha)$ are the iterations of the mapping i.

Let X is closed sub-set of E. the set X call i-bounded if

$$M_\alpha = \sup\{p_{i^n(\alpha)}(x - y) : n = 0, 1, 2, ...\} < \infty; \qquad x, y \in X, \qquad \alpha \in A$$

Definition 2.2 ([38])**.** *E is say convex semi-normed space, if for any u, v in E (with $u \neq v$), there exists a point w in E ($u \neq w \neq v$) such as*

$$P_K(u - w) + P_K(w - v) = P_K(u - v). \tag{2.2}$$

General form (see [57])

Let E be hlctvs, complete, where its topology provided with a family of semi-norms $\mathfrak{p} = \{p_\alpha : \alpha \in A\}$, such that A is the index set. Let $X \subset E$ be a sub-set of E and $T : X \longrightarrow E$ be a non linear function. the map T is non-self mapping.

Theorem 2.2. *Let x be i-bounded set and $T : X \to E$ est a contractive function such that:*

(a) $\sum_{n=0}^{\infty} 2^{2n} K_{\alpha}.K_{i(\alpha)}...K_{i^n(\alpha)} < \infty$ *for all*$\alpha \in A$*;*

(b) For all $x \in \partial X$, $Tx \in X$. So T has a unique fixed point in$X$.

We can see its applications in [25], where they proved the existence and uniqueness of the following problem:

$$^{C}D_{0^+}^{\alpha} u\left(t\right) = f\left(t, \sup_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\tau_1(t)),...,u(t-\zeta_N(t))\right), t\in J_k, k\in\mathbb{N} \tag{2.3}$$

$$\bigtriangledown u(t_k) = I_k\left(u\left(t_k - \rho_k(t_k)\right)\right), \ k \in \mathbb{N}^* \tag{2.4}$$

with

$$u\left(t\right) = \phi\left(t\right), \quad t \leq 0, \tag{2.5}$$

After that, we based on its perspectives, we extend the results of that theory, specific the notion of contaraction in specific type.

2.3 Extension of principle of contraction to non-metric case

The main results of this paragraph are in [41].

particular form

In the case of A be the set of all the ideal in $\mathbb{R}$.

The assumptions on the family of semi-norms defined on E:

1- The notion of contraction:

For all $h \in H \subset \mathbb{R}$, $\forall \alpha \in A$

$$\begin{cases} p_\alpha(Tx - Ty) \le K_\alpha.\dfrac{[h]}{h}.p_{i(\alpha+h)}(x-y) & \text{si} \quad h > 1, \\ p_\alpha(Tx - Ty) \le K_\alpha.h.p_{i(\alpha+h)}(x-y) & \text{si} \quad 0 < h \le 1, \\ p_\alpha(Tx - Ty) \le K_\alpha.p_{i(\alpha)}(x-y) & \text{si} \quad h = 0, \\ p_\alpha(Tx - Ty) \le K_\alpha.\dfrac{h}{[h]}.p_{ii(\alpha+h)}(x-y) & \text{si} \quad h < 0. \end{cases}$$

Where i is an application defined on subset $S \subset A$ such as

$$\begin{array}{rcccl} i & : & S & \longrightarrow & S \\ & & k & \longrightarrow & i(k) \end{array}$$

With: $\forall h \in H, \forall k \in S$, then $k + h \in S$ and equality $i(k+h) = i(k) + i(h)$ is true.

2- $M_\alpha(h) = \sup\{p_{(i^n(\alpha)+\sum_{s=1}^n i^s(h))}(x-y), n = 1, 2, 3, ...\} < \infty$

3- $h \in H \setminus \{0\}$, we notice:

$$N_h = \min\left\{\frac{[h]}{h}, \frac{h}{[h]}, |h|\right\}$$

Such as $0 < M_h < 1$.

And

$$h_\alpha \doteqdot \inf\left\{h \in \mathbb{R}^* : p_\alpha(Tx - Ty) \le K_\alpha.N_h.p_{i(\alpha+h)}(x-y)\right\}$$

Under the Under the above hypotheses,we prove that the following theorem :

Theorem 2.3. *Let $X \subset E$ a i-bounded set and $T : X \to E$ is a contractive function such as:*

(a) $\sum_{n=0}^{\infty} 2^{2n} K_\alpha.K_{i(\alpha)}...K_{i^n(\alpha)} < \infty$ *for all* $\alpha \in A$*;*

(b) $\forall x \in \partial X$*,* $Tx \in X$*.*

Then, T has a unique fixed point in X.

Proof. Let $x_0 \in X$. We construct two sequences $(x_n)_{n=1}^{\infty}$ and $(x'_n)_{n=1}^{\infty}$ in the following manner:

- We put $x'_1 = Tx_0$ such as so that

  * $x_1 = x'_1$ if $x'_1 \in X$

* We choose $x_1 \in \partial X$ (where ∂X is the boundary of $\mathbb{X}$) such that

$$p_\alpha(x_0 - x_1) + p_\alpha(x_1 - x_1') = p_\alpha(x_0 - x_1'), \forall\alpha \in A$$

- We put $x_2' = Tx_1$ such as so that
  * $x_2 = x_2'$ if $x_2' \in X$
  * We choose $x_2 \in \partial X$ such that

$$p_\alpha(x_1 - x_2) + p_\alpha(x_2 - x_2') = p_\alpha(x_1 - x_2'), \forall\alpha \in A$$

- In the same way we obtain two sequences $(x_n)_{n=1}^{\infty}$ and $(x_n')_{n=1}^{\infty}$ which verify the following properties:
  (i) $x_{n+1}' = Tx_n$;
  (ii) $x_{n+1} = x_{n+1}'$ if $x_{n+1}' \in X$;
  (iii) $x_{n+1} \in \partial X$, and $p_\alpha(x_n - x_{n+1}) + p_\alpha(x_{n+1} - x_{n+1}') = p_\alpha(x_n - x_{n+1}')$ for all $\alpha \in A$.

Let $\mathbb{P} = \{x_i \in (x_n)_{n=1}^{\infty} : x_i = x_i'\}$ and $\mathbb{D} = \{x_i \in (x_n)_{n=1}^{\infty} : x_i \neq x_i'\}$. $x_n \in \mathbb{D}$ implies $x_{n-1} \in \mathbb{P}$ and $x_{n+1} \in \mathbb{P}$ (according to condition (b)of the theorem).
We must estimate $p_\alpha(x_n - x_{n+1})$ for all $\alpha \in A$. Let $\alpha \in A$ fixed.
Then, the possible cases are as follows:

Case 1: $x_n, x_{n+1} \in \mathbb{P}$. As T is contractive on a:

$$p_\alpha(x_n - x_{n+1}) = p_\alpha(T_{x-1} - Tx_n) \leq k_\alpha.N_{h_\alpha}.p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n)$$

Case 2: $x_n \in \mathbb{P}, x_{n+1} \in \mathbb{D}$. So

$$\begin{aligned} p_\alpha(x_n - x_{n+1}) &\leq p_\alpha(x_n - x_{n+1}) + p_\alpha(x_{n+1} - x_{n+1}') \\ &= p_\alpha(x_n - x_{n+1}') \\ &= p_\alpha(Tx_{n-1} - Tx_n) \\ &\leq k_\alpha.N_{h_\alpha}.p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n) \end{aligned}$$

Case 3: $x_n \in \mathbb{D}, x_{n+1} \in \mathbb{P}$. Since $x_n \in \mathbb{D}$ is a convex linear combination of x_{n-1} and x_n' then

$$p_\alpha(x_n - x_{n+1}) \leq \max\{p_\alpha(x_{n-1} - x_{n+1}); p_\alpha(x_n' - x_{n+1})\}$$

for every $\alpha \in A$. If $p_\alpha(x_{n-1} - x_{n+1}) \le p_\alpha(x'_n - x_{n+1})$. We have

$$p_\alpha(x_n - x_{n+1}) \le p_\alpha(x'_n - x_{n+1}) = p_\alpha(Tx_{n-1} - Tx_n) \le k_\alpha.N_{h_\alpha}.p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n).$$

Other wise, we have

$$p_\alpha(x_n - x_{n+1}) \le p_\alpha(x_{n-1} - x_{n+1}) = p_\alpha(Tx_{n-2} - Tx_n) \le k_\alpha.N_{h_\alpha}.p_{i(\alpha+h_\alpha)}(x_{n-2} - x_n).$$

So, for all natural number $n \ge 2$ we obtain

$$p_\alpha(x_n - x_{n+1}) \le k_\alpha.N_{h_\alpha}.p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n)$$

or

$$p_\alpha(x_n - x_{n+1}) \le k_\alpha.N_{h_\alpha}.p_{i(\alpha+h_\alpha)}(x_{n-2} - x_n)$$

As

$$\begin{aligned} p_{i(\alpha+h_\alpha)}(x_{n-2} - x_n) &\le p_{i(\alpha+h_\alpha)}(x_{n-2} - x_{n-1}) + p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n) \\ &\le 2\max\{p_{i(\alpha+h_\alpha)}(x_{n-2} - x_{n-1}), p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n)\}, \end{aligned}$$

we have

$$p_\alpha(x_n - x_{n+1}) \le 2k_\alpha.N_{h_\alpha}.\max\{p_{i(\alpha+h_\alpha)}(x_{n-2} - x_{n-1}), p_{i(\alpha+h_\alpha)}(x_{n-1} - x_n)\} \forall n \ge 2, \forall \alpha \in A$$

Subsequently, we will show that

$$p_\alpha(x_{2s} - x_{2s+1}) \le 2^{2s-1}.\left(N_{h_\alpha}\right)^{2s-1}.k_\alpha.k_{i(\alpha)}.....k_{i^{s-1}(\alpha)}.M_\alpha(h_\alpha) \tag{2.6}$$

and

$$p_\alpha(x_{2s+1} - x_{2s+2}) \le 2^{2s-1}.\left(N_{h_\alpha}\right)^{2s-1}.k_\alpha.k_{i(\alpha)}.....k_{i^{s-1}(\alpha)}.M_\alpha(h_\alpha) \tag{2.7}$$

for all natural number s and for every $\alpha \in A$.
we have

$$p_\alpha(x_2 - x_3) \le 2.N_{h_\alpha}.k_\alpha.p_{i(\alpha+h_\alpha)}(x_1 - x_2) \le 2.N_{h_\alpha}.k_\alpha.M_\alpha(h_\alpha)$$

or

$$p_\alpha(x_2 - x_3) \le 2.N_{h_\alpha}.k_\alpha.p_{j(\alpha+h_\alpha)}(x_0 - x_2) \le 2.N_{h_\alpha}.k_\alpha.M_\alpha(h_\alpha),$$

i.e. $p_\alpha(x_2 - x_3) \le 2.N_{h_\alpha}.k_\alpha.M_\alpha(h_\alpha)$, $\forall \alpha \in A$ Also,

$$p_\alpha(x_3 - x_4) \le 2.N_{h_\alpha}.k_\alpha.p_{i(\alpha+h_\alpha)}(x_2 - x_3) \le 2.N_{h_\alpha}.k_\alpha.M_\alpha(h_\alpha)$$

or

$$p_\alpha(x_3 - x_4) \leq 2.N_{h_\alpha}.k_\alpha.p_{i(\alpha+h_\alpha)}(x_1 - x_3) \leq 2.N_{h_\alpha}.k_\alpha.M_\alpha(h_\alpha),$$

i.e. $p_\alpha(x_3 - x_4) \leq 2.N_{h_\alpha}.k_\alpha.M_\alpha(h_\alpha)$ for all $\alpha \in A$.

Suppose that for every$s \geq 2$ (s natural number)the inequalities (2.6) and (2.7) are true. In view of $M_{i(\alpha)}(h_\alpha) \leq M_\alpha(h_\alpha)$ we obtain that

$$\begin{aligned} p_\alpha(x_{2s+2} - x_{2s+3}) &\leq 2.N_{h_\alpha}.k_\alpha.p_{i(\alpha+h_\alpha)}(x_{2s+1} - x_{2s+2}) \\ &\leq 2.N_{h_\alpha}.k_\alpha.2^{2s}.\left(N_{h_\alpha}\right)^{2s}.k_{i(\alpha)}.k_{i^2(\alpha)}.....k_{i^s(\alpha)}.M_{i(\alpha)}(h_\alpha) \\ &\leq 2^{2(s+1)-1}.\left(N_{h_\alpha}\right)^{2(s+1)-1}.k_\alpha.k_{i(\alpha)}.....k_{i^{(s+1)-1}(\alpha)}.M_\alpha(h_\alpha) \end{aligned}$$

or

$$\begin{aligned} p_\alpha(x_{2s+3} - x_{2s+4}) &\leq 2.k_\alpha.N_{h_\alpha}.p_{j(\alpha+h_\alpha)}(x_{2s+1} - x_{2s+2}) \\ &\leq 2.N_{h_\alpha}.k_\alpha.2^{2s}.\left(N_{h_\alpha}\right)^{2s}.k_{i(\alpha)}.k_{i^2(\alpha)}.....k_{i^s(\alpha)}.M_{i(\alpha)}(h_\alpha) \\ &\leq 2^{2(s+1)-1}.\left(N_{h_\alpha}\right)^{2(s+1)-1}.k_\alpha.k_{i(\alpha)}.....k_{i^{(s+1)-1}(\alpha)}.M_\alpha \end{aligned}$$

Then, we prove that $(x_n)_{n=1}^\infty$ is a Cauchy sequence. Because of 1.1 and 1.6 we have

$$p_\alpha(x_{2s} - x_{2s+1}) + p_\alpha(x_{2s+1} - x_{2s+2}) \leq 2^{2s}.\left(N_{h_\alpha}\right)^{2s-1}.k_\alpha.k_{i(\alpha)}.....k_{i^{s-1}(\alpha)}.M_\alpha(h_\alpha), \qquad \alpha \in A.$$

The sequence $\sum_{s=0}^{\infty} 2^{2s}.\left(N_{h_\alpha}\right)^{2s}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$ is convergent.
Indeed, we have $\mid 2^{2s}.\left(N_{h_\alpha}\right)^{2s}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)} \mid \leq 2^{2s}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$
and the sequence $\sum_{s=0}^{\infty} 2^{2s}k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$ is convergent.
So, $\sum_{s=0}^{\infty} 2^{2s}.\left(N_{h_\alpha}\right)^{2s}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$ is absolutely convergent sequence. Consequently, it is convergent.
Thus $\sum_{s=N+1}^{\infty} 2^{2s}.\left(N_{h_\alpha}\right)^{2s}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$ tends to 0 when n tends to $+\infty$.
In other wise say, for all $\varepsilon > 0$ from a certain $N > 1$ we have

$$4.M_\alpha(h_\alpha).\sum_{s=N-1}^{\infty} 2^{2(s-1)}.\left(N_{h_\alpha}\right)^{2(s-1)}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)} < \varepsilon.$$

Let $n \geq 2N$ and let $r \geq 1$. So there exists a sufficiently large integer M such that

$$\begin{aligned} p_\alpha(x_n, x_{n+r}) &\leq p_\alpha(x_n, x_{n+1}) + p_\alpha(x_{n+1}, x_{n+2}) + ... + p_\alpha(x_{n+r-1}, x_{n+r}) \\ &\leq \sum_{s=N}^{M} [p_\alpha(x_{2s}, x_{2s+1}) + p_\alpha(x_{2s+1}, x_{2s+2})] \\ &\leq \sum_{s=N}^{\infty} 2^{2s}. (N_{h_\alpha})^{2s} .k_\alpha.k_{i(\alpha)}.....k_{i^{s-1}(\alpha)}.M_\alpha(h_\alpha) \\ &= 4.M_\alpha(h_\alpha). (N_{h_\alpha}). \sum_{s=N}^{\infty} 2^{2(s-1)}. (N_{h_\alpha})^{2(s-1)} .k_\alpha.k_{i(\alpha)}.....k_{i^{s-1}(\alpha)} < \varepsilon. \end{aligned}$$

Therefore,$\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since X is being complete, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x_* \in X$. there exists a subsequence $\{x_{n_s}\}_{s=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_s}\}_{s=1}^{\infty} \subset P$, and $x_{n_s+1} = Tx_{n_s}$ for every natural number s.

So, for all $\alpha \in A$ we have

$$\begin{aligned} p_\alpha(x_* - Tx_*) &\leq p_\alpha(x_* - x_{n_s}) + p_\alpha(x_{n_s} - Tx_*) \\ &= p_\alpha(x_* - x_{n_s}) + p_\alpha(Tx_{n_s-1} - Tx_*) \\ &\leq p_\alpha(x_* - x_{n_s}) + k_\alpha.p_{i(\alpha+h_\alpha)}(x_{n_s-1} - x_*) \underset{s\longrightarrow\infty}{\longrightarrow} 0. \end{aligned}$$

Consequently, $P_\alpha(x_* - Tx_*) = 0$ for all $\alpha \in A$, i.e.,x_* is necessarily a fixed point of T. Let $x, y \in X$ such that $x = Tx$ and $y = Ty$. Then for all $\alpha \in A$ the inequalities following are true.

$$\begin{aligned} p_\alpha(x - y) &= p_\alpha(Tx - Ty) \leq N_{h_\alpha}k_\alpha.p_{i(\alpha+h_\alpha)}(x - y) = N_{h_\alpha}.k_\alpha.p_{i(\alpha+h_\alpha)}(Tx - Ty) \\ &\leq (N_{h_\alpha})^2 .k_\alpha.k_{i(\alpha)}.p_{i^2(\alpha)+i^2(h_\alpha)+i(h_\alpha)}(x - y) \leq ... \leq (N_{h_\alpha})^n k_\alpha.k_{i(\alpha)}...k_{i^n(\alpha)}.M_\alpha(h_\alpha) \underset{n\longrightarrow\infty}{\longrightarrow} 0. \end{aligned}$$

Thus,$P_\alpha(x - y) = 0$ for all $\alpha \in A$, i.e. the fixed point of T is unique. □

Remark 2.1. *1. If we assume in the previous theorem that $\{(N_h)^n\}_{n\in\mathbb{N}}$ is a bounded sequence, then we can prove in the same way that the result of the theorem remains true. Indeed, we have*

$$| \; 2^{2s}. (N_{h_\alpha})^{2s} .k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)} \; | \leq C.2^{2s}.k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}.$$

Since the sequence $\sum_{s=0}^{\infty} 2^{2s}k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$ is convergent.

So, the sequence $\sum_{s=0}^{\infty} 2^{2s}. (N_{h_\alpha})^{2s} .k_\alpha.k_{i(\alpha)}.....k_{i^s(\alpha)}$ is absolutely convergent. Thus, it is convergent.

2. If the set H is reduced to the singleton $\{0\}$, then the result of the theorem 3.3 coincides *with that of theorem 1,p10,[22].*

2.4 Application 1

We know that the functional differential equations whose right part depend on the extreme values of the controlled quantity appear in many technological processes.
Some initial value problems associated to some fractional differential equations with implusive conditions and maxima, witch can be formulated as follows:

$$^{C}D_{0^+}^{\alpha}u\left(t\right)=f\left(t,\sup_{\sigma\in[a(t),\,b(t)]}u(\sigma),u(t-\zeta_1(t)),\ldots,u(t-\zeta_N(t))\right),t\in J_k,k\in\mathbb{N} \tag{2.8}$$

$$\triangle\, u(t_k)=I_k\left(u\left(t_k-\rho_k(t_k)\right)\right),\;\; k\in\mathbb{N}^* \tag{2.9}$$

$$u\left(t\right)=\phi\left(t\right),\quad t\leq 0, \tag{2.10}$$

where $^{C}D_{0^+}^{\alpha}$ is the notion of Caputo's derivation of operator in order $\alpha\in\,]0,1[$, $N\in\mathbb{N}$; a, b, ζ_i (with $0\leq i\leq N$) are continuous functions defined on $\mathbb{R}_+=[0,+\infty[$, $f:\mathbb{R}_+\times\mathbb{R}^{1+N}\longrightarrow\mathbb{R}$ is a non-linear continuous function, $\triangle u(t_k)=u(t_k^+)-u(t_k^-)$, where $u(t_k^+)=\lim\limits_{\substack{t\to t_k\\ t>t_k}}$, $u(t_k^-)=\lim\limits_{\substack{t\to t_k\\ t<t_k}}$, and the sequence $\{t_k\}\in\mathbb{R}_+$ such as $t_k<t_{k+1}$ $\forall k\in\mathbb{N}^*$ and $\lim\limits_{k\to\infty}t_k=\infty$, $I_0=[0,t_1]$ and $I_k=\,]t_k,t_{k+1}]$ $(k\in\mathbb{N}^*)$, the condition of initial: $\Psi:]-\infty,\,0]\longrightarrow\mathbb{R}$ is a continuous function such that $\Psi\left(0\right)=\Psi_0>0$.
In the following, we proved that this problem had a unique solution.

Preliminaries

Now, we present the notion of derivation in fractional.
Let $n\in\mathbb{N}$, $\alpha\in\mathbb{R}_+$ such that $n-1<\alpha\leq n$,and d^n/dt^n the ordinary derivation in order n. [[23, 3]] Let u be a function on $\mathbb{R}_+$, we can define respectively the Riemann-Liouville's fractional integral and fractional derivation of order α by:

$$I_{0^+}^{\alpha}u\left(t\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}u\left(s\right)ds,\qquad t>0,$$

$$D_{0^+}^{\alpha}u\left(t\right):=\frac{d^n}{dt^n}I_{0^+}^{n-\alpha}u\left(t\right):=\frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^n}{dt^n}\int_0^t\left(t-s\right)^{n-\alpha-1}u\left(s\right)ds,\qquad t>0,$$

When the right sides exist, we have $\Gamma(.)$ be the Gamma function. [[23, 3]] Let u be a function on $\mathbb{R}_+$, we define the Caputo fractional derivative of order α as follow, where it noted by ${}^{C}D_{0^+}^{\alpha}$.

$$ {}^{C}D_{0^+}^{\alpha}u(t) := \left(D_{0^+}^{\alpha}\left[u - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!}(.)^k \right] \right)(t), \qquad t > 0, $$

Concept of solutions for (2.8)-(2.10)

Due to the non-locality of the fractional differential operators, there is a fundamental dependency between the fractional derivative considered in its definitions and the limit of the integral of the definition. This therefore leads to a different solution concept for impulsive fractional differential equations.

There are currently two main methods (see [46, 47]). In the following, we detail the methodology employed in this work in the definition 2.3 .

Let E be hlctvs of real valued continuous functions defined on $\mathbb{R}$:

$$ E = \Big\{ \quad u : \mathbb{R} \to \mathbb{R} : u\mid_{\mathbb{R}_-} \in \mathcal{C}(\mathbb{R}_-) \text{ and for every } k \in \mathbb{N} : \\ u\mid_{J_k} \in \mathcal{C}(J_k) \text{ with } u(t_k^+) \text{ finite} \Big\}. \tag{2.11} $$

and $\{P_K : K \in \mathcal{K}\}$ be the family of semi-norms on E, defined by

$$ P_K(u) = \sup_{t \in K}\left\{ e^{-\lambda t}|u(t)| \right\}, \tag{2.12} $$

where K is the set of all real compact, denoted by $\mathcal{K}$,and $\lambda \in \mathbb{R}_+$ to be specified later.

In the following definition of solutions, the approach A_1 in [47] is adapted to (2.8)-(2.10).

Definition 2.3 ([47])**.** *A function $u \in E$ is said to be a solution of* (2.8) − (2.10)*, if it satifies* (2.10)*, moreover $u = u_0 \mid_{J_0}$, and for every $k \in \mathbb{N}^*$: $u = u_k \mid_{J_k}$, where $u_0 \in \mathcal{C}(]-\infty, t_1])$ is the solution of*

$$ \begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) = f\left(t, \sup\limits_{\sigma \in [a(t),\, b(t)]} u(\sigma), u(t-\zeta_1(t)), \ldots, u(t-\zeta_N(t)) \right), & t \in]0, t_1] \\ \\ u(t) = \Psi(t), & t \leq 0 \end{cases} $$

and for $k \in \mathbb{N}^*$: $u_k \in \mathcal{C}\left(]-\infty, t_{k+1}]\right)$ is the solution of

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} u\left(t\right) = f\left(t, \sup\limits_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\zeta_1(t)), \ldots, u(t-\zeta_N(t))\right), & t\in[0,t_{k+1}] \\ u(t_k^+) = u(t_k^-) + I_k\left(u\left(t_k^- - \rho_k(t_k^-)\right)\right), \\ u\left(t\right) = \Psi\left(t\right), & t<0 \end{cases}$$

Equivalent integral equation of (2.8)-(2.10)

By the following hypothesis, we can formulate the problem $(2.8)-(2.10)$ to integral equation equivalent.

(H_1) a, b are continuous functions such that

(i) $\forall t \geq 0$: $0 \leq \overline{a} \leq a(t) \leq b(t)$, with $\overline{a} = \inf a(t)$, $\exists \overline{b}$ such that $\overline{a} < \overline{b}$, $a(t) = \overline{a}$, $b(t) = \overline{b} : \forall t \in \left[0, \overline{b}\right]$ and $b(t) \leq t$ for $t > \overline{b}$.

(ii) $\overline{b} < t_1$

(H_2) ζ_i $(1 \leq i \leq N)$ is a continuous function such that

(i) $\exists \zeta > 0 : \ \zeta_i(t) > t - \zeta, \ \ \forall t > 0.$

(ii) $\zeta < t_1$, and if $\zeta_i(t) \in \{t_k\}_{k=1}^{\infty}$, then $t - \zeta_i(t) \in \{t_k\}_{k=1}^{\infty}$.

(iii) $\exists h_i > 0$ such that: $\zeta_i(t) \geq t$, $\forall t \in [0, h_i]$ and $\zeta_i(t) < t$, $\forall t \in]h_i, +\infty[$.

(H_3) For every $k \in \mathbb{N}^* : \rho_k(t) > 0, \quad \forall t > 0$

In the next lemma, we present the integral equation equivalent to (2.8)-(2.10).

Lemma 2.1. *Let f be a continuous function and $(H_1), (H_2.(i), (ii)), (H_3)$ are verified. So, $u \in X$ is a solution of (2.8)-(2.10) iif u is a solution of the following integral equation*

$$u(t) = \begin{cases} \Psi_0 + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \sup\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), \ldots, u(s-\zeta_N(s))\right) ds, & t\in J_0, \\ \Psi_0 + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \sup\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), \ldots, u(s-\zeta_N(s))\right) ds & \\ \quad + \displaystyle\sum_{i=1}^{k} I_i(u(t_i^- - \rho_i(t_i^-))), & t\in J_k : k\in\mathbb{N}^* \\ \Psi(t), & t\leq 0 \end{cases} \tag{2.13}$$

Proof. If $t \in [0, t_1]$,

$$\begin{cases} {}^{C}D^{\alpha} u(t) = f\left(t, \max_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\zeta_1(t)), ..., u(t-\zeta_N(t))\right) \\ u(0) = \Psi_0 \end{cases}$$

then:

$$u(t) = \Psi_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \tag{2.14}$$

If $t \in]t_1, t_2]$

for all $t \in [0, t_2]$

$$\begin{cases} {}^{C}D^{\alpha} u(t) = f\left(t, \max_{\sigma\in[a(t),\, b(t)]} u(\sigma), u(t-\zeta_1(t)), ..., u(t-\zeta_N(t))\right) \\ \\ u(t_1^+) = \Psi_0 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \end{cases}$$

So:

$$u(t) = \Psi_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds$$

We have:

$$\Psi_0 = u(t_1^+) - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds$$

And:$u(t_1^+) = I_1(u(t_1^- - \rho(t_1^-))) + u(t_1^-)$. So by (2.14):

$$u(t_1^+) = I_1(u(t_1^- - \rho(t_1^-))) + \Psi_0 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds$$

Then:

$$\begin{aligned} u(t) &= I_1(u(t_1^- - \rho(t_1^-))) + \Psi_0 + \int_0^{t} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \\ &\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \\ &\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \end{aligned}$$

$$u(t)=\Psi_0+\int_0^{t_1}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds+I_1(u(t_1^- -\rho(t_1^-)))\quad(2.15)$$

If $t\in]t_2,t_3]$

for all $t\in[0,t_3]$

$$\begin{cases} {}^CD^\alpha u\left(t\right)=f\left(t,\max\limits_{\sigma\in[a(t),\,b(t)]}u\left(\sigma\right),u\left(t-\zeta_1\left(t\right)\right),...,u\left(t-\zeta_N\left(t\right)\right)\right)\\ \\ u(t_2^+)=\Psi_0+\displaystyle\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds\end{cases}$$

So:

$$u(t)=\Psi_0+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds$$

We have:

$$\Psi_0 \;=\; u(t_2^+)-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds$$

And: $u(t_2^+)=I_2(u(t_2^- -\rho(t_2^-)))+u(t_2^-)$. So by (2.15):

$$u(t_2^+)=I_2(u(t_2^- -\rho(t_2^-)))+\Psi_0+\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds+I_1(u(t_1^- -\rho(t_1^-)))$$

Then:

$$\begin{aligned} u(t) \;=\;& I_2(u(t_2^- -\rho(t_2^-)))+\Psi_0+\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds\\ &+I_1(u(t_1^- -\rho(t_1^-)))-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds\\ &+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds\end{aligned}$$

So:

$$u(t)=\Psi_0+\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\Big(s,\max_{\sigma\in[a(s),b(s)]}u(\sigma),u(s-\zeta_1(s)),...,u(s-\zeta_N(s))\Big)ds+\sum_{i=1}^{2}I_i(u(t_i^- -\rho(t_i^-)))\quad(2.16)$$

In the same way: If $t\in]t_k,t_{k+1}]$

$\forall t \in [0, t_{k+1}]$

$$\begin{cases} {}^{C}D^{\alpha}u\left(t\right) = f\left(t, \max\limits_{\sigma\in[a(t),\, b(t)]} u\left(\sigma\right), u\left(t-\zeta_1\left(t\right)\right), ..., u\left(t-\zeta_N\left(t\right)\right)\right) \\ \\ u(t_k^+) = \Psi_0 + \displaystyle\int_0^{t_k} \frac{\left(t_k - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \end{cases}$$

So:

$$u(t) = \Psi_0 + \int_0^{t} \frac{\left(t - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds$$

We have:

$$\Psi_0 \;\; = \;\; u(t_k^+) - \int_0^{t_2} \frac{\left(t_k - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds$$

And: $u(t_k^+) = I_k(u(t_k^- - \rho(t_k^-))) + u(t_k^-)$. So:

$$\begin{aligned} u(t_k^+) \;\; &= \;\; I_k(u(t_k^- - \rho(t_k^-))) + \Psi_0 + \int_0^{t_k} \frac{\left(t_k - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \\ &\quad + \sum_{i=1}^{k-1} I_i(u(t_i^- - \rho(t_i^-))) \end{aligned}$$

Then:

$$\begin{aligned} u(t) \;\; &= \;\; I_k(u(t_k^- - \rho(t_k^-))) + \Psi_0 + \int_0^{t_k} \frac{\left(t_k - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \\ &\quad + \sum_{i=1}^{k-1} I_i(u(t_i^- - \rho(t_i^-))) - \int_0^{t_2} \frac{\left(t_k - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \\ &\quad + \int_0^{t} \frac{\left(t - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds \end{aligned}$$

$$u(t) = \Psi_0 + \int_0^{t} \frac{\left(t - s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\Big(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-\zeta_1(s)), ..., u(s-\zeta_N(s))\Big) ds + \sum_{i=1}^{k} I_i(u(t_i^- - \rho(t_i^-))) \quad (2.17)$$

$\square$

The proof of existence and uniqueness

For the prove of main results, we put the following hypothesis.

(H_4) There exist positive real valued functions L_1 and L_2 defined on $\mathbb{R}_+$, satisfying

(i) $|f(t,\xi,x_1,...,x_N) - f(t,\eta,y_1,...,y_N)| \leq L_1(t)|\xi-\eta| + L_2(t)\sum_{i=1}^{N}|x_i-y_i|$ whenever the left hand side is defined.

(ii) For $\lambda > 0$ and $\nu := 1 + 1/\alpha$, we have

$$R_\lambda := \int_0^{+\infty} L_1^\nu(s)\, e^{-\nu\lambda(s-b(s))}ds < \infty, \text{ and } S_\lambda = \int_0^{+\infty} L_2^\nu(s)\, e^{-\nu\lambda s}ds < \infty.$$

(H_5) $\forall k \in \mathbb{N}^*$, $\exists l_k > 0$ such that: $|I_k(x) - I_k(y)| \leq l_k|x-y|$, $\forall x, y \in \mathbb{R}$

(H_6) There exists a positive constant $M > \Psi_0$ such that

$$\frac{1}{\Gamma(\alpha)\alpha} f(t, M, x_1, ..., x_N) \leq \frac{M - \Psi_0}{\overline{b}^\alpha}, \ \forall (t,x_1,...,x_N) \in [0,\overline{b}] \times \mathbb{R}^N.$$

(H_7) f is a non negative function and moreover, $\exists \iota \in]\Psi_0, M[$ such that

$$\frac{1}{\Gamma(\alpha)\alpha} f(t, \iota, x_1, ..., x_N) > \frac{M - \Psi_0}{|\overline{b} - \max\limits_{1\leq i\leq N} h_i|^\alpha}, \forall (t,x_1,...,x_N) \in [0,\overline{b}] \times \left(]\Psi_0, \Psi_0 + \tfrac{\iota-\Psi_0}{\overline{b}}\zeta[\right)^N.$$

Remark 2.2. *Note that if L_1 and L_2 are bounded functions, the requirement specified in $(H_4.(ii))$ are satisfied (remember $b(t) \leq t$ for $t > \overline{b}$). However, we point out that with (H_4), not only L_1 and L_2 but even f is may increase infinitely compared to t (there is an example in the previous section). Therefore, the hypothesis (H_4) contains a larger range of functions than the classical Lipschitz condition.*

We denote by $\mathbf{X}_{\Psi,M}$ the subset of E defined by

$$\mathbf{X}_{\Psi,M} = \left\{u \in E : u(t) = \Psi(t) \text{ for } t \leq 0, \text{ and } u(t) \leq M \text{ for } t \in [\overline{a}, \overline{b}]\right\},$$

where $\overline{a}, \overline{b}$ and M are the constants given by (H_1) and (H_6). $\mathbf{X}_{\Psi,M}$ is closed and its boundary is:

$$\partial\mathbf{X}_{\Psi,M} = \left\{u \in X : u(t) = \Psi(t) \text{ for } t \leq 0 \text{ and } \max_{t\in[\overline{a},\overline{b}]} u(t) = M\right\}.$$

We observe that the boundary of $\mathbb{X}$, on the sup-function take, a and b depend on the time t. This studied by the neutral functional differential equations theory for applications in [58, 59, 57], but it also used in the showing of contractivity of corresponding operator under the definition **??**. For that, we define the map i on $\mathcal{K}$ into it self by;

$$i(K) := \begin{cases} K & \text{if } K_+ = \emptyset, \\ \left[0,\ \max\{K_m,\ \zeta,\ \overline{b}\}\right] & \text{if } K_+ \neq \emptyset, \end{cases} \tag{2.18}$$

where $K_+ := K \cap \left]0,\ +\infty\right[$, $K_m = \sup K$.

Lemma 2.2. *[25, Remark 2] For i defined by (2.34), the set $\mathbf{X}_{\Psi,M}$ is i-bounded. However, $i(K) = i^k(K), \forall k \in \mathbb{N}$*

In this work, F denotes the operator defined on $\mathbf{X}_{\Psi,M}$ by the right hand side of (2.13). Under some appropriate hypotheses, the following lemma asserts the non-self nature of this operator.

Lemma 2.3. *Under the hypotheses $(H_1) - (H_3)$, F maps $\mathbf{X}_{\Psi,M}$ into $\mathbb{X}$ and moreover, (H_7) ensure that it does non-self mapping, provided that*

$$\max_{1 \leq i \leq N} h_i < \overline{b} \tag{2.19}$$

Proof. Recall first that the functions f, ζ_i $(1 \leq i \leq N)$, a, b and ρ_k $(k \in \mathbb{N}^*)$ are all continuous. Let now $u \in \mathbf{X}_{\Psi,M} \subset X$, then $u \mid_{J_k} \in \mathcal{C}(J_k)$, so in view of $(H_1) - (H_3)$, $Fu \in \mathcal{C}(J_k)$ as a composition of continuous functions on J_k. Since for every $k \in \mathbb{N}^*$, $u(t_k^+)$ exists, then $Fu(t_k^+)$ exists also as a limite of a composotion of functions, which admit each of them a limit in t_k^+. Hence $Fu \in X$.
Let now $v_0 \in \mathbf{X}_{\Psi,M}$ be a function defined by means of the constant ι given in (H_7) such that for $t \in [0, \overline{b}]$:

$$v_0(t) = \Psi_0 + \frac{(\iota - \Psi_0)t}{\overline{b}}$$

Under the hypothesis (H_1), $(H_2.(i), (iii))$, (H_7) and (**??**), we get

$$\begin{aligned} Fv_0\left(\overline{b}\right) &= \Psi_0 + \int_0^{\overline{b}} \frac{\left(\overline{b} - s\right)^{\alpha-1}}{\Gamma(\alpha)} f\left(s,\iota,v_0(s-\zeta_1(s)),\ldots,v_0(s-\zeta_N(s))\right) ds \\ &> \Psi_0 + \alpha\ \frac{M-\Psi_0}{(\overline{b}-\max\limits_{1\leq i\leq N} h_i)^\alpha} \int_{\max\limits_{1\leq i\leq N} h_i}^{\overline{b}} (\overline{b} - s)^{\alpha-1} ds = M \end{aligned}$$

it means $Fv_0 \notin \mathbf{X}_{\Psi,M}$ and the proof finishes. $\square$

Proposition 2.1. *Let $(H_1) - (H_5)$ are satisfies. Then, for each $u, v \in \mathbf{X}_{\Psi,M}$ and $\forall K \in \mathcal{K}$ the following inequality is true:*

$$P_K\left(Fu - Fv\right) \leq S_\lambda P_{i(K)}\left(u - v\right), \tag{2.20}$$

where

$$S_\lambda = \frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)\,\lambda^{\frac{\alpha^2}{\alpha+1}}}\left(R_\lambda^{\frac{\alpha}{\alpha+1}} + N\,e^{\lambda\zeta}\,S_\lambda^{\frac{\alpha}{\alpha+1}}\right) + \sum_{k\geq 1} l_k$$

Proof. For $K_+ = \emptyset$, we have

$$P_K\left(Fu - Fv\right) = P_K\left(\Psi_0 - \Psi_0\right) = 0,$$

and thus (2.20) is satisfied.

For $K_+ \neq \emptyset$, let $t \in K_+$. We use (H_1), $(H_2.(iii))$, (H_3), $(H_4.(i))$ and (H_5), we obtained

$$\left|Fu\left(t\right) - Fv\left(t\right)\right| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)} L_1\left(s\right)\left|\sup_{\sigma\in[a(s),b(s)]} u\left(\sigma\right) - \sup_{\sigma\in[a(s),b(s)]} v\left(\sigma\right)\right| ds$$

$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)} L_2\left(s\right)\sum_{i=1}^{N}\left|u\left(s-\zeta_i\left(s\right)\right) - v\left(s-\zeta_i\left(s\right)\right)\right| ds$$

$$+\sum_{t_k<t} l_k\left|u(t_k^- - \rho(t_k^-)) - v(t_k^- - \rho(t_k^-))\right|$$

$$\leq \int_0^t L_1\left(s\right)\frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)}\sup_{\sigma\in[a(s),b(s)]}\left|u\left(\sigma\right) - v\left(\sigma\right)\right| ds$$

$$+\sum_{i\in\{1,\ldots,N:h_i\leq t\}}\int_{h_i}^t L_2\left(s\right)\frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left|u\left(r_i\left(s\right)\right) - v\left(r_i\left(s\right)\right)\right| ds$$

$$+\sum_{t_k<t} l_k\left|u(t_k^- - \rho(t_k^-)) - v(t_k^- - \rho(t_k^-))\right|$$

$$\leq \int_0^t L_1\left(s\right)\frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)} e^{\lambda b(s)}\max_{\sigma\in[\overline{a},\max\{K_m,\overline{b}\}]} e^{-\lambda\sigma}\left|u\left(\sigma\right) - v\left(\sigma\right)\right| ds$$

$$+\sum_{i\in\{1,\ldots,N:h_i\leq t\}}\int_{h_i}^t L_2\left(s\right)\frac{(t-s)^{\alpha-1}}{\Gamma\left(\alpha\right)} e^{\lambda r_i(s)} e^{-\lambda r_i(s)}\left|u\left(r_i\left(s\right)\right) - v\left(r_i\left(s\right)\right)\right| ds$$

$$+\sum_{t_k<t} l_k . e^{\lambda t}\max_{\delta\in[0,\,t]} e^{-\lambda\delta}\left|u(\delta) - v(\delta)\right|,$$

where $r_i\left(s\right) = s - \zeta_i\left(s\right)$. Taking into consideration (2.34) ,and $(H_2.(i),(iii))$, we get:

$$|Fu\,(t)-Fv\,(t)| \leq \left[\int_0^t L_1\,(s)\,\frac{(t-s)^{\alpha-1}}{\Gamma\,(\alpha)}e^{\lambda b(s)}ds\right.$$

$$+\sum_{i\in\{1,\ldots,N:h_i\leq t\}}\int_{h_i}^t L_2\,(s)\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{\lambda r_i(s)}ds$$

$$\left.+\sum_{t_k<t} l_k.e^{\lambda t}\right] P_{j(K)}\,(u-v)$$

Now, we multiply the both sides of the above inequality by $e^{-\lambda t}$, we get:

$$e^{-\lambda t}\,|Fu\,(t)-Fv\,(t)| \leq \left[\int_0^t L_1\,(s)\,\frac{(t-s)^{\alpha-1}}{\Gamma\,(\alpha)}e^{-\lambda(t-b(s))}ds\right.$$

$$+\sum_{i\in\{1,\ldots,N:h_i\leq t\}}\int_{h_i}^t L_2\,(s)\,\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda(t-r_i(s))}ds$$

$$\left.+\sum_{t_k<t} l_k\right] P_{j(K)}\,(u-v)$$

We change the variable $x=\lambda\,(t-s)$, lead to:

$$e^{-\lambda t}\,|Fu\,(t)-Fv\,(t)| \leq$$

$$\left[\tfrac{1}{\lambda^\alpha\Gamma(\alpha)}\int_0^{\lambda t} L_1(t-\frac{x}{\lambda})\,x^{\alpha-1}e^{-x}e^{-\lambda((t-\frac{x}{\lambda})-b(t-\frac{x}{\lambda}))}dx\;+\right.$$

$$\tfrac{1}{\lambda^\alpha\Gamma(\alpha)}\sum_{i\in\{1,\ldots,N:h_i\leq t\}}\int_0^{\lambda(t-t_i)} L_2(t-\frac{x}{\lambda})\,x^{\alpha-1}e^{-x}e^{-\lambda\zeta_i\left(t-\frac{x}{\lambda}\right)}dx$$

$$\left.+\sum_{t_k<t} l_k\;\right] P_{j(K)}\,(u-v)$$

Accord to ($H_4.(ii)$), Hölder's inequality ,and ($H_2.(i)$), we give:

$$
\begin{aligned}
e^{-\lambda t}\left|Fu\left(t\right)-Fv\left(t\right)\right| \;&\leq \left[\frac{1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\left(\int_{0}^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\right.\\
&\left(\int_{0}^{\lambda t}L_{1}^{\nu}(t-\frac{x}{\lambda})e^{-\nu\lambda((t-\frac{x}{\lambda})-b(t-\frac{x}{\lambda}))}dx\right)^{\frac{1}{\nu}}\\
&+\frac{1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\sum_{i\in\{1,\ldots,N:h_{i}\leq t\}}\left(\int_{0}^{\lambda t}x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\\
&\left.\left(\int_{0}^{\lambda t}L_{2}^{\nu}(t-\frac{x}{\lambda})e^{-\nu\lambda\zeta_{i}\left(t-\frac{x}{\lambda}\right)}dx\right)^{\frac{1}{\nu}}+\sum_{k\geq 1}l_{k}\right]P_{j(K)}\left(u-v\right)\\
&\leq\left[\frac{\Gamma(\alpha^{2})^{\frac{1}{\mu}}}{\Gamma(\alpha)\,\lambda^{\frac{\alpha^{2}}{\alpha+1}}}\left(R_{\lambda}^{\frac{1}{\nu}}+N\,e^{\lambda\zeta}\,S_{\lambda}^{\frac{1}{\nu}}\right)+\sum_{k\geq 1}l_{k}\right]P_{j(K)}\left(u-v\right)
\end{aligned}
$$

In the last, by the application of supremum function on K, this yields immediately to (2.20) and the proof is finish. $\square$

The following theorem provides sufficient conditions for the existence, uniqueness result of (2.8)-(2.10).

Theorem 2.4. *Let the conditions* $(H_1)-(H_7)$ *and condition (2.19) hold true. If the following relation is satisfied:*

$$
\max\{\zeta,\,\bar{b}\}^{\frac{\alpha^{2}}{\alpha+1}}\,\frac{\Gamma(\alpha^{2})^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)}\,\left(R_{1/\max\{\zeta,\bar{b}\}}^{\frac{\alpha}{\alpha+1}}+N\,e\,S_{1/\max\{\zeta,\bar{b}\}}^{\frac{\alpha}{\alpha+1}}\right)+\sum_{k\in\mathbb{N}^{*}}l_{k}<\frac{1}{4}, \tag{2.21}
$$

then the problem (2.8)-(2.10) has a unique global solution in $\mathbf{X}_{\Psi,M}$.

Proof. Recall that, in view of Lemma 2.1, the solutions of (2.8)-(2.10) are the fixed points of the operator F. Hence it is sufficient to show that all conditions of Theorem 2.2 are fulfilled.

For any u,v in $\mathbb{X}$ (with $u\neq v$), let $\beta\in\,]0,1[$, then $w_{\beta}=(1-\beta)u+\beta v$ is such that $u\neq w\neq v$. A direct arithmetic results in (2.2) and Therefore E be a convex with respect to each semi-norm.

Note first that the series (**??**) in our case turns into

$$
\sum_{n=0}^{\infty}2^{2n}C_{\lambda}^{n+1} \tag{2.22}
$$

Let us now choose $\lambda = 1/\max\{\zeta, \overline{b}\}$ in (2.12). By the use of Proposition 2.1, condition (2.21), get (2.20) is true with $C_\lambda < 1/4$.

Consequently F is i-contractive ,and the series (2.22) converges.

Let now $u \in \partial \mathbf{X}_{\Psi,M}$, then for $\overline{a} \leq t \leq \overline{b}$, (H_1) and (H_6) give:

$$\begin{aligned} Fu(t) &= \Psi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, M, u(s-\zeta_1(s)), \ldots, u(s-\zeta_N(s))\right) ds \\ &\leq \Psi_0 + \frac{\alpha(M-\Psi_0)}{\overline{b}^\alpha} \int_0^t (t-s)^{\alpha-1} ds \leq M, \end{aligned}$$

which means that $F(\partial \mathbf{X}_{\Psi,M}) \subset \mathbf{X}_{\Psi,M}$. The proof is finish. $\square$

Remark 2.3.

(i) We emphasize here that the operator F is non-self- map under the assumptions of Theorem 2.4 and Lemma 2.3. This means that classical variants of fixed point theories often used in the literature can't be directly applied.

(ii) In addition to the existence, uniqueness result of the solutions of (2.8)-(2.10) *theorem 2.4 (with appropriate adaptation) also provides [25, Theorem 1] given by the following corollary 2.1 of this given. The generalization is due to the assumption that* (H_1) *allows taking the supremum of the required necessity over evry long time intervals. The first aspect of improvement comes from* (H_4) *(see remark 2.2), the second comes from the condition* (2.25)*, which is more restrictive than (21) in less [25]. In the special case where* L_1 *and* L_2 *in* (H_4) *are constants, the value of* α *close to* 1.

Corollary 2.1. *By the use of condition* $(H_1.(i)), (H_2.(i), (iii)), (H_4), (H_6), (H_7)$ *and (2.19), the problem*

$$^{C}D_{0^+}^{\alpha} u(t) = f\left(t, \max_{\sigma \in [a(t),\, b(t)]} u(\sigma), u(t-\zeta_1(t)), \ldots, u(t-\zeta_N(t))\right), \qquad t > 0 \tag{2.23}$$

$$u(t) = \Psi(t), \quad t \leq 0, \tag{2.24}$$

has a unique continuous solution in $\mathbf{A}_{\Psi,M}$ *provided that*

$$\max\{\zeta, \overline{b}\}^{\frac{\alpha^2}{\alpha+1}} \frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)} \left(R_{1/\max\{\zeta, \overline{b}\}}^{\frac{\alpha}{\alpha+1}} + N\, e\, S_{1/\max\{\zeta, \overline{b}\}}^{\frac{\alpha}{\alpha+1}} \right) < \frac{1}{4}. \tag{2.25}$$

Where

$$\mathbf{A}_{\Psi,M} = \left\{u \in \mathcal{C}(\mathbb{R}) : u(t) = \Psi(t) \ \textit{for } t \leq 0, \ \ \textit{and } u(t) \leq M \ \textit{for } t \in [\overline{a},\, \overline{b}]\right\}$$

Proof. By omitting the impulsive condition (2.9), i.e. if for each $k \in \mathbb{N}^* :\ I_k = 0$, the problem (2.8)-(2.10) reduced to (2.23)-(2.24). Furthermore, the assumptions of theorem 2.4 boil down to the assumptions of corollary 2.1.
Note also that in this particular case the condition (2.9) with $I_k = 0$, means:

$$\Delta u(t_k) := u(t_k^+) - u(t_k^-) = I_k\left(u\left(t_k - \rho_k(t_k)\right)\right) = 0.$$

This means that the space E defined in (2.11) becomes $\mathcal{C}(\mathbb{R})$ and thus subset $\mathbf{X}_{\Psi,M}$ will be $\mathbf{A}_{\Psi,M}$.
Since the condition (2.21) includes as a special case the condition (2.25) where $l_k = 0$ ($k \in \mathbb{N}^*$), the result follows Theorem 2.4 immediately. $\square$

2.4.1 Examples

We end with two examples illustrate our results including the improvements explain in remark 2.3.

Example 2.1. *We consider the following problem*

$$\begin{aligned} {}^{C}D^{0.5}u(t) = & \frac{4.25\left(\frac{t}{3}+1\right)^{\frac{1}{3}}}{3\times10^{-1}+2\times10^{-3}\left|\sup\limits_{\sigma\in[a(t),\,b(t)]} u(\sigma)\right|} \\ & + \frac{5.43\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}}{1+3\times10^{-4}\left|u\left(3\times10^{-4}-\frac{3\times10^{-4}+12\times10^{-8}}{1+t}\right)\right|},\ t>0,\ t\neq10^k,\ k\in\mathbb{N}^* \end{aligned} \tag{2.26}$$

$$\Delta u(10^k) = \frac{1}{4ek(k+1)\left(1+\left|u\left(-\frac{1}{5k}-\frac{3\times10^k}{4+4\times10^k}\right)\right|\right)}, \qquad k \in \mathbb{N}^* \tag{2.27}$$

$$u(t) = t + 1, \qquad t \leq 0 \tag{2.28}$$

where

$$a(t) = \begin{cases} 10^{-1} : & 0\leq t\leq\frac{1}{2} \\ 2\times10^{-1}t : & t\geq\frac{1}{2} \end{cases} \quad \textit{and} \quad b(t) = \begin{cases} 1/2 : & 0\leq t\leq\frac{1}{2} \\ (2t+1)/4 : & t\geq\frac{1}{2} \end{cases}$$

The problem (2.26)-(2.28) identify to (2.8)-(2.10) with:

$$\begin{aligned} &\alpha = 0.5, \quad N = 1, \quad \zeta_1(t) = t - 3 \times 10^{-4} + \tfrac{3\times10^{-4}\left(1+4\times10^{-4}\right)t}{(1+t)}, \\ &f(t,x,y) = \tfrac{4.25\left(\frac{t}{3}+1\right)^{\frac{1}{3}}}{3\times10^{-1}+2\times10^{-3}|x|} + \tfrac{5.43\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}}{1+3\times10^{-4}|y|}, \quad \Psi(t) = t+1 \end{aligned} \tag{2.29}$$

and

$$\{t_k\}_{k\in\mathbb{N}^*} = \{10^k\}_{k\in\mathbb{N}^*}, \quad I_k(x) = \frac{1}{4ek(k+1)\ (1+|x|)}, \quad \rho_k(t) = t + \frac{1}{5k} + \frac{3t}{4+4t}$$

It is easy to observe that assumptions $(H_1) - (H_3)$ *are verified with*

$$\overline{a} = 10^{-1}, \ \overline{b} = \frac{1}{2}, \ \zeta = 3\times10^{-4} \ \textit{and} \ h_1 = 4\times10^{-4}$$

Now, we verify (H_4).

$$\begin{aligned} |f(t,\xi,x) - f(t,\eta,y)| \leq \ & \frac{2\times4.25\left(\frac{t}{3}+1\right)^{\frac{1}{3}}\times10^{-3}\left(|\eta|-|\xi|\right)}{\left(3\times10^{-1}+2\times10^{-3}|\xi|\right)\left(3\times10^{-1}+2\times10^{-3}|\eta|\right)} \\ & + \frac{3\times5.43\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}\times10^{-4}\left(|y|-|x|\right)}{\left(1+3\times10^{-4}|x|\right)\left(1+3\times10^{-4}|y|\right)} \\ & \leq 9.44\times10^{-2}\left(\tfrac{t}{3}+1\right)^{\frac{1}{3}}|\xi-\eta| \\ & +16.29\times10^{-4}\left(\tfrac{2t}{15}+\tfrac{1}{3}\right)^{\frac{1}{3}}|x-y| \end{aligned}$$

So, $(H_4.(i))$ *is satisfies with*

$$L_1(t) = 9.44\times10^{-2}\left(\frac{t}{3}+1\right)^{\frac{1}{3}} \quad \textit{and} \quad L_2(t) = 16.29\times10^{-4}\left(\frac{2t}{15}+\frac{1}{3}\right)^{\frac{1}{3}}$$

The last computations leader to

$$R_\lambda = 8.42421124\times10^{-4}\left(\tfrac{63\lambda+18}{162\lambda^2}+\tfrac{9\lambda+1}{27\lambda^2}e^{\frac{3\lambda}{2}}\right) \quad \textit{and} \quad S_\lambda = 4.32278119\times10^{-9}\left(\tfrac{15\lambda+2}{135\lambda^2}\right)$$

Which means that $(H_4.(ii))$ *is verified also. After that,* $\forall k \in \mathbb{N}^*$ *we have:*

$$
\begin{aligned}
|I_k(x) - I_k(y)| &\leq \frac{1}{4ek(k+1)} \left| \frac{1}{1+|x|} - \frac{1}{1+|y|} \right| \\
&\leq \frac{1}{4ek(k+1)} \frac{|x-y|}{(1+|x|)(1+|y|)} \\
&\leq \frac{1}{4ek(k+1)} |x-y|
\end{aligned}
$$

So (H_5) *is satisfies with* $l_k = \dfrac{1}{4ek(k+1)}$.

Also, $\exists M = 15.2 > \Psi_0 = 1$*, such* $\forall (t,x) \in [0, \frac{1}{2}] \times \mathbb{R}$*, we have*

$$
f(t, M, x) \leq \frac{4.25 \left(\frac{7}{6}\right)^{\frac{1}{3}}}{3 \times 10^{-1} + 2 \times 10^{-3} \times M} + 5.43 \left(\tfrac{2}{5}\right)^{\frac{1}{3}} \simeq 17.5422869253
$$

and

$$
\alpha.\Gamma(\alpha).\frac{M - \Psi_0}{\bar{b}^{\alpha}} = 0,5.\sqrt{\pi}\frac{15.2-1}{0.5^{0.5}} \simeq 17.7970607499
$$

Then (H_6) *is true.*
For the checking of (H_7)*, we choose* $\iota = \Psi_0 + 10^{-9}$*, so we have* $\forall (t,x) \in [0, \frac{1}{2}] \times \left[\Psi_0, \Psi_0 + \frac{\iota - \Psi_0}{\bar{b}}\zeta\right]$*:*

$$
\begin{aligned}
f(t, \iota, x) &> \frac{4.25}{3 \times 10^{-1} + 2 \times 10^{-3} \times \iota} \\
&+ \frac{5.43 \left(\frac{1}{3}\right)^{\frac{1}{3}}}{1 + 3 \times 10^{-4} \left(\Psi_0 + \frac{\iota - \Psi_0}{\bar{b}}\zeta\right)} \simeq 17.8377994017
\end{aligned}
$$

and

$$
\alpha.\Gamma(\alpha)\frac{M - \Psi_0}{|\bar{b} - h_1|^{\alpha}} = 0,5.\sqrt{\pi}.\frac{15.2-1}{(0,4996)^{\frac{1}{2}}} \simeq 17.8041838483
$$

Since moreover f *is non negative the hypothesis* (H_7) *is true.*

Therefore, (2.19) is clearly verified and

$$
\max\{\zeta, \bar{b}\}^{\frac{\alpha^2}{\alpha+1}} \frac{\Gamma(\alpha^2)^{\frac{1}{1+\alpha}}}{\Gamma(\alpha)} \left(R_{1/\max\{\zeta,\bar{b}\}}^{\frac{\alpha}{\alpha+1}} + NeS_{1/\max\{\zeta,\bar{b}\}}^{\frac{\alpha}{\alpha+1}} \right) + \sum_{k \in \mathbb{N}^*} l_k \simeq 0.0928213
$$

Then,all conditions of Theorem 2.4 are fulfilled, and in consequent (2.26)-(2.28) admits a unique solution in $\mathbf{X}_{\Psi,M}$*, where*

$$\mathbf{X}_{\Psi,M}=\Big\{u\in X:u\left(t\right)=t+1 \quad for \quad t\leq 0 \quad and$$
$$u\left(t\right)\leq 15.2 \quad for\ \ t\in[10^{-1},\,1/2]\Big\}$$

Denote f increases indefinitely with respect to t as it is pointed out in remark 2.2.

Example 2.2. *Let $(\mathcal{P})$ is a special form of (2.23)-(2.24) to correspond: α, N, ζ_1, f, and Ψ given by (2.29) with*

$$a(t)=10^{-1} \qquad and \qquad b(t)=\begin{cases}\frac{1}{2} & 0\leq t\leq\frac{1}{2}\\ \\ 0.2-\frac{1}{0.999\,t} & t\geq\frac{1}{2}\end{cases} \tag{2.30}$$

By using example 2.1, we conclude that all assumptions of corollary 2.1 are verified and then the problem on question has a unique continuous solution in $\mathbf{A}_{\Psi,M}$, where

$$\mathbf{A}_{\Psi,M}=\Big\{u\in\mathcal{C}\left(\mathbb{R}\right):u\left(t\right)=t+1 \quad for \quad t\leq 0 \quad and$$
$$u\left(t\right)\leq 15.2 \quad for\ \ t\in\left[10^{-1},\,1/2\right]\Big\}$$

We see that $\overline{b}=\frac{1}{2}$ is the boundary of $b(t)$ in (2.30), that is [25, Theorem 1] covers $(\mathcal{P})$, but it doesn't applicable, because f doesn't satisfies the Lipschitz condition with constant arguments.

2.5 Application 2

Now, we will study the existence, uniqueness and stability from the order of problem more general from the previous work, witch been initial value problems for fractional order differential equation; Where we replace the term memory $t-\zeta_i(t)$ by $g_i(t)$ such that the function g is non-bounded. And defined by the following equation:

$${}^{C}D^{\alpha}u\left(t\right)=f\left(t,\max_{\sigma\in[a(t),\,b(t)]}u\left(\sigma\right),u\left(g_1\left(t\right)\right),...,u\left(g_N\left(t\right)\right)\right), \qquad t>0, \tag{2.31}$$

with the initial condition function

$$u\left(t\right)=\phi\left(t\right), \quad t\leq 0, \tag{2.32}$$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative operator of order $\alpha \in]0,1[$ defined by (**??**), N is a positive integer, a, b and g_i (with $0 \leq i \leq N$) are real continuous functions defined on $\mathbb{R}_+ = [0, +\infty[$ subject to conditions which will be specified later, $\phi :]-\infty, 0] \longrightarrow \mathbb{R}$ is a continuous function such that $\phi(0) = \phi_0 > 0$, and $f : \mathbb{R}_+ \times \mathbb{R}^{1+N} \longrightarrow \mathbb{R}$ is a nonlinear continuous function.

Analytic study

we applied the Riemann-Liouville fractional integral operator I^{α} of order α to both and sides of (2.31) and using its properties together with the initial condition (2.32), we get the following lemma.

Where f, a, b and g_i (with $0 \leq i \leq N$) are continuous.

Lemma 2.4. *Let $0 < \alpha < 1$ and $f : \mathbb{R}_+ \times \mathbb{R}^{1+N} \longrightarrow \mathbb{R}$ be continuous. A function u is a solution of the fractional integral equation*

$$u(t) = \begin{cases} \phi_0 + \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\Big(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(g_1(s)), \ldots, u(g_N(s))\Big) ds & \text{if } t > 0, \\ \phi(t) & \text{if } t \leq 0 \end{cases} \tag{2.33}$$

Where $k \geq 1$ if and only if u is a solution of the fractional IVP (2.31)-(2.32)

Let $E = C(\mathbb{R})$ provided with the family of semi norms $\mathfrak{P} = \{p_k; k \in \mathfrak{K}\}$ such that $p_k(g) = \sum\limits_{t\in k} e^{-\lambda t} \mid g(t) \mid, \forall k \in \mathfrak{K}, \forall g \in E$ with $\mathfrak{K} = \{$ the set of all the compact in $\mathbb{R}\}$. And let $X \subseteq E$ such that:

$$X = \left\{u \in E/u(t) = \phi(t) \text{ for } t \leq 0, \text{ and } u(t) \leq M \text{ for } t \in [\overline{a}, \overline{b}]\right\}$$

So:

$$\partial X = \left\{u \in E/u(t) = \phi(t) \text{ for } t \leq 0, \text{ and } \max_{t\in[\overline{a},\overline{b}]} u(t) = M\right\}$$

we define the application ii by:

$$i(k) = \begin{cases} k & \text{if} \quad k_+ = \emptyset \\ \\ \left[0, \max\{k_m + e^{-H(k_m)}, \overline{b}\}\right] & \text{if} \quad k_+ \neq \emptyset \end{cases} \tag{2.34}$$

with $k_+ = k \cap [0, +\infty[$ and $k_m = \max\limits_{t \in k} t$.
We supposed the following conditions:

(H_1) $\forall t \geq 0$: $0 \leq \overline{a} \leq a(t) \leq b(t)$,with $\overline{a} = \inf a(t); \exists \overline{a} < \overline{b}$ such that: $a(t) = \overline{a}$; $b(t) = \overline{b}$,$\forall t \in \left[0, \overline{b}\right]$ and $b(t) \leq t$ for $t > \overline{b}$.

(H_2) $\exists H \in C([1, +\infty[)$ such that:

* the function H is creasing.
* for all $x, y \in [1, +\infty[$, we have $H(x+y) \leq H(x) + H(y)$.
* $\forall x \in [1, +\infty[, H(H(x)) \geq H^2(x)$.

(H_3) For all $t \geq 0$, there exists $H \in C([1, +\infty[)$ such that $\forall i = \overline{1, N}$

$$t \leq g_i(t) \leq t + e^{-H(t)}$$

Where H verified $((H_3))$.

(H_4) There exist two real constants L_1 and L_2, satisfying :

(i) $\left|f\left(t, \xi, x_1, ..., x_N\right) - f\left(t, \eta, y_1, ..., y_N\right)\right| \leq L_1 \left|\xi - \eta\right| + L_2 \sum\limits_{i=1}^{N} \left|x_i - y_i\right|$, whenever the left hand side is defined.

(ii) For $\lambda > 0$ and $\nu := 1 + 1/\alpha$, we have

$$R_{\alpha,\lambda} := \int_0^{+\infty} e^{-\lambda\nu(s-b(s))} ds < \infty,$$

$$S_{\alpha,\lambda} := \int_0^{+\infty} e^{\lambda\nu e^{-H(s)}} ds < \infty,$$

(H_5) There exists a positive constant $M > \phi_0$ such that

$$\frac{1}{\Gamma\left(\alpha\right)\alpha} f\left(t, M, x_1, ..., x_N\right) \leq \frac{M - \phi_0}{\overline{b}^{\alpha}}, \ \ \forall\left(t, x_1, ..., x_N\right) \in \left[0\,, b^*\right] \times \mathbb{R}^N$$

(H_6) f is a non negative function and moreover, $\exists \iota \in]\phi_0, M[$ such that

$$\frac{1}{\Gamma\left(\alpha\right)\alpha} f\left(t, \iota, x_1, ..., x_N\right) > \frac{M - \phi_0}{|\overline{b}}|^{\alpha}, \ \forall\left(t, x_1, ..., x_N\right) \in [0\,,\overline{b}] \times \left(]\phi_0, \phi_0 + \tfrac{\iota - \phi_0}{\overline{b}}(\overline{b} + c)[\right)^N$$

Lemma 2.5. *The set E is $i-$ bounded.*

Proof. When $\overline{b} \leq k_m + e^{-H(k_m)}$ so by the hypotheses (H_2) we have :

$$i(K) = [0, k_m + e^{-H(k_m)}]$$

And

$$\begin{aligned} i^2(K) &= \left[0, k_m + e^{-H(k_m)} + e^{-H(k_m + e^{-H(k_m)})}\right] \\ &\subset \left[0, k_m + e^{-H(k_m)} + e^{-H(k_m)} * e^{-H(-H(k_m))}\right] \\ &\subset \left[0, k_m + e^{-H(k_m)} + e^{-H^2(k_m)}\right] \end{aligned}$$

Then:

$$\begin{aligned} i^3(K) &\subset \left[0, k_m + e^{-H(k_m)} + e^{-H^2(k_m)} + e^{k_m + e^{-H(k_m)} + e^{-H^2(k_m)}}\right] \\ &\subset \left[0, k_m + e^{-H(k_m)} + e^{-H^2(k_m)} + e^{-H(k_m)} * e^{-H^2(k_m)} * e^{-H^3(k_m)}\right] \\ &\subset \left[0, k_m + e^{-H(k_m)} + e^{-H^2(k_m)} + e^{-H^3(k_m)}\right] \end{aligned}$$

by the same means we get:

$$i^n(K) \subset \left[0, k_m + \sum_{i=1}^{n} e^{-H^i(k_m)}\right]$$

We know that $\sum_{i=1}^{n} e^{-H^i(k_m)}$ is convergent.

indeed: on a

$$\begin{aligned} \lim_{n\to+\infty} \frac{e^{-H^{n+1}(k_m)}}{e^{-H^n(k_m)}} &= \lim_{n\to+\infty} e^{-H^{n+1}(k_m) + H^n(k_m)} \\ &= \lim_{n\to+\infty} e^{-H^n(k_m)(H(k_m)-1)} \\ &= 0 < 1 \end{aligned}$$

Under Dalampher ...., we have $\sum_{i=1}^{+\infty} e^{-H^j(k_m)} = S'$.

consequently, $\forall n \in N : i^n \subset [0, k_m + S']$. $\square$

Lemma 2.6. *F maps X into E and moreover, it is non-self mapping (under hypotheses (H_1) and(H_6).*

Proof. firstly, we know that functions f, a and b are continuous.

We have if $u \in X$, so $\max_{\sigma\in[a(s),b(s)]} u(\sigma) \leq M$. And then; $Fu \in E$.

Now, we suppose that (H_1) and (H_6) are verified.

And,let $u \in X$ defined by means:

for $t \in [0, \overline{b}]$:

$$u(t) = \phi_0 + \frac{(\iota - \phi_0)t}{\overline{b}}$$

where ι given in (H_6).

Under hypothesis (H_1); it is clear that:

$$\text{for } 0 \leq s \leq \overline{b} : \max_{\sigma \in [a(s), b(s)]} u(\sigma) = \iota$$

And; for every $i = \overline{1, m}$:

$$\phi_0 < u(g_i(s)) < \phi_0 + \frac{h - \phi_0}{\overline{b}}(\overline{b} + c)$$

such that $c = \max_{s \in [0, \overline{b}]} e^{-H(t)}$.

since: $0 < g_i(s) < \overline{b} + c$.

Then; When hypotheses (H_1) and(H_6) verified, so:

$$\begin{aligned}
Fu\left(\overline{b}\right) &= \phi_0 + \int_0^{\overline{b}} \frac{\left(\overline{b} - s\right)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, \iota, u\left(g_1(s)\right), ..., u\left(g_N(s)\right)\right) ds \\
&> \phi_0 + \alpha \frac{M - \phi_0}{\overline{b}^{\alpha}} \int_0^{\overline{b}} \left(\overline{b} - s\right)^{\alpha-1} ds \\
&= \phi_0 + \frac{M - \phi_0}{\overline{b}^{\alpha}} \overline{b}^{\alpha} = M
\end{aligned}$$

□

Now, we proved that F is contraction function. Let $u, v \in C(\mathbb{R})$.Then:

$$
\begin{aligned}
|Fu(t)-Fv(t)| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f\left(s, \max_{\sigma\in[a(s),b(s)]} u(\sigma), u(s-g_1(s)),\dots,u(g_N(s))\right) - f\left(s, \max_{\sigma\in[a(s),b(s)]} v(\sigma), v(g_1(s)),\dots,v(g_N(s))\right)\right. \\
&\le L_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|\max_{\sigma\in[a(s),b(s)]} u(\sigma) - \max_{\sigma\in[a(s),b(s)]} v(\sigma)\right| ds + L_2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=1}^{N} |u(g_i(s)) \\
&\le L_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \max_{\sigma\in[a(s),b(s)]} |u(\sigma)-v(\sigma)| ds + L_2 \sum_{i=1}^{N} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(g_i(s)) - v(g_i(s \\
&\le \int_0^t L_1(s) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} \max_{\sigma\in[a(s),b(s)]} e^{-\lambda\sigma} |u(\sigma)-v(\sigma)| ds \\
&\quad + L_2 \sum_{i=1}^{N} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda g_i(s)} . e^{-\lambda g_i(s)} |u(g_i(s)) - v(g_i(s))| ds
\end{aligned}
$$

We have $g_i(s) \le s + e^{H(s)}$ (seeing (H_3)) so it is clear that (by the definition of function j): $\forall k \in \mathfrak{K}$ with $K_+ \neq \emptyset$, we have $\left[0, \max\{K_m, \bar{b}\}\right] \subset j(K)$; And $g_i(s) \in j(K)$.

Hence:

$$
|Fu(t)-Fv(t)| \le \left[L_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda b(s)} ds + L_2 \sum_{i=1}^{N} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda g_i(s)} ds \right] P_{j(K)}(u-v)
$$

We multiply the both side by $e^{-\lambda t}$:

$$
e^{-\lambda t}|Fu(t)-Fv(t)| \le \left[L_1 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t-b(s))} ds + L_2 \sum_{i=1}^{N} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda(t-g_i(s))} ds \right] P_{j(K)}(u-v)
$$

We change $x = \lambda(t-s)$ so:

$$
\begin{aligned}
e^{-\lambda t}|Fu(t)-Fv(t)| &\le \left[\frac{L_1}{\lambda^\alpha \Gamma(\alpha)} \int_0^{\lambda t} x^{\alpha-1} e^{-\lambda\left(t-b\left(t-\frac{x}{\lambda}\right)\right)} dx + \frac{L_2}{\lambda^\alpha \Gamma(\alpha)} \sum_{i=1}^{N} \int_0^{\lambda t} x^{\alpha-1} e^{-\lambda\left(t-g_i\left(t-\frac{x}{\lambda}\right)\right)} dx \right] P_{j(K} \\
&\le \left[\frac{L_1}{\lambda^\alpha \Gamma(\alpha)} \int_0^{\lambda t} x^{\alpha-1} e^{-x} e^{-\lambda\left(\left(t-\frac{x}{\lambda}\right)-b\left(-\frac{x}{\lambda}\right)\right)} dx \right. \\
&\quad \left. + \frac{L_2}{\lambda^\alpha \Gamma(\alpha)} \sum_{i=1}^{N} \int_0^{\lambda t} x^{\alpha-1} e^{-x} e^{-\lambda\left(\left(t-\frac{x}{\lambda}\right)-g_i\left(t-\frac{x}{\lambda}\right)\right)} dx \right] P_{j(K)}(u-v)
\end{aligned}
$$

Let $\mu = 1 + \alpha$ and $\nu = 1 + \frac{1}{\alpha}$, so when we use Hölder's inequality, we obtain:

$$
\begin{aligned}
e^{-\lambda t}\left|Fu\left(t\right) - Fv\left(t\right)\right| &\leq \left[\frac{L_1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\left(\int_0^{\lambda t} x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\left(\int_0^{\lambda t} e^{-\lambda\nu\left(\left(t-\frac{x}{\lambda}\right)-b\left(t-\frac{x}{\lambda}\right)\right)}dx\right)^{\frac{1}{\nu}}\right.\\
&\left.+\frac{L_2}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\sum_{i=1}^{N}\left(\int_0^{\lambda t} x^{\mu(\alpha-1)}e^{-\mu x}dx\right)^{\frac{1}{\mu}}\left(e^{-\lambda\nu\left(\left(t-\frac{x}{\lambda}\right)-g_i\left(t-\frac{x}{\lambda}\right)\right)}dx\right)^{\frac{1}{\nu}}\right]P_{j(K)}\left(u-v\right)\\
&\leq \left[\frac{\Gamma(\alpha^2)^{\frac{1}{\mu}}.L_1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\left(\lambda\int_0^{t} e^{-\lambda\nu(s-b(s))}ds\right)^{\frac{1}{\nu}} + \frac{\Gamma(\alpha^2)^{\frac{1}{\mu}}.L_2}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\sum_{i=1}^{N}\left(\lambda\int_0^{t} e^{-\lambda\nu(s-g_i(s))}ds\right)^{\frac{1}{\nu}}\right]P_j\\
&\leq \left[\frac{\Gamma(\alpha^2)^{\frac{1}{\mu}}.L_1}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\left(\lambda\int_0^{t} e^{-\lambda\nu(s-b(s))}ds\right)^{\frac{1}{\nu}} + \frac{\Gamma(\alpha^2)^{\frac{1}{\mu}}.L_2}{\lambda^{\alpha}\Gamma\left(\alpha\right)}\sum_{i=1}^{N}\left(\lambda\int_0^{t} e^{\lambda\nu e^{-H(s)}}ds\right)^{\frac{1}{\nu}}\right]P_{j(K)}\\
&\leq \left[\frac{\Gamma(\alpha^2)^{\frac{1}{\mu}}}{\lambda^{\frac{\alpha^2}{\alpha+1}}\Gamma\left(\alpha\right)}\left(L_1R_{\alpha,\lambda}^{\frac{1}{\nu}} + NL_2S_{\alpha,\lambda}^{\frac{1}{\nu}}\right)\right]P_{j(K)}\left(u-v\right)
\end{aligned}
$$

Let $u \in \partial\mathbf{E}_{\phi,M}$, so $\max\limits_{t\in[\overline{a},\overline{b}]} u\left(t\right) = M$ and in view of hypothesis (H_1),
for $0 \leq t \leq \overline{b}$ we have $\max\limits_{\sigma\in[a(t),\,b(t)]} u\left(\sigma\right) = \max\limits_{t\in[\overline{a},\overline{b}]} u\left(t\right)$.

Then for $\overline{a} \leq t \leq \overline{b}$, hypothesis (H_5) gives:

$$
\begin{aligned}
Fu\left(t\right) &= \phi_0 + \frac{1}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1} f\left(s, M, u\left(g_1\left(s\right)\right), ..., u\left(g_N\left(s\right)\right)\right)ds\\
&\leq \phi_0 + \frac{\alpha(M-\phi_0)}{\overline{b}^{\alpha}}\int_0^t \left(t-s\right)^{\alpha-1}ds \leq M,
\end{aligned}
$$

The study by order α of stability of last equation : Let

$$
F_{\alpha}(U)(t) = \begin{cases} \phi_0 + \displaystyle\int_0^t \frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)} f\left(s, \max\limits_{\sigma\in[a(s),b(s)]} u(\sigma), u(g_1(s)), ..., u(g_N(s))\right)ds & \textbf{if } t > 0,\\ \phi(t) & \textbf{if } t \leq 0 \end{cases} \tag{2.35}
$$

Definition 2.4. *We call that an equation is stable if its function equivalent F verified :*

$$
\forall\varepsilon > 0, \exists\delta > 0 : |\alpha_1 - \alpha_2| < \delta \Rightarrow |F_{\alpha_1}(U)(t) - F_{\alpha_2}(U)(t)| < \varepsilon \tag{2.36}
$$

(H_7) **for all** $t > 0$**, we have**

$$\frac{1}{\alpha^2\Gamma(\alpha)} f\left(t, \max_{\sigma\in[a(t),\, b(t)]} u(\sigma), x_1, ..., x_N\right) \leq \frac{M}{t^{\alpha}}, \ \ \forall (x_1, ..., x_N) \in \mathbb{R}^N$$

such that $M = \max_{\sigma\in[a(t),\, b(t)]} u(\sigma)$

Chapter 3

Fixed Point in Complete Ordered Semi-Normed Space

In the third chapter, we extend the fixed point theory of monotone functions in an ordered Banach space to complete semi-normed spaces. We generalise some properties related to the notion of order, and we weaken some conditions on the functions, in particular the compactness. Finally, we give an example of application of our results.

For the rest of this work, we assume that the semi-normed real vector space $(X, \{\rho_\alpha\}_{\alpha\in I})$ is endowed with the normal order $\leq$ defined by K, and we say that $(X, \{\rho_\alpha\}_{\alpha\in I}, \leqslant)$ is an ordered space. It is important to note here, that the relation $\leq$ between elements of Y refers to the order defined by K in X, while its use between real numbers means the natural order in $\mathbb{R}$.

Definition 3.1 (see [55]). *Let $(X, \{\rho_\alpha\}_{\alpha\in I})$ be hlctvs provided with semi normed structure, $y \in X$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X. So:*

* *We call that (y_n) converge to y if for all $\xi \in \mathbb{R}_+$ with $\xi \gg 0$:*

$$\exists n_0 \in \mathbb{N} \text{ such that } \forall \alpha \in I, \forall n \geq n_0, \rho_\alpha(y_n - y) \ll \xi$$

and we denote $\lim\limits_{n\longrightarrow+\infty} \rho_\alpha(y_n - y) = 0$for all $\alpha \in I$.

* *We call that (y_n) be Cauchy sequence if for all $\xi \in \mathbb{R}_+$ with $\xi \gg 0$:*

$$\exists n_0 \in \mathbb{N} \text{ such that } \forall \alpha \in I, \forall n, m \geq n_0, \rho_\alpha(y_n - y_m) \ll \xi$$

* *We call that $(X, \{\rho_\alpha\}_{\alpha \in I})$ is complete if all Cauchy sequence is converging.*

.

In the following proposition, we present another consequence of a normal order. Its result will be useful for obtaining our main result, it concerns monotone sequences.

Lemma 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a increasing (decreasing) sequence and weakly converge to x so it bounded above (below) by x.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a increasing sequence So $\forall m > n$, we have $x_n \leq x_m$, i.e : $x_m - x_n \in K$. K is a convex closed set in X (locale convex topology vector space), so necessary is weakly closed, see[40].
By hypothesis, we have $x_m - x_n \rightharpoonup x - x_n$, so $x - x_n \in K$; Then $x_n \leq x$.
By the same way, we can prove for $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence and $x_n \rightharpoonup x$ so $x_n \geq x$. □

Proposition 3.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a increasing (decreasing) sequence and converges weakly to x so it converge to x.*

Proof. (see [40]) Firstly, under the lemma 3.1 we have : $x_n \leq x, \forall n \geq 1$.
Then

$$\mathbf{o} \leq x - x_n \qquad\qquad \forall n \geq 1 \tag{3.1}$$

we supposed that $x_n \not\longrightarrow x$ in $\mathbb{X}$. So, $\exists \varepsilon_0 > 0$, $\exists \alpha_0 \in A$, such that: $\forall N \geq 1$, we have $n \geq N$ and $p_{\alpha_0}(x - x_n) \geq \varepsilon_0$.
Let $F_n \doteqdot \{v \in X : v \leq x_n\}$ and $F \doteqdot \bigcup\limits_{n \geq N} F_n$. it is clear that F and $\overline{F}$ are convex.
$\forall v \in F, \exists n \geq N,$ such that $v \in F_n$, so $v \leq x_n$.
And, under 3.1 we have $\mathbf{o} \leq x - x_n \leq x - v$.
K be normal order, so we obtained $\varepsilon_0 \leq p_{\alpha_0}(x - x_n) \leq N_{\alpha_0} p_{\alpha_0}(x - v)$.
So, $p_{\alpha_0}(x - v) \geq \dfrac{\varepsilon_0}{N_{\alpha_0}}, \forall v \in F$, then $x \in \overline{F}$.
Under the separated convex theory (see [40]), $\exists f \in X'$ and $\lambda \in \mathbb{R}$, such that: $f(x) < \lambda$ and $f(v) > \lambda, \forall v \in \overline{F}$. car, $x_n \in F_{n+1} \subset F, \forall n \geq N$, we have $f(x_n) > \lambda, \forall n \geq N$.
$f(x) = \lim\limits_{n \longrightarrow +\infty} f(x_n) \geq \lambda$. Which contradicts $f(x) < \lambda$. Consequently, the sequence x_n be converge to $\mathbb{X}$. □

Remark 3.1. *we can proved the same result of proposition in decreasing sequence.*

3.1 Generalisation of fixed point theories of monotonic functions

In this section, we assume that $(X, \{p_\alpha\}_{\alpha\in A}, \leq)$ is a complete ordered semi-normed space, and $F : X \longrightarrow X$ be an operator on $\mathbb{X}$. The existence of fixed points for F, returns to find solutions for the equation $u = F(u)$, and in this purpose we consider the sequence of iterations defined by $u_n = F(u_{n-1})$ for $n \geq 1$, and u_0 a starting element in $\mathbb{X}$.

The fixed point theorem of increasing operator

Before stating our first main result, recall that $u_0 \in X$ is called a sub-solution (super-solution) of the equation $u = F(u)$, if $u_0 \leq F(u_0)$ $(F(u_0) \leq u_0)$.

Theorem 3.1. *Let u_0 be a sub-solution (v_0 be an super-solution) of $u = F(u)$.*
Suppose that $F : X \longrightarrow X$ is a continuous, weakly compact and increasing operator. Then, the iterative sequence $\{u_n\}_{n\geq 1}$ ($\{v_n\}_{n\geq 1}$) converges if and only if it is bounded above (below).
When it converges its limit u^* (v^*) is the smallest fixed point of F such that $u_0 \leq u^*$ (v^* is the largest fixed point of F such that $v^* \leq v_0$).

Proof. Consider the case of a sub-solution. F being increasing, we have $u_0 \leq F(u_0) = u_1 \Rightarrow u_1 \leq u_2$, and so on by successive steps we obtain for all $n \geq 1$

$$u_0 \leq u_1 \leq u_2 \leq u_3 \leq ... \leq u_{n-1} \leq u_n$$

Thus, if u_n has a limit $u^* \in X$, so it converges weakly to u^*, Under the lemma 3.1 u_n bounded above by u^*.
Conversely, if $\{u_n\}_{n\geq 1}$ is bounded by u^*, all the elements of the sequence $u_n = F(u_{n-1})$ are in the interval $[u_0, u^*]$, and since F is weakly compact, the set $\{u_n : n \geq 1\}$ is weakly relatively compact, there exists a subsequence of $\{u_n : n \geq 1\}$ weakly converging to $u^* \in X$, by the proposition 3.1 the sequence u_n converge to u^*.
Since $u_{n+1} = F(u_n)$,when $n \longrightarrow +\infty$, we have $u^* = F(u^*)$. Suppose that $w = F(w)$ with $u_0 \leq w$.Thus $u_1 = F(u_0) \leq F(w) = w, ...$ Hence,$u^* \leq w$. □

Monotonic iteration method

Corollary 3.1. *$F : [u_0; v_0] \subset X \longrightarrow X$ continuous, weakly compact and monotonically increasing. If u_0 is a sub-solution and v_0 a super-solution of $F(u) = u$, such that $u_0 \leq v_0$, Therefore*

$\{u_n\}_{n\geq 1}$ converges to the smallest fixed point u of F in $[u_0; v_0]$, and $\{v_n\}_{n\geq 1}$ converges to the largest fixed point v of F in $[u_0; v_0]$.
Moreover, we have $u_n \leq u \leq v \leq v_n; \forall n \geq 0$.

Proof. $\{u_0 \leq F(u_0); F(v_0) \leq v_0;$ et $u_0 \leq v_0\} \Rightarrow u_0 \leq u_1 \leq v_0$, and in the same way we get $u_n \leq v_0; \forall n \geq 0$. Then, $\{u_n\}_{n\geq 1}$ is increased. So, the result is a direct consequence of the theorem. □

3.1.1 The fixed point of decreasing operator theorem

Theorem 3.2. *Suppose that :*

(i) k be normal order and the operator $F : k \longrightarrow k$ is completely continuous decreasing (i.e: continuous, compact and decreasing);

(ii) $F(\mathbf{o}) > \mathbf{o}$ and $F^2(\mathbf{o}) > \varepsilon_0 F(\mathbf{o})$, where $\varepsilon_0 > 0$;

(iii) $\forall x > \lambda F(\mathbf{o})$ $(\lambda = \lambda(x))$ and $0 < t < 1, \exists \eta = \eta(x,t) > 0$ such that $F(tx) \leq [t(1+\eta)]^{-1} Fx$

So, F has a fixed point x^* positive $x^* > \mathbf{o}$. Moreover, we can successively construct the sequence $x_n = Ax_{n-1}, n \in \mathbb{N}$, such that for all initial point $x_0 \in k$, we have

$$p_\alpha(x_n - x^*) \xrightarrow[n\longrightarrow+\infty]{} 0, \alpha \in A \tag{3.2}$$

proof. we define the recurrent sequence $u_0 = \mathbf{o}$; $u_n = Fu_{n-1}, n \in \mathbb{N}$.
Since F is decreasing operator, we can show by the recurrence that:

$$\mathbf{o} = u_0 \leq u_2 \leq ... \leq u_{2n} \leq ... \leq u_{2n-1} \leq u_3 \leq u_1 = F(\mathbf{o}). \tag{3.3}$$

$$u_{2n} = F^2(u_{2n-2}); u_{2n+1} = F^2(u_{2n-1}), n \in \mathbb{N} \tag{3.4}$$

and

$$u_{2n} = F(u_{2n-1}); u_{2n+1} = F(u_{2n}), n \in \mathbb{N} \tag{3.5}$$

Indeed, by recurrence :
for $n = 0$: we have $u_0 = \mathbf{o} \leq F(\mathbf{o}) = Fu_0 = u_1$ is true.
We suppose that 3.3 is true for n, and we proved that it is true for $n + 1$
We have $u_0 \leq u_2 \leq ... \leq u_{2n} \leq ... \leq u_{2n-1} \leq u_3 \leq u_1$ and since F is decreasing so F^2 is increasing.
Thus,

$$F^2(u_0) \leq F^2(u_2) \leq ... \leq F^2(u_{2n}) \leq ... \leq F^2(u_{2n-1}) \leq F^2(u_3) \leq F^2(u_1)$$

$$\mathbf{o} = u_0 \leq u_2 \leq u_4 \leq ... \leq u_{2n} \leq u_{2(n+1)} \leq ... \leq u_{2(n+1)-1} \leq ... \leq u_5 \leq u_3 \leq u_1 = F(\mathbf{o})$$

is true.

Since k is a normal order, the ordered interval $[\mathbf{o}; F(\mathbf{o})]$ is bounded. From (3.3), we get that $u_{2n} \in [\mathbf{o}; F(\mathbf{o})]$ for $n \in \mathbb{N}$ and since F^2 is a continuous, weakly compact and increasing operator,the sequence (u_{2n}) therefore converge to $u^* \in [\mathbf{o}; F(\mathbf{o})]$ (under the theorem 3.1) .

In the same way, we show that the sequence (u_{2n+1}) is convergent sequence ; so there exists $v^* \in [\mathbf{o}; F(\mathbf{o})] : u_{2n+1} \longrightarrow v^*$ when $n \longrightarrow +\infty$. the hypothesis (ii) and the assertion (3.3) entail that

$$\mathbf{o} < \varepsilon_0 F(\mathbf{o}) \leq F^2(\mathbf{o}) = u_2 \leq u_{2n} \leq u^* \leq v^* \leq u_{2n-1}, n \in \mathbb{N}^* \tag{3.6}$$

By crossing the limit in (3.4) and (3.5), we will have:

$$u^* = F^2(u^*); v^* = F^2(v^*); v^* = F(u^*); u^* = F(v^*) \tag{3.7}$$

According to (3.6), we see that $u^* \geq \varepsilon_0 F(\mathbf{o}) = \varepsilon_0 u_1 \geq \varepsilon_0 v^*$.

So, if we put $t_0 = \sup\{t > 0 : u^* \geq t v^*\}$, we find $0 < \varepsilon_0 \leq t_0 < \infty$ and $u^* \geq t_0 v^*$ with $t_0 \leq 1$ because $u^* \leq v^*$.

Let's show, now that $t_0 = 1$; by the absurd, we suppose $0 < t_0 < 1$; in this case the hypothesis (iii)entail that there exists $\eta_0 > 0$ such that

$$v^* = F(u^*) \leq F(t_0 v^*) \leq [t_0(1+\eta_0)]^{-1} F(v^*) = [t_0(1+\eta_0)]^{-1} u^*$$

According to (3.6) and the decease of F. so we get $u^* \geq t_0(1+\eta_0)v^*$, which contradicts the definition of t_0 ; hence $t_0 = 1$ that it is say $u^* \geq v^*$.

$$u^* = v^* \tag{3.8}$$

Consequently, since (3.7), we deduce that u^* is a positive fixed point of F.

Now, denote the unique fixed point of F by x^*.

We proved that successively we can construct a sequence (x_n) defined by : $x_n = F(x_{n-1}), n \in \mathbb{N}^*$ such that

$$\forall x_0 \in k, p_\alpha(x_n - x^*) \xrightarrow[n \longrightarrow +\infty]{} 0.$$

As F be decreasing, $x_0 \geq \mathbf{o}$ implies that $\mathbf{o} \leq F(x_0) \leq F(\mathbf{o})$ i.e. $u_0 \leq x_1 \leq u_1$. Apply the previous inequality to F, we find

$$u_2 \leq x_2 \leq u_1$$

By continuing this process, we more generally get :

$$u_{2n} \leq x_{2n} \leq u_{2n-1} \text{ et } u_{2n} \leq x_{2n+1} \leq u_{2n+1} \leq u_{2n-1} n \in \mathbb{N}^* \tag{3.9}$$

In the same way, we get that

$$u_{2n} \leq x^* \leq u_{2n-1}, n \in \mathbb{N} \tag{3.10}$$

By following, as k be a normal order, (3.9) et (3.10) cause that $\forall \alpha \in A$

$$\begin{array}{rcl} p_\alpha(x_{2n} - x^*) & \leq & p_\alpha(x_{2n} - u_{2n}) + p_\alpha(u_{2n} - x^*) \\ & \leq & N_\alpha p_\alpha(u_{2n-1} - u_{2n}) + N_\alpha p_\alpha(u_{2n-1} - u_{2n}) \\ & = & 2N_\alpha p_\alpha(u_{2n-1} - u_{2n}) \\ & \leq & 2N_\alpha \left(p_\alpha(u_{2n} - u^*) + p_\alpha(u_{2n-1} - u^*)\right) \longrightarrow 0 \text{ qu. } n \longrightarrow +\infty \end{array}$$

(because $u_{2n} \longrightarrow u^*$; $u_{2n+1} \longrightarrow v^* = u^*$; $n \longrightarrow +\infty$). and by the same way, we deduce that $p_\alpha(x_{2n-1} - x^*) \longrightarrow 0, \forall \alpha \in A$ when $n \longrightarrow +\infty$. the relation (3.2) is true for all sequence $(x_n)_n$, then the proof is complete. □

Application

We consider the following boundary problem:

$$\begin{cases} u^{(3)}(t)u'(t) + u"(t)^2 + (2m+1)u'(t)^2 = f(t, u'(t)) \textbf{ sur } \mathbb{R}, \forall m \leq -\dfrac{3}{4} \\ u'(0) = a \in \mathbb{R} \\ u"(0) = c < 0 \\ u'(b) = \lim_{t \longrightarrow b} u'(t) = 0 \end{cases}$$

Such that $f : [0, b[\times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at t and continuously differentiable at u. We proved that there exist two points u_0 and v_0 are respectively a sub-solution and a sup-solution for the operator $w = F(w)$ defined by the solution of the following problem:

$$\begin{cases} -w"(t) - (4m+2)w(t) = -2f\left(t, \sqrt{w(t)}\right) \textbf{ sur } \mathbb{R} \\ w(0) = a^2 \in \mathbb{R}+ \\ w(b) = \lim_{t \longrightarrow b} w(t) = 0 \end{cases}$$

However, we put $w(t) = u'(t)^2$ so $w'(t) = 2.u'(t).u"(t)$ and $w"(t) = 2\left(u"(t)^2 + u^{(3)}(t)u'(t)\right)$. Let $E = C(\mathbb{R})$ provided with the family of semi norms $\mathfrak{P} = \{p_k; k \in \mathcal{K}\}$ such that

$p_k(g) = \sum_{t\in k} e^{-\lambda t} \mid g(t) \mid, \forall k \in \mathcal{K}, \forall g \in E$,**with** $\mathcal{K} = \{$ **the set of all the compact in** $\mathbb{R}\}$.
And $C^2(\mathbb{R})$ **provided with the family of semi norms** $\rho = \{\rho_k; k \in \mathcal{K}\}$ **such that** $\rho_k(g) = p_k(g) + p_k(g') + p_k(g")$, $\forall k \in \mathcal{K}, \forall g \in E$
$F : C^2(\mathbb{R}_+) \subset E \longrightarrow E$ **is well defined, because the problem:**

$$\begin{cases} -\partial^2 w(t) - 2(2m+1)w(t) = -2f\left(t, \sqrt{w(t)}\right) \ \textbf{sur } \mathbb{R} \\ w(0) = a^2 \in \mathbb{R} \\ w(b) = \lim_{t\longrightarrow b} w(t) = 0 \end{cases}$$

$$V \doteqdot F(w) = (-\partial^2 + \lambda)^{-1} g_w$$

such as $\lambda = -2(2m+1)$, $g_w(t) = -2f(t, \sqrt{w(t)})$ **and** $f(t,u)$ **is negative** $\forall t \in [0, b[; u \in C^2([0, b[)$

$$\begin{cases} -V" + \lambda V = g_w(t) \\ V(0) = a^2 \\ V(b) = \lim_{t\longrightarrow b} V(t) = 0 \end{cases}$$

has unique solution $\forall F \in E$.

Remark 3.2. *Let* $(\mathbb{X}, \{\rho_\alpha\}_{\alpha\in I})$ *and* $(\mathbb{Y}, \{\rho'_\alpha\}_{\alpha\in I})$ *be hlctvs,* $T : \mathbb{X} \longrightarrow \mathbb{Y}$ *be a mapping.*

1. *We call that* T *is sequentially continuous if:*

$$\forall \alpha \in I, \rho_\alpha(x-y) \longrightarrow 0 \Longrightarrow \rho'_\alpha(Tx - Ty) \longrightarrow 0$$

2. *We call that* T *is continuous (by the definition 3.3) if there exists* $\mathfrak{J} \subset I$ *such that:*

$$\forall \alpha \in \mathfrak{J}, \rho_\alpha(x-y) \longrightarrow 0 \Longrightarrow \rho'_\alpha(Tx - Ty) \longrightarrow 0$$

Theorem 3.3 (Finite increases theory). *Let* T *be a continuous function on the interval* $[\zeta, \eta]$ *into* $\mathbb{R}$*, and it differentiable on* $]\zeta, \eta[$.
So there exists $\beta \in]\zeta, \eta[$ *such that:*

$$T(\eta) - T(\zeta) = T'(\beta)(\eta - \zeta)$$

Or $\exists\Theta \in]0,1[$ *such that:*

$$T(\eta) - T(\zeta) = T'(\zeta + \Theta(\eta - \zeta))(\eta - \zeta)$$

However, $F : C^2(\mathbb{R}_+) \longrightarrow E$ **verified**

* **Since the operator** F **be strictly positive so monotonous increasing.**

* **The operator** F **be continuous (clear)because**

$$P_\alpha\left(F(.,w) - F(.,w_0)\right) \;=\; \sup_{t\in[0,b[}\left\{e^{-\lambda t}|F(t,w(t)) - F(t,w_0(t))|\right\}$$

And by the finite increases theory, we have

$$P_\alpha\left(F(.,w) - F(.,w_0)\right) \leq \sup_{t\in[0,b[}\left\{e^{-\lambda t}|w(t) - w_0(t)|f(c)\right\} \longrightarrow 0$$

such that $c \in [w(t), w_0(t)], \forall t \in [0,b[$ **so** $P_\alpha\left(F(.,w) - F(.,w_0)\right) \longrightarrow 0$**, then** F **is continuous.**
However, $w \longrightarrow w_0$ **(i.e.** $\rho_\alpha(w - w_0) \longrightarrow 0$**),**
so

$$\begin{aligned}\rho_\alpha(w - w_0) &= p_\alpha(w - w_0) + p_\alpha(w' - w_0') + p_\alpha(w" - w"_0)\\ &= \sup_{t\in[0,b[}\left\{e^{-\lambda t}|w(t) - w_0(t)|\right\} + \sup_{t\in[0,b[}\left\{e^{-\lambda t}|w'(t) - w_0'(t)|\right\}\\ &\quad + \sup_{t\in[0,b[}\left\{e^{-\lambda t}|w"(t) - w"_0(t)|\right\}\end{aligned}$$

$$\rho_\alpha(w - w_0) \geq \sup_{t\in[0,b[}\left\{e^{-\lambda t}|w(t) - w_0(t)|\right\} \longrightarrow 0$$

* **The operator** F **be weakly compact, because it is the inverse of Laplace function(weakly Compact).**

By the theorem 3.1, the operator F **has two points** u_0 **and** v_0 **such that**

* u_0 **is sup-solution;** $u_0 \geq Fu_0$**.**

* v_0 **is sub-solution;** $v_0 \leq Fv_0$**.**

So, F **admits solution.**

Example

Let $C_0^1(\mathbb{R})$ **provided with the family of semi norms** $\xi = \{\xi_k; k \in \mathcal{K}\}$ **such that** $\xi_k(g) = p_k(g) + p_k(g), \forall k \in \mathcal{K}, \forall g \in E$**,with** $\mathcal{K} = \{$ **the set of all the compact in** $\mathbb{R}\}$**.**
And let g **be a continuous function, we put** $f(t, u'(t)) = g(t)$**, so** $F(w(t)) = -2\left(-\partial^2 + \lambda\right)^{-1} g(t)$**.**
By the previous application the problem has a solution u^* **such that** $u_0 \leq u^* \leq v_0$**.**
Now, we proved that this solution is unique (we use lax-Milgram theorem).
We have:

$$-w"(t) - (4m+2)w(t) = -g(t), t \in [0, b[, m \leq -\frac{3}{4}$$

so $\forall v \in C_0^1([0, b[)$

$$\begin{aligned} -\int_0^b w"(t)v(t)\,dt - (4m+2)\int_0^b w(t)v(t)\,dt &= -\int_0^b g(t)v(t)\,dt \\ \int_0^b w'(t)v'(t)\,dt - (4m+2)\int_0^b w(t)v(t)\,dt &= -\int_0^b g(t)v(t)\,dt \end{aligned}$$

we take $a : C^2([0, b[) \times C_0^1([0, b[) \longrightarrow \mathbb{R}$**,** $L : C_0^1([0, b[) \longrightarrow \mathbb{R}$ **such as**

$$a(w, v) = \int_0^b w'(t)v'(t)\,dt - (4m+2)\int_0^b w(t)v(t)\,dt$$

and

$$L(v) = -\int_0^b g(t)v(t)\,dt$$

***** a **bilineair:**

1.

$$\begin{aligned} a(\gamma w_1 + \beta w_2, v) &= \int_0^b (\gamma w_1'(t) + \beta w_2'(t))v'(t)\,dt \\ &\quad -(4m+2)\int_0^b \gamma w_1(t) + \beta w_2(t))v(t)\,dt \\ &= \gamma\left[\int_0^b w_1'(t)v'(t)\,dt - (4m+2)\int_0^b w_1(t)v(t)\,dt\right] \\ &\quad +\beta\left[\int_0^b w_2'(t)v'(t)\,dt - (4m+2)\int_0^b w_2(t)v(t)\,dt\right] \\ &= \gamma a(w_1, v) + \beta a(w_2, v) \end{aligned}$$

2.

$$\begin{aligned}
a(w,\gamma v_1+\beta v_2) &= \int_0^b w'(t)(\gamma v_1'(t)+\beta v_2'(t))\,dt\\
&\quad -(4m+2)\int_0^b w(t)(\gamma v_1(t)+\beta v_2(t))\,dt\\
&= \gamma\left[\int_0^b w'(t)v_1'(t)\,dt-(4m+2)\int_0^b w(t)v_1(t)\,dt\right]\\
&\quad +\beta\left[\int_0^b w'(t)v_2'(t)\,dt-(4m+2)\int_0^b w(t)v_2(t)\,dt\right]\\
&= \gamma a(w,v_1)+\beta a(w,v_2)
\end{aligned}$$

*** a continuous:**

$$\begin{aligned}
a(w,v) &= \int_0^b w'(t)v'(t)\,dt-(4m+2)\int_0^b w(t)v(t)\,dt\\
&= \int_0^b e^{2\lambda t}e^{-2\lambda t}w'(t)v'(t)\,dt-(4m+2)\int_0^b e^{2\lambda t}e^{-2\lambda t}w(t)v(t)\,dt\\
&\leqslant \sup_{t\in 0,b[}\{e^{-\lambda t}w'(t)\}\sup_{t\in 0,b[}\{e^{-\lambda t}v'(t)\}\int_0^b e^{2\lambda t}\,dt\\
&\quad -(4m+2)\sup_{t\in 0,b[}\{e^{-\lambda t}w(t)\}\sup_{t\in 0,b[}\{e^{-\lambda t}v(t)\}\int_0^b e^{2\lambda t}\,dt\\
&\leqslant \left[(-4m-1)\int_0^b e^{2\lambda t}\,dt\right]\rho_\alpha(w).\xi_\alpha(v)\\
&= (-4m-1)\left(\frac{1}{2\lambda}(e^{2\lambda b}-1)\right)\rho_\alpha(w).\xi_\alpha(v)
\end{aligned}$$

*** a coercive:**

$$\begin{aligned}
a(w,w) &= \int_0^b w'(t)^2\,dt-(4m+2)\int_0^b w(t)^2\,dt\\
&= \int_0^b e^{2\lambda t}\left[e^{-\lambda t}w'(t)\right]^2\,dt-(4m+2)\int_0^b e^{2\lambda t}\left[e^{-\lambda t}w(t)\right]^2\,dt\\
&\geq \int_0^b e^{\lambda t}\left[e^{-\lambda t}w'(t)\right]^2\,dt-(4m+2)\int_0^b e^{\lambda t}\left[e^{-\lambda t}w(t)\right]^2\,dt\\
&\geq \min\left(1;(-4m-2)\right)\left[\int_0^b e^{\lambda t}\left[e^{-\lambda t}w'(t)\right]^2\,dt+\int_0^b e^{\lambda t}\left[e^{-\lambda t}w(t)\right]^2\,dt\right]\\
&\geq -\min\left(1;(-4m-2)\right)\rho_\alpha(w)^2\left(-2\int_0^b e^{\lambda t}\,dt\right)\\
&\geq \left(\frac{2e^{\lambda b}-2}{\lambda}\right)\min\left(1;(-4m-2)\right)\rho_\alpha(w)^2
\end{aligned}$$

*** L linear:**

$$\begin{aligned}
L(\gamma v_1+\beta v_2) &= \int_0^b g(t)\left(\gamma v_1(t)+\beta v_2(t)\,dt\right)\\
&= \gamma\int_0^b g(t)v_1(t)\,dt+\beta\int_0^b g(t)v_2(t)\,dt\\
&= \gamma L(v_1)+\beta L(v_2)
\end{aligned}$$

* L **continuous:**

$$\begin{aligned} L(v) &= \int_0^b g(t)v(t)\,dt \\ &= \int_0^b e^{-\lambda t} g(t).e^{-\lambda t} v(t).e^{2\lambda t}\,dt \\ &\leq \int_0^b e^{2\lambda t}\,dt.p_\alpha(g).\rho_\alpha(v) \end{aligned}$$

So

$$p_\alpha(L) \leq \frac{\sinh(b)}{\lambda} p_\alpha(g).\rho_\alpha(v)$$

General Conclusion

In Our thesis, we extend the fixed point theories in Banach space to semi-normed space. This type of results is very useful in functional analysis, because the most of non-linear problems are defined in non-metric topological space.
We study two extension, the first one based to the generalisation of contraction of map and the index-set of semi-norms family, also we applied that in initial value problem with maxima deviate impluse condition ;and we intend to prove the stability of solutions when we replace the term memory $t - \zeta_i(t)$ by $g_i(t)$ where the function g is non-bounded. The second concerns the weakening of the conditions usually used in ordered structures. Our perspective is to further weaken these conditions, in particular we intend and hope to achieve at least the same results, assuming only that the function is bounded.

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