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Theme

Some Problems Of Fractional Partial Differential Equations and Systems.

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Thanks

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A big thank to all the people we have consul understanding and assimilation of this work.

Finally it is a big thank to our parents and all our friends.

Dedication

.... I dedicate this modest work to:

Allah who gives me
the power and strength to challenge the difficulties.

To my beloved parents,
who have been my source of inspiration and who supported
me when I thought of giving up, who continually
provide their moral, spiritual,emotional and
financial support.

To my brothers, sisters, relatives, friends, and classmates
who shared their words of advice and encouragement to finish
this study.

* * *

ABADI ISRA

Dedication

.... I dedicate this modest work to:

Allah who gives me
the power and strength to challenge the difficulties.

To my father's pur soul
that I wished he was beside me in this special moment in
my life.

To my beloved mother
who have been my source of inspiration and who supported
me when I thought of giving up, who continually
provide their moral, spiritual, emotional and
financial support.

To my brothers, sisters, relatives, friends, and classmates
who shared their words of advice and encouragement to finish
this study.

★ ★ ★

HADA MAROUA

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Notations

•Γ	Gamma function.
•β	Beta function.
•E_α	one-parameter Mittag-Leffler function.
•$E_{(\alpha,\beta)}$	two-parameter Mittag-Leffler function.
•$\mathcal{F}$	Fourier Transforms.
•$\mathcal{F}^{-1}$	inverse Fourier Transforms.
•${}^c_aD^\alpha_t$	The fractional derivative on the left .
•${}^c_tD^\alpha_b$	The fractional derivative on the right.
•$\Re e$	real part.
•$suppf$	Support of the function f.
•$D'(\Omega)$	Space of all distributions on the open set $\Omega \subseteq \mathbb{R}$.
•$\langle .,. \rangle$	The scalar product.
•$S(\Omega)$	Schwartz space.
•$\mathbb{R}$	Set of real numbers.
•$\mathbb{C}$	Set of complex numbers.
•$\mathbb{N}$	Set of natural numbers.
•$a.e.$	Almost everywhere.
• $:=$	By definition.
•PV	The main value.

Introduction

THE subject of fractional calculus (a e calculating integrals and arbitrary derivatives, real or complex) has gained considerable popularity and importance over the past three decades, mainly due to its demonstrated applications in many fields of science and engineering. It indeed provides several potentially useful tools for solving differential and integral equations, as well as various other problems involving special functions of mathematical physics, as well as their extensions and generalizaions.

Fractional differential calculus has been associated with many of the most famous scientists, Hospital.G.W.L, Lepnitz, Riemann.B and Liouville J who gave those of the basic derivative fractional definitions, After these, research continued in this field and in different directions. The researcher expanded Bassam.M.A definition Holmgren-M.Riesz and applied the results obtained to some existential theories of ordinary differential equations.

The aim of this work is to study some problems nonexistence of solution to fractional differential equations.
This work is divided into four chapter :

- In the first chapter, we presented some definitions and theorems that we will use in this memoir.
- In the secend chapter, mainly introduces definitions and basic properties of fractional derivatives, Riemann-Liouville fractional derivative, Caputo fractional derivative and fractional Laplace operator, etc.
- In the third chapter, we will review the sufficient conditions and necessary conditions of non-existance of a solution.of the problem:

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t}(u) + (-\Delta)^{\beta/2}(u) = h(x,t) \mid u \mid^{1+\tilde{p}} & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ =: Q, \\ u(x,0) = u_0(x) \geq 0 & \text{for } x \in \mathbb{R}^n. \end{cases}$$

The results presented in this chapter are from the article [7].

- In the fourth chapter, we will present some results of nonexistence of solutions for a nonlocal nonlinear differential problem of the form :

$$\begin{cases} u_t + (-\Delta)^{\beta/2}(\mid u \mid^p) = \mid u \mid^q & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

  The subject of this chapter is taken from the article [3].

Chaptre 1

Preliminaries

Let (Ω, M, μ) denote a measure space, i.e., Ω is a set and

1. M is a σ-algebra in Ω, i.e., M is a collection of subsets of Ω such that:
   - $\phi \in M$.
   - $A \in M \Rightarrow A^c \in M$.
   - $\bigcup\limits_{n=1}^{\infty} A_n \in M$ whenever $A_n \in M \quad \forall n$.
2. μ is a measure, i.e., $\mu : M \to [0, \infty[$ satisfies
   - $\mu(\phi) = 0$.
   - $\begin{cases} \mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \bigcup\limits_{n=1}^{\infty} \mu(A_n) whenever (A_n) \text{ is a disjoint} \\ \text{countable family of members of M.} \end{cases}$

   The members of M are called the measurable sets. Sometimes we shall write $|A|$ instead of $\mu(A)$. We shall also assume even though this is not essential that.
3. Ω is σ-finite, i.e., there exists a countable family (Ω_n) in M such that $\Omega_n = \bigcup\limits_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty \ \forall n$.

The sets $E \in M$ with the property that $\mu(E) = 0$ are called the null sets. We say that a property holds a.e. (or for almost all $x \in \Omega$) if it holds everywhere on Ω except on a null set.

1.1 Some Results about Integration that everyone must know

•Let E be a measurable set, $\Omega \subset E$.

Theorem 1.1. *[2] **(Lebesgue dominated convergence theorem)***
Let $\{f_n\}$ be a sequence of measurable functions such that $\lim\limits_{n\longrightarrow\infty} f_n(x) = f(x)$ a.e. in E, and for every $n \in \mathbb{N}$, $\mid f_n(x) \mid \leq g(x)$ a.e. in E, where g is integrable on E. Then

$$\lim_{n\longrightarrow\infty} \int\limits_E f_n(x) = \int\limits_E f(x)dx.$$

Lemma 1.1. *[2]**(Fatou's lemma).***
Let $\{f_n\}$ be a sequence of functions in $L^1(E)$ that satisfy

1. *for all n, $f_n \geq 0$ a.e.*
2. $\sup\limits_n \int f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf\limits_{n\longrightarrow\infty} f_n \leq +\infty$. Then $f \in L^1(E)$ and

$$\int f \leq \liminf_{n\longrightarrow\infty} \int f_n.$$

Theorem 1.2. *[2]**(Tonelli).***
Let $F(x, y) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function satisfying

1. $\int\limits_{\Omega_2} F(x, y)d\mu_2 < \infty$ *for a.e. $x \in \Omega_1$*
   and
2. $\int\limits_{\Omega_1} d\mu_1 \int\limits_{\Omega_2} F(x, y)d\mu_2 < \infty$
   Then $F \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.3. *[2]**(Fubini).***
Assume that $F \in L^1(\Omega_1\times\Omega_2)$. Then for a.e. $x \in \Omega_1$, $F(x, y) \in L^1_y(\Omega_2)$ and $\int\limits_{\Omega_2} F(x, y)d\mu_2 \in$

$L^1_x(\Omega_1)$. *Similarly, for a.e.* $y \in \Omega_2, F(x,y) \in L^1_x(\Omega_1)$ *and* $\int\limits_{\Omega_1} F(x,y)d\mu_1 \in L^1_y(\Omega_1)$. *Moreover, one has*

$$\int\limits_{\Omega_1} d\mu_1 \int\limits_{\Omega_2} F(x,y)d\mu_2 = \int\limits_{\Omega_2} d\mu_2 \int\limits_{\Omega_1} F(x,y)d\mu_1 = \iint_{\Omega_1\times\Omega_2} F(x,y)d\mu_1 d\mu_2.$$

1.2 Some Results from functional analysis

1.2.1 Definition and elementary properties of L^p spaces

Definition 1.1. *[2]*

Let $p \in \mathbb{R}$ *with* $1 < p < \infty$, *we set*

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f \textit{ is measurable and } \mid f \mid^p \in L^1(\Omega)\}$$

with

$$\parallel f \parallel_{L^p} = \parallel f \parallel_p = \left[\int\limits_{\Omega} \mid f(x) \mid^p d\mu\right]^{\frac{1}{p}}.$$

$\parallel \cdot \parallel_p$ *is a norm.*

Definition 1.2. *[2]*

Let

$$L^\infty(\Omega) = \left\{ \mathrm{f} : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \mathrm{f} \textit{ is measurable and there is a constant } C \\ \textit{suth that } \mid f \mid \leq C \textit{ a.e. on } \Omega \end{array} \right\}$$

with

$$\parallel f \parallel_{L^\infty} = \parallel f \parallel_\infty = \inf\{C \mid f(x) \mid \leq C \textit{ a.e. on } \Omega\}.$$

$\parallel \cdot \parallel_\infty$ *is a norm.*

Remark 1.1.

If $f \in L^\infty$ *then we have*

$$\mid f(x) \mid \leq \parallel f \parallel_\infty \textit{ a.e. on } \Omega.$$

Indeed, there exists, a sequence C_n *such that* $C_n \longrightarrow \parallel f \parallel_\infty$ *and for each* n, $\mid f(x) \mid \leq C_n$ *a.e. on* Ω. *Therefore* $\mid f(x) \mid \leq C_n$ *for all* $x \in \Omega \setminus E_n$, *with* $\mid E_n \mid = 0$. *We set* $E = \bigcup\limits_{n=1}^{\infty} E_n$, *so that* $| E |= 0$ *and*

$$| f(x) |\leq C_n \ \forall n, \ \forall x \in \Omega \setminus E;$$

it follows that $| f(x) |\leq \| f \|_{\infty} \forall x \in \Omega \setminus E$.

Notation. Let $1 \leq p \leq \infty$; we denote by p' the conjugate exponent,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Lemma 1.2. ***(Young inequality)*** *[2]*

Let a, b *two real positive, and* $1 \leq p, p' < \infty$, *so*

$$ab \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'}.$$

Lemma 1.3. ***(ϵ-Young inequality)*** *[2]*

Let a, b *two real positive, and* $1 \leq p, p' < \infty$, *so for all* $\epsilon > 0$

$$ab \leq \epsilon a^p + C_\epsilon b^q.$$

Lemma 1.4. ***(Hölder inequality)*** *[2]*

Let Ω *be an open set in* $\mathbb{R}^n$, $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$, *with* $1 \leq p \leq \infty$. *Then* $f \cdot g \in L^1(\Omega)$ *and*

$$\int | fg |\leq \| f \|_p^p \cdot \| g \|_{p'}^{p'}.$$

1.2.2 Sobolev spaces

[13] We refer to Cazenave and Haraux, 1998, for the definitions and results given below. Consider an open subset Ω of $\mathbb{R}^n$. $D(\Omega)$ is the space of C^∞ (real-valued or complex valued) functions with compact support in Ω and $D'(\Omega)$ is the space of distributions on Ω. A distribution $T \in D'(\Omega)$ is said to belong to $L^p(\Omega)(1 \leq p \leq \infty)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T, \phi \rangle = \int\limits_{\Omega} f(x)\phi(x)dx,$$

for all $\phi \in D(\Omega)$. In that case, it is well known that f is unique. Let $m \in \mathbb{N}$ and let $p \in [1, \infty]$. Define

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \setminus D^\alpha f \in L^p(\Omega) \ \text{ for all } \alpha \in \mathbb{N}^n \text{ such that } \ | \alpha |\leq m\}.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm

$$\| f \|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \| D^\alpha f \|_{L^p},$$

for all $f \in W^{m,p}(\Omega)$. For all m, p as above, we denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$. If $p = 2$, one sets $W^{m,2}(\Omega) = H^m(\Omega), W_0^{m,2}(\Omega) = H_0^m(\Omega)$ and one equips $H^m(\Omega)$ with the following equivalent norm:

$$\| f \|_{H^m} = \left(\sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Then $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle_{H^m} = \sum_{|\alpha| \leq m} \int\limits_\Omega D^\alpha u . D^\alpha v dx.$$

If Ω is bounded, there exists a constant $C(\Omega)$ such that

$$\| u \|_{L^2} \leq C(\Omega) \| \nabla u \|_{L^2},$$

for all $u \in H_0^1(\Omega)$ (this is Poincaré inequality). It may be more convenient to equip $H_0^1(\Omega)$ with the following scalar product

$$\langle u, v \rangle = \int\limits_\Omega \nabla u . \nabla v dx,$$

which defines an equivalent norm to $\| \, . \, \|_{H^1}$ on the closed space $H_0^1(\Omega)$.

1.2.3 Concept of test function

Distributions are prticular linear functionls on the space of test functions. Let's start by defining these test functions:

Definition 1.3. *[8]*

We call a test function, or basic function, a function $\varphi \in C^\infty(\mathbb{R}, \mathbb{C})$*.The support of* φ *is group*

$$supp(\varphi) = \overline{\{x \in \Omega : \varphi \neq 0\}}.$$

General properties

1. The set D of test functions is a vector subspace and an ideal of $C^\infty(\mathbb{R}, C)$.
2. If φ is a test function, the same is true for φ'.
3. If φ is a test function of $\mathbb{R}$ in $\mathbb{R}$ and f a function $C^\infty(\mathbb{R}, C)$, $f \circ \varphi$ is a test function.
4. Any test function φ is uniquely written in the form $\varphi(x) = \psi(x) + x\xi(x)$, where ψ and ξ are even test function. Just note that :

$$\varphi(x) = \frac{\varphi(x)+\varphi(-x)}{2} + \frac{\varphi(x)-\varphi(-x)}{2} = \frac{\varphi(x)+\varphi(-x)}{2} + x\frac{\varphi(x)-\varphi(-x)}{2x},$$

or

$$\frac{\varphi(x)-\varphi(-x)}{2x} = \frac{1}{2x}\int_{-x}^{+x} \varphi'(t)dt = \frac{1}{2}\int_{-1}^{+1} \varphi'(xu)du,$$

function C^∞ by virtue of the theorem of derivation of integrals with parameters on the segments.

Exemple 1.1.

Consider the function

$$\varphi(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } \mid x \mid < 1, \\ 0 & \text{if } \mid x \mid \geq 1. \end{cases}$$

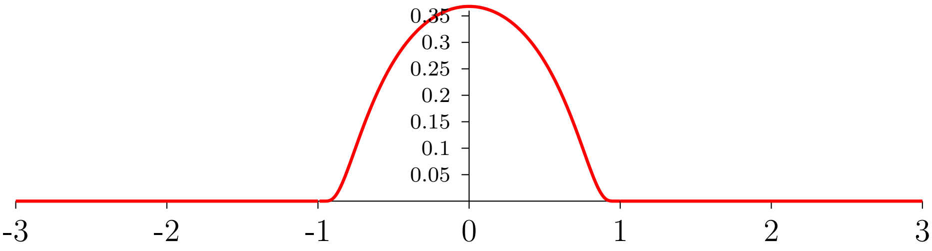
It is even, continuous, increasing on $[-1, 0]$, *decreasing on* $[0, 1]$

φ *is class* C^∞ *on* $\mathbb{R}$, *because* $\varphi = f_h$, *where* $h(x) = 1 - x^2$.

NB: we also have

$$e^{\frac{1}{x^2-1}} = e^{\frac{1}{2}(\frac{1}{x-1}-\frac{1}{x+1})} = e^{(\frac{1}{2x-2}-\frac{1}{2x+2})} = f(2x+2).f(2-2x).$$

We can easily verify that, for all x, $\varphi(x) = f_0(2x+2).f_0(2-2x)$.



1.2.4 Schwartz space

When working with the Fourier transformation we require $\Omega = \mathbb{R}^n$. However, $D(\mathbb{R}^n)$ is, in a sense, too small for the Fourier transform, which also makes $D'(\mathbb{R}^n)$ too large. For the purpose of the Fourier transformation, the Schwartz spaces, $S(\mathbb{R}^n)$ (named in honour of Laurent Schwartz) and $S'(\mathbb{R}^n)$, as introduced below, are optimally adapted in the sense that they are both closed under the Fourier transform.

Definition 1.4. *[4] For $n \in \mathbb{N}$,*

$$S(\mathbb{R}^n) = \left\{ \phi \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} (1+ \mid x \mid^2)^{\frac{k}{2}} \sum_{|\alpha|\leq l} \mid D^\alpha \phi(x) \mid < \infty, \quad \textit{for all } k, l \in \mathbb{N}^* \right\} \tag{1.1}$$

with the natation D^α, for $\alpha \in \mathbb{N}^n - \{0\}$ is defined as

$$D^\alpha = \frac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_1}} \frac{\partial^{\alpha_2}}{(\partial x_2)^{\alpha_2}} \frac{\partial^{\alpha_3}}{(\partial x_3)^{\alpha_3}} \cdots \frac{\partial^{\alpha_n}}{(\partial x_n)^{\alpha_n}}, \; |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$$

is called the Schwartz space of all rapidly decreasing infinitely differentiable functions, or Schwartz space for short.

A sequence $\{\phi_j\}_{j=1}^\infty \subset S(\mathbb{R}^n)$ is said to converge in $S(\mathbb{R}^n)$ to $\phi \in S(\mathbb{R}^n)$ if

$$\| \phi_j - \phi \|_{k,l} \to 0 \textit{ for } j \to \infty \textit{ and all } k; l \in \mathbb{N}^*.$$

Where

$$\| \phi \|_{k,l} = \sup_{x\in\mathbb{R}^n} (1+ \mid x \mid^2)^{\frac{k}{2}} \sum_{|\alpha|\leq l} \mid D^\alpha \phi(x) \mid .$$

Proposition 1.1. *[4] The Schwartz space, $S(\mathbb{R}^n)$ is a subspace of $L^p(\mathbb{R}^n)$ for every p in* $\mathbb{N}$.

Proof. For every function ϕ in $S(\mathbb{R}^n)$, there exists a constant K such that

$$\mid \phi(x) \mid \leq \frac{K}{1+ \mid x \mid^2}$$

for every x in $\mathbb{R}^n$. Thus,

$$\int_{\mathbb{R}^n} \mid \phi(x) \mid^p dx \leq K^p \int_{\mathbb{R}^n} \frac{1}{(1+ \mid x \mid^2)^p} dx < \infty$$

for every $p < \infty$. For $p = \infty$, it follows from the definition, (1.1), that

$$\sup_{x\in\mathbb{R}^n} \mid \phi(x) \mid < \infty,$$

and we conclude that $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for every p in $\mathbb{N}$. ☐

Proposition 1.2. *If $\phi \in S(\mathbb{R}^n)$, so are both $x^\alpha\phi$ and $D^\alpha\phi$ for $\alpha \in \mathbb{N}^n - \{0\}$.*

Proof. This follows directly from the definition of $S(\mathbb{R}^n)$, (1.1). All functions $\phi \in S(\mathbb{R}^n)$ are rapidly decreasing (i.e. go to zero when multiplied with an arbitrary polynomial), and so do all of their derivatives. ☐

1.3 Fourier transforms

Definition 1.5. *[11]*

The exponential Fourier transforms of a continuous function $h(t)$ absolutely integrable in $(-\infty, +\infty)$ is defined by

$$\mathcal{F}\{h(t);\omega\} = \int\limits_{-\infty}^{+\infty} e^{i\omega t} h(t)dt,$$

and the original $h(t)$ can be restored from its Fourier Transforms, $\mathcal{H}(t)$.

- ***The inverse Fourier Transforms:***

$$\mathcal{F}^{-1} = h(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \mathcal{H}(\omega)e^{-i\omega t}h(t)d\omega.$$

- ***The Fourier transform of the convolution***

$$h(t) * g(t) = \int\limits_{-\infty}^{+\infty} h(t-\tau)g(\tau)d\tau,$$

of the two functions $h(t)$ and $g(t)$. Which are defined in $(-\infty, +\infty)$, is equal to the product of the Fourier transform:

$$\mathcal{F}\{h(t) * g(t);\omega\} = \mathcal{H}(\omega)\mathcal{G}(\omega)$$

under assumption that both $\mathcal{H}(\omega)$ and $\mathcal{G}(\omega)$ exist.

- ***The Fourier transform n-th derivative of $h(t)$:***

$$\mathcal{F}\{h^{(n)}(t);\omega\} = (-i\omega)^{(n)}\mathcal{H}(\omega).$$

Chaptre 2

Fractional calculus

This chapter is devoted to the definitions of the special functions, and some definitions and properties of Riemann-Liouville, Caputo fractional derivatives and fractional Laplacian operator.

2.1 Special functions

2.1.1 Gamma function

[12] Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int\limits_0^\infty t^{z-1}e^{-t}dt \ (\mathfrak{Re}(z) > 0)$$

where $t^{z-1} = e^{(z-1)log(t)}$. This integral is convergent for all complex $z \in \mathbb{C}$ $(\mathfrak{Re}(z) > 0)$.

Proposition 2.1.

1. $\Gamma(z+1) = z\Gamma(z)$ $(\mathfrak{Re}(z) > 0)$, $\Gamma(n+1) = n!, \ \forall n \in \mathbb{N}$.
2. $\Gamma(1) = 1$ *and* $\Gamma(-m) = \mp\infty$ *for all* $m \in \mathbb{N}$.
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4 $\Gamma(z) = \lim\limits_{n \to \infty} \dfrac{n! n^z}{z(z+1)...(z+n)}, \ \Re e(z) > 0.$

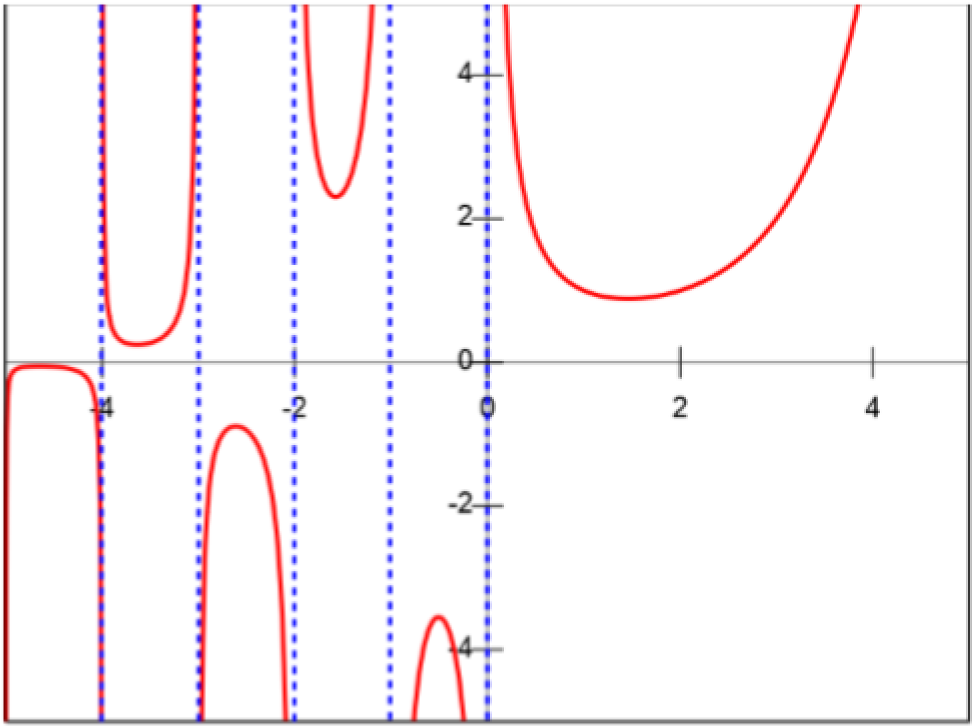


Figure 2.1: Gamma function.

2.1.2 Beta function

Definition 2.1. *[12]*

Here we consider Beta function, denoted (B) *or the first order Euler function, can be defined as*

$$B(u,v) = \int\limits_0^1 t^{u-1}(1-t)^{v-1} dt, \Re e \ u > 0, \Re e \ v > 0.$$

the symmetry

$$B(u,v) = B(v,u).$$

Proposition 2.2.

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \Re e \ u > 0, \Re e \ v > 0.$$

2.1.3 Mittag-Leffler function

[12] We introduce the one and two-parameter Mittag-Leffler functions, denoted as $E_\alpha(.)$ and $E_{(\alpha,\beta)}(.)$, respectively.

Definition 2.2.

One parameter Mittag-Leffler function (E_α)*, is defined as:*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\quad Re(\alpha)>0.$$

Two parameter Mittag-Leffler function $E_{(\alpha,\beta)}$*, is defined as:*

$$E_{(\alpha,\beta)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\quad \Re e(\alpha)>0\ ,\Re e(\beta)>0\ ,\beta\in\mathbb{C}.$$

.

Exemple 2.1.

$$E_{(1,1)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z$$

$$E_{(1,2)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{e^z-1}{z}$$

$$E_{(1,3)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!}=\frac{e^z-1-z}{z^2}.$$

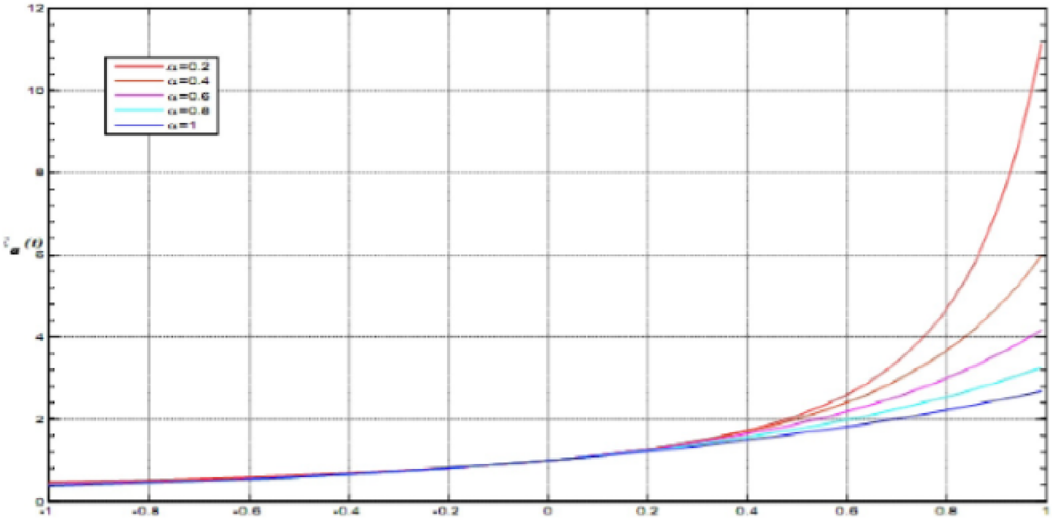


Figure 2.2: One-parameter Mittag-Leffler function.

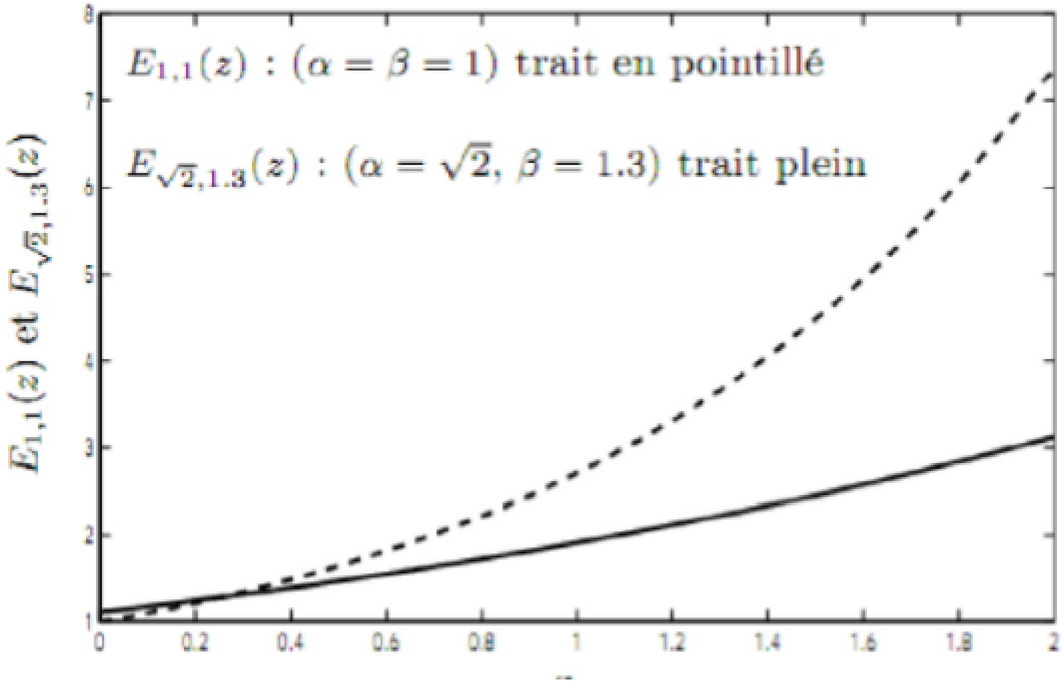


Figure 2.3: Two-parameter Mittag-Leffler function.

2.2 Some fractional derivatives

Lemma 2.1. *Integrals of arbitrary order If f is locally integrable on $(c;+\infty)$, then the n-th integral of f is given by:*

$$\begin{aligned} {}_cD_t^{-n}[f](t) &= \int_c^t ds_1 \int_c^{s_1} ds_2 \dots \int_c^{s_{n-1}} f(s_n)ds_n \\ &= \frac{1}{(n-1)!}\int_c^t (t-s)^{n-1} f(s_n)ds \end{aligned}$$

For almost not everything t where $-\infty\ c < t < \infty$ and $n \in \mathbb{N}$, since $\Gamma(n) = (n-1)!$. We have one immediate generalization which is the integral of fractional order f $\alpha > 0$.

2.2.1 Riemann-Liouville fractional integrals

[6] To introduce R-L fractional integral, consider first the following iteration integrals

$$\begin{aligned} D^{-1}[f](t) &= \int\limits_0^t f(\tau)d\tau \\ D^{-2}[f](t) &= \int\limits_0^t d\tau_1 \int\limits_0^{\tau_1} f(\tau)d\tau, \\ &\dots \\ D^{-n}[f](t) &= \int\limits_0^t d\tau_1 \int\limits_0^{\tau_1} d\tau_2 \cdots \int\limits_0^{\tau_{n-1}} f(\tau)d\tau. \\ &\dots \end{aligned}$$

These multiple iteration integrals can all be expressed as

$$\int\limits_0^t K_n(t,\tau)f(\tau)d\tau,$$

for a certain kernel function $K_n(t,\tau)$ obviously, $K_1(t,\tau) = 1$. When $n = 2$, then

$$\begin{aligned} \int\limits_0^t d\tau \int\limits_0^{\tau_1} f(\tau_1)d\tau_1 &= \int\limits_0^t f(\tau)d\tau \int\limits_0^{\tau} d\tau_1 \\ &= \int\limits_0^t (t-\tau)f(\tau)d\tau, \end{aligned}$$

thus $K_1(t,\tau) = (t-\tau)$. When $n = 3$,

$$\begin{aligned}\int_0^t d\tau \int_0^\tau d\tau_1 \int_0^{\tau_1} f(\tau_2) d\tau_2 &= \int_0^t d\tau \int_0^\tau (\tau - \tau_1) f(\tau_1) d\tau_1 \\ &= \int_0^t f(\tau) d\tau \int_\tau^t (\tau_1 - \tau) d\tau_1 \\ &= \int_0^t f(\tau) \frac{(t-\tau)^2}{2} d\tau,\end{aligned}$$

hence $K_2(t,\tau) = \frac{(t-\tau)^2}{2}$. Generally, $K_n(t,\tau) = \frac{(t-\tau)^{n-1}}{(n-1)!}$ by induction, yielding

$$D^{-n}[f](t) = \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau,$$

where $\Gamma(n) = (n-1)!$. Assume $f \in C[0,T]$, the space of continuous functions on $[0,T]$, then for arbitrary $t \in C[0,T]$, the integral exists in the sense of Riemann integral for any $n \geqslant 1$. Certainly, this idea can be extended to the situation $0 < n < 1$, where the integral exists as a generalized integral. Extending n to a general complex number, one obtains the definition of the R-L integral.

Definition 2.3.

Suppose that f is piecewise continuous in $(0,\infty)$, and integrable in any finite subinterval of $[0,\infty)$. For any $t > 0$ and any complex number ν with $\mathfrak{Re}\nu > 0$, the ν-th R-L fractional integral of f is defined by

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau. \tag{2.1}$$

Below, $\boldsymbol{C}$ will denote the class of functions of f such that 2.1 makes sense.

Exemple 2.2.

Let $f(t) = t^\mu$ and $\mu > -1$, then obviously $f \in \boldsymbol{C}$. By definition,

$$\begin{aligned}{}_0D_t^{-\nu} t^\mu &= \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{v-1} \tau^\mu d\tau \\ &= \frac{B(\nu,\mu+1)}{\Gamma(\nu)} t^{\nu+\mu} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{\nu+\mu}, \mathfrak{Re}\nu > 0, t > 0.\end{aligned}$$

Where B and Γ are the Beta function and the Gamma function, respectively. When μ and ν are integers, it reduces to the classical situation, and is consistent to the multiple iteration integrals above.

2.2.2 Riemann-Liouville fractional derivative

Definition 2.4. *The derivative on the left and the derivative on the right in the sense of Riemann-Liouville of order $\alpha > 0$ of the function $f \in L^1(0,T)$ are defined by*

$$ {}_a^RD_t^{\alpha} f(t) = \frac{d^n}{dt^n}\left[{}_aD_t^{n-\alpha} f(t)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int\limits_a^t (t-\tau)^{(n-\alpha-1)} f(\tau)d\tau, \quad n = [\alpha]+1. $$

And

$$ {}_t^RD_b^{\alpha} f(t) = (-1)^n\frac{d^n}{dt^n}\left[{}_tD_b^{n-\alpha} f(t)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int\limits_t^b (t-\tau)^{(n-\alpha-1)} f(\tau)d\tau, \quad n = [\alpha]+1. $$

Respectively.

In particular, when $\alpha = n \in \mathbb{N}$ so

$$ {}_a^RD_t^0 f(t) ={}_t^R D_b^0 f(t) = f(t) $$

$$ {}_a^RD_t^n f(t) = f^{(n)}(t) \quad \text{and} \quad {}_t^RD_b^n f(t) = (-1)^n f^{(n)}(t) $$

or $f^n(t)$ the classical derivative of $f(t)$ of order n. If $0 < \alpha < 1$ we have

$$ {}_a^RD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_a^t (t-\tau)^{(-\alpha)} f(\tau)d\tau, \quad t > a. $$

And

$$ {}_t^RD_b^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int\limits_t^b (t-\tau)^{(-\alpha)} f(\tau)d\tau, \quad t < b. $$

Remark 2.1.

We can see that the fractional Riemann-Liouville derivative and the classical derivative of order n only commute if $f^{(k)}(a) = 0$, for all $k = 0, 1, 2, \cdots, n-1$.

Fractional Derivative of $(\mathbf{t}-\mathbf{a})^{\beta}$ [12]

Let us now evaluate the Riemann Liouville fractionial derivative ${}_aD_t^p f(t)$ of the power function

$$f(t) = (t-a)^{\nu},$$

where ν is a real number.

For this purpose let us assume that $n-1 \leq p < n$ and recall that by the definition of the Riemann Liouville derivative

$$ {}_aD_t^p f(t) = \frac{d^n}{dt^n}\left({}_aD_t^{-(n-p)} f(t)\right), \quad (n-1 \leq p < n). \tag{2.2}$$

Substituting into the formula (2.2) the fractional integral of order $\alpha = n-p$ of this function, which we have evaluated earlier.

$$ {}_aD_t^{-\alpha}\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}(t-a)^{\nu+\alpha},$$

wc obtain:

$$ {}_aD_t^{p}\left((t-a)^{\nu}\right) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}(t-a)^{\nu-p},$$

and the only restriction for $f(t) = (t-a)^{\nu}$ is its integrability, namely $\nu > -1$.

2.2.3 Caputo fractional derivatives

Riemann-Liouville derivatives have drawbacks when they try to shape real phenomena by fractional differential equations. We will therefore now approach a modified concept of fractional derivative. We will show that the second option is more suitable for such tasks.

Definition 2.5. *[13]*
If $\alpha \in \mathbb{R}_+$ and $n = [\alpha]+1$, for all $f \in C^n(a,b)$, then The fractional derivative on the left and the fractional derivative on the right in the sense of Caputo of order $\alpha > 0$ of the function f in $t \in (a,b)$, are defined respectively by:

$$ {}_a^cD_t^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{(n-\alpha-1)} f^{(n)}(\tau)d\tau \tag{2.3}$$

and

$$ {}_t^cD_b^{\alpha}(f)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{(n-\alpha-1)} f^{(n)}(\tau)d\tau. \tag{2.4}$$

In particular, if $0 < \alpha < 1$ *,one have*

$$ {}_a^cD_t^\alpha(f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau \tag{2.5}$$

and

$$ {}_t^cD_b^\alpha(f)(t) = \frac{-1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \tag{2.6}$$

Remark 2.2.

We can see that if $\alpha = m \in \mathbb{N}$*, so*

$$ {}_a^cD_t^0 f(t) ={}_t^c D_b^0 f(t) = f(t)$$

$$ {}_a^cD_t^m f(t) = f^{(m)}(t) \quad and \quad {}_t^cD_b^m f(t) = (-1)^m f^{(m)}(t)$$

where $f^m(t)$ *the classical derivative of* $f(t)$ *order* m*.*

Exemple 2.3.

- *The derivative of a constant in the sense of Caputo is null. Because for all* $\alpha > 0$ *so* $n = [\alpha] + 1 \neq 0$*, then* $D^n C = 0$ *that is*

$$ {}_a^cD_t^\alpha C = \frac{-1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{(n-\alpha-1)} 0 d\tau = 0.$$

- *If* $f(t) = (t-a)^\nu$ *end* $n = [\alpha] + 1$ *with* $\nu > n-1$*, then*

$$ f^n(t) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)} (t-a)^{\nu-n} \tag{2.7}$$

so

$$ {}_a^cD_t^\alpha (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(n-\alpha)\Gamma(\nu-n+1)} \int_a^t (t-\tau)^{(n-\alpha-1)} (\tau-a)^{\nu-n} d\tau \tag{2.8}$$

by performing the change of variable $\tau = a + s(t-a)$*, we obtain*

$$
\begin{aligned}
{}_a^cD_t^\alpha (t-a)^\nu &= \frac{\Gamma(\nu+1)}{\Gamma(n-\alpha)\Gamma(\nu-n+1)} \int\limits_a^t (t-\tau)^{(n-\alpha-1)}(\tau-a)^{\nu-n} d\tau \\
&= \frac{\Gamma(\nu+1)}{\Gamma(n-\alpha)\Gamma(\nu-n+1)} (t-a)^{\nu-\alpha} \int\limits_0^1 (1-s)^{(n-\alpha-1)}(s)^{\nu-n} ds \\
&= \frac{\Gamma(\nu+1)B(n-\alpha,\nu-n+1)}{\Gamma(n-\alpha)\Gamma(\nu-n+1)} (t-a)^{\nu-\alpha} \\
&= \frac{\Gamma(\nu+1)\Gamma(n-\alpha)\Gamma(\nu-n+1)}{\Gamma(n-\alpha)\Gamma(\nu-n+1)\Gamma(\nu-\alpha+1)} (t-a)^{\nu-\alpha} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} (t-a)^{\nu-\alpha}.
\end{aligned}
$$

2.3 Calcul properties

Proposition 2.3. ***(Composition with whole order derivatives)***
If f *is a fairly regular function over the interval* $[a,b]$*, and* $m-1 \leq \alpha \leq m$*,* $n \in \mathbb{R}^*$*. So*

1 $\frac{d^n}{dt^n}\left({}_a^RD_t^\alpha f(t)\right) = {}_a^R D_t^{n+\alpha} f(t).$

2 ${}_a^RD_t^\alpha \left(\frac{d^n}{dt^n} f(t)\right) = {}_a^R D_t^{\alpha+n} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(1+k-\alpha-n)}.$

Proposition 2.4. ***(Composition with fractional derivatives)***
If f *is a fairly regular function over the interval* $[a,b]$*, for all* $m-1 \leq p < m, n-1 \leq q < n$*. We have*

1 ${}_a^RD_t^p \left({}_a^RD_t^q f(t)\right) = {}_a^R D_t^{p+q} f(t) - \sum_{k=1}^{n} \left[{}_a^RD_t^{q-k} f(t)\right]_{t=a} \frac{(t-a)^{-p-k}}{\Gamma(1-p-k)}.$

2.3.1 Relationship between fractional derivatives with in the sense of Caputo and those of Riemann-Liouville

Proposition 2.5. *If* $\alpha \in \mathbb{R}_+$ *and* f *is a fairly regular function over the interval* $[a,b]$*, then the fractional derivative to the left (resp- to the right) of Caputo is linked to Riemann-Liouville's the fractional derivative to the left (resp- to the right) by*

$$
{}_a^CD_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \quad n = [\alpha]+1 \tag{2.9}
$$

resp

$$ {}_{t}^{C}D_{b}^{\alpha}f(t) ={}_{t}^{R} D_{b}^{\alpha}f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}, \quad n=[\alpha]+1. \tag{2.10}$$

In particular, when $0<\alpha<1$ *then*

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) ={}_{a}^{R} D_{t}^{\alpha}f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-\alpha)^{-\alpha}, \quad n=[\alpha]+1$$

and

$$ {}_{t}^{C}D_{b}^{\alpha}f(t) ={}_{t}^{R} D_{b}^{\alpha}f(t) - \frac{f(b)}{\Gamma(1-\alpha)}(t-\alpha)^{-\alpha}, \quad n=[\alpha]+1.$$

Proof. See [12] □

2.3.2 Integration by parts

[1] This paragraph is linked to the integration formula by parts for the fractional derivatives of Riemann-Liouville and those of Caputo.

Theorem 2.1. *Let* $0<\alpha<1$ *and* f,g *be two fairly regular functions on* $[a,b]$*, for all* $t\in(a,b)$*. We have*

$$\int_{a}^{t} [{}_{a}^{R}D_{\tau}^{\alpha}f(\tau)]g(\tau)d\tau = \int_{a}^{t} f(\tau)[{}_{\tau}^{R}D_{t}^{\alpha}g(\tau)]d\tau$$

$$\int_{t}^{b} [{}_{\tau}^{R}D_{b}^{\alpha}f(\tau)]g(\tau)d\tau = \int_{t}^{b} f(\tau)[{}_{t}^{R}D_{\tau}^{\alpha}g(\tau)]d\tau \tag{2.11}$$

In particular

$$\int_{a}^{b} [{}_{a}^{R}D_{\tau}^{\alpha}f(\tau)]g(\tau)d\tau = \int_{a}^{b} f(\tau)[{}_{\tau}^{R}D_{b}^{\alpha}g(\tau)]d\tau.$$

Proof.

$$\begin{aligned}
\int_a^t [{}_a^R D_\tau^\alpha f(\tau)] g(\tau) d\tau =& \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{d}{d\tau} \left(\int_a^\tau (\tau - s)^{-\alpha} f(s) ds \right) g(\tau) d\tau \\
=& -\frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\int_a^\tau (\tau - s)^{-\alpha} f(s) ds \right) g'(\tau) d\tau \\
&+ \left[\frac{1}{\Gamma(1-\alpha)} g(\tau) \int_a^\tau (\tau - s)^{-\alpha} f(s) ds \right]_{\tau=a}^{\tau=t} \\
=& -\frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\int_s^t (\tau - s)^{-\alpha} g'(s) ds \right) f(s) ds \\
&+ g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f(s) ds \\
=& \int_a^t f(s) [{}_s^C D_t^\alpha g(s)] ds \\
&+ g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f(s) ds \\
=& \int_a^t f(s) \left[{}_s^R D_t^\alpha g(s) - g(t) \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \right] ds \\
&+ g(t) \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f(s) ds \\
=& \int_a^t f(s) [{}_s^R D_t^\alpha g(s)] ds \\
=& \int_a^t f(\tau) [{}_\tau^R D_t^\alpha g(\tau)] d\tau
\end{aligned}$$

The formul (2.11) can be demonstrated in the same way. □

Theorem 2.2. *Let* $0 < \alpha < 1$ *and* f, g *be two fairly regular functions on* $[a, b]$, *for all* $t \in (a, b)$. *We have*

$$\begin{aligned}
\int\limits_a^t [{}_a^C D_\tau^\alpha f(\tau)] g(\tau) d\tau =& \int\limits_a^t f(\tau) [{}_\tau^C D_t^\alpha g(\tau)] d\tau \\
&+ g(t) {}_a D_t^{1-\alpha} f(t) - f(a) {}_a D_t^{1-\alpha} g(a).
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\int\limits_t^b [{}_\tau^C D_b^\alpha f(\tau)] g(\tau) d\tau =& \int\limits_t^b f(\tau) [{}_t^C D_\tau^\alpha g(\tau)] d\tau \\
&+ g(t) {}_t D_b^{1-\alpha} f(t) - f(b) {}_t D_b^{1-\alpha} g(b).
\end{aligned} \tag{2.13}$$

In particular

$$\begin{aligned}
\int\limits_a^b [{}_a^C D_\tau^\alpha f(\tau)] g(\tau) d\tau =& \int\limits_a^b [{}_\tau^C D_b^\alpha f(\tau)] g(\tau) d\tau \\
&+ g(b) {}_a D_b^{1-\alpha} f(b) - f(a) {}_a D_b^{1-\alpha} g(a).
\end{aligned} \tag{2.14}$$

Proof.

$$\begin{aligned}\int_a^t [{}^C_a D^\alpha_\tau f(\tau)]g(\tau)d\tau &= \int_a^t [{}^R_a D^\alpha_\tau f(\tau)]g(\tau)d\tau - \frac{f(a)}{\Gamma(1-\alpha)}\int_a^t (\tau-a)^{-\alpha}g(\tau)d\tau \\ &= \int_a^t f(\tau)[{}^R_\tau D^\alpha_t g(\tau)]d\tau - f(a)_a D_t^{1-\alpha}g(a) \\ &= \int_a^t f(\tau)[{}^C_\tau D^\alpha_t g(\tau)]d\tau + \frac{g(t)}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha}f(\tau)d\tau \\ &\quad -f(a)_a D_t^{1-\alpha}g(a) \\ &= \int_a^t f(\tau)[{}^C_\tau D^\alpha_t g(\tau)]d\tau + g(t)_a D_t^{1-\alpha}f(t) \\ &\quad -f(a)_a D_t^{1-\alpha}g(a)\end{aligned}$$

The same is true for formula (2.13). □

2.4 Fractional Laplacian operator

The basic operator involved in this kind of problems is the so-called fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$.This section is devoted to the definition of this operator and to its properties.

2.4.1 Definition and properties

Can be viewed as a pseudo differential operator of symbol $\mid \xi \mid^{2s}$.

Definition 2.6. *[9]*
Let $s \in (0,1)$, and define the operator $(-\Delta)^s : S \to L^2(\mathbb{R}^{\ltimes})$ by

$$(-\Delta)^s u(x) := C(n,s) \lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}} dy \quad x \in \mathbb{R}^n, \tag{2.15}$$

where $B(x,\epsilon)$ is the ball centered at $x \in \mathbb{R}^n$ with radius ϵ, and $C(n,s)$is the following (positive) normalization constant

$$C(n,s) := \left(\int\limits_{\mathbb{R}^n} \frac{1-cos(\zeta_1)}{\mid \zeta \mid^{n+2s}} d\zeta \right)^{-1} \tag{2.16}$$

with $\zeta = (\zeta_1, \zeta'), \zeta' \in \mathbb{R}^{n-1}$.
The operator defined in (2.15) is the fractional Laplacian

we can write

$$(-\Delta)^s u(x) := C(n,s) P.V. \int\limits_{\mathbb{R}^n} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}} dy \quad x \in \mathbb{R}^n. \tag{2.17}$$

Proposition 2.6. *[9]*

Let $s \in (0,1)$*. Then, for any* $u \in S$*,*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n,s) \int\limits_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{\mid y \mid^{n+2s}} dy \quad x \in \mathbb{R}^n. \tag{2.18}$$

Proof.

By (2.17), we have that

$$(-\Delta)^s u(x) = -C(n,s) P.V. \int\limits_{\mathbb{R}^n} \frac{u(y)-u(x)}{\mid x-y \mid^{n+2s}} dy \quad x \in \mathbb{R}^n, \tag{2.19}$$

for every $x \in \mathbb{R}^n$. Hence, substituting $z = y - x$ in (2.19), it follows that

$$(-\Delta)^s u(x) = -C(n,s) P.V. \int\limits_{\mathbb{R}^n} \frac{u(z+x)-u(x)}{\mid z \mid^{n+2s}} dz \quad x \in \mathbb{R}^n, \tag{2.20}$$

for every $x \in \mathbb{R}^n$. However, by putting $z^* = -z$, one has

$$P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{\mid z \mid^{n+2s}} dz = P.V. \int\limits_{\mathbb{R}^n} \frac{u(x-z^*)-u(x)}{\mid z^* \mid^{n+2s}} dz^*. \tag{2.21}$$

So, after relabeling z^* as z, the following equalities hold:

$$\begin{aligned} 2P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{\mid z \mid^{n+2s}} dz =& P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{\mid z \mid^{n+2s}} dz \\ &+ P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{\mid z \mid^{n+2s}} dz \\ &+ P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+z)+u(x-y)-2u(x)}{\mid y \mid^{n+2s}} dy. \end{aligned} \tag{2.22}$$

Finally, a second-order Taylor expansion yields

$$\frac{u(x+y)+u(x-y)-2u(x)}{\mid y \mid^{n+2s}} \leq \frac{\parallel D^2 u \parallel_{L^\infty(\mathbb{R}^n)}}{\mid y \mid^{n+2s-2}},$$

and since $s \in (0,1)$, one has

$$\frac{u(x+y)+u(x-y)-2u(x)}{\mid y \mid^{n+2s}} \in L^1(\mathbb{R}^n).$$

Thus, for any $u \in S$, we have that

$$P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{\mid y \mid^{n+2s}} dy = \int\limits_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{\mid y \mid^{n+2s}} dy. \tag{2.23}$$

In conclusion, relation (2.18) holds thanks to (2.21)-(2.23). □

Remark 2.3. *Let $s \in (0, 1/2)$. Observe that for any $u \in S$ and for a fixed $x \in \mathbb{R}^n$, we have that*

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}} dy \leq & C \int_{B(x,R)} \frac{\mid x-y \mid}{\mid x-y \mid^{n+2s}} dy \\ & + \parallel u \parallel_{l^\infty(\mathbb{R}^n)} \int\limits_{\mathbb{R}^n \;\; B(x,\mathbb{R})} \frac{1}{\mid x-y \mid^{n+2s}} dy \\ & \leq C \left(\int\limits_0^R \frac{1}{\rho^{2s}} d\rho + \int\limits_R^{+\infty} \frac{1}{\rho^{2s+1}} d\rho \right) < +\infty, \end{aligned}$$

where C is a positive constant depending only on the dimension n and the L^∞-norm of the function u. Hence, in the case $s \in (0, 1/2)$, the integral

$$\int\limits_{\mathbb{R}^n} \frac{u(x)-u(y)}{\mid x-y \mid^{n+2s}} dy$$

is not singular near the point x, so one can get rid of the P.V. in (2.17).
Moreover, very recently, a new nonlocal and nonlinear operator (the fractional p-Laplacian $(-\Delta)^s_p$) was considered. Namely, for $p \in (1, +\infty), s \in (0, 1)$, and u smooth enough, it is defined as

$$(-\Delta)^s_p u(x) = P.V. \int\limits_{\mathbb{R}^n} \frac{\mid u(x)-u(y) \mid^{p-2} (u(x)-u(y))}{\mid x-y \mid^{n+sp}} dy \quad x \in \mathbb{R}^n.$$

Up to some normalization constant depending on n,p, and s, this definition is consistent with the one of the fractional Laplacian $(-\Delta)^s$ in the case $p = 2$.

2.4.2 The constant C(n, s)

Here we recall some properties of the constant $C(n, s)$.

Lemma 2.2.

[9] Let $s \in (0, 1)$ and $C(n, s)$ be the constant defined in (2.16), and let $A(n, s)$ and $B(n, s)$

be as follows:

$$A(n,s) := \begin{cases} 1 & if\ \ n=1 \\ \int\limits_{\mathbb{R}^{n-1}} \dfrac{1}{(1+\mid \eta' \mid^2)^{n+2s/2}} d\eta' & if\ \ n \geq 2, \end{cases}$$

and

$$B(s) := s(1-s) \int\limits_{\mathbb{R}} \frac{1-cost}{\mid t \mid^{1+2s}} dt.$$

Then

$$C(n,s) = \frac{s(1-s)}{A(n,s)B(s)}. \tag{2.24}$$

Proof.

The proof is based on direct computations. Let $n \geq 2$ and (ζ_1, ζ'), with $\zeta' \in \mathbb{R}^{n-1}$. By using the change of variables $\eta' = \zeta' / \mid \zeta_1 \mid$, one has

$$\begin{aligned} \int\limits_{\mathbb{R}^n} \frac{1-cos(\zeta_1)}{\mid \zeta \mid^{n+2s}} d\zeta &= \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^{n-1}} \frac{1-cos(\zeta_1)}{\mid \zeta_1 \mid^{n+2s}} \frac{1}{(1+\mid \zeta' \mid^2 / \mid \zeta_1' \mid^2)^{\frac{n+2s}{2}}} d\zeta' \right) d\zeta_1 \\ &= \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^{n-1}} \frac{1-cos(\zeta_1)}{\mid \zeta_1 \mid^{n+2s}} \frac{1}{(1+\mid \eta' \mid^2)^{\frac{n+2s}{2}}} d\eta' \right) d\zeta_1 \\ &= \frac{A(n,s)B(s)}{s(1-s)}. \end{aligned}$$

This completes the proof. □

Remark 2.4.

Note that by (2.16) and (2.4.2) it follows that

$$C(n,s)^{-1} \int\limits_{\mathbb{R}^n} \frac{1-cos(\zeta_1)}{\mid \zeta \mid^{n+2s}} d\zeta = A(n,s) \int\limits_{\mathbb{R}^n} \frac{1-cost}{\mid t \mid^{1+2s}} dt.$$

For $n \geq 3$, *denoting by* S_{n-2} *the Lebesgue measure of the unit sphere in* $\mathbb{R}^{n-1}$, *one has*

$$A(n,s) = \int\limits_{\mathbb{R}^{n-1}} \frac{1}{(1+\mid \eta' \mid^2)^{\frac{n+2s}{2}}} d\eta' < S_{n-2} \left(\frac{\pi}{4} + \frac{1}{1+2s} \right)$$

and

$$\int\limits_{\mathbb{R}^n} \frac{1-cost}{\mid t \mid^{1+2s}} dt < \frac{1}{(1-s)} + \frac{2}{s}.$$

Consequently, in this setting it is easy to see that

$$C(n,s)^{-1} < S_{n-2}\left(\frac{\pi}{4}+\frac{1}{1+2s}\right)\left(\frac{1}{2(1-s)}+\frac{2}{s}\right).$$

However, we also have $C(n,s)^{-1}<$

$$\begin{cases} \dfrac{1}{2(1-s)}+\dfrac{2}{s} & \textit{if } n=1 \\ \left(\dfrac{\pi}{4}+\dfrac{1}{1+2s}\right)\left(\dfrac{1}{2(1-s)}+\dfrac{2}{s}\right) & \textit{if } n=2. \end{cases}$$

The following property will be used in what follows.

Proposition 2.7.

Let $s \in (0,1)$ *and* $C(n,s)$ *be the constant defined in (2.16). Then, for any* $\zeta \in \mathbb{R}^n$, *the equality*

$$\int\limits_{\mathbb{R}^n} \frac{1-cos(\zeta.y)}{\mid y\mid^{n+2s}}dy = C(n,s)^{-1}\mid \zeta\mid^{2s} \tag{2.25}$$

holds.

Proof.

First, we observe that, for $\zeta=(\zeta_1,...,\zeta_n)$, we have

$$\frac{1-cos(\zeta_1)}{\mid \zeta\mid^{n+2s}} \leq \frac{\mid \zeta_1\mid^2}{\mid \zeta\mid^{n+2s}} \leq \frac{1}{\mid \zeta\mid^{n-2+2s}}$$

near the origin. As a consequence,

$$\int\limits_{\mathbb{R}} \frac{1-cos(\zeta_1)}{\mid \zeta\mid^{n+2s}}d\zeta$$

is finite and positive, also thanks to the choice of s.

Now let us define the map $J:\mathbb{R}^n \to \mathbb{R}$ as follows:

$$J(\zeta) := \int\limits_{\mathbb{R}^n} \frac{1-cos(\zeta.y)}{\mid y\mid^{n+2s}}dy, \tag{2.26}$$

for every $\zeta \in \mathbb{R}^n$. We claim that J is rotationally invariant; that is,

$$J(\zeta) = J(\mid \zeta\mid e_1) \quad \zeta \in \mathbb{R}^n, \tag{2.27}$$

where e_1 stands for the first direction vector on the space $\mathbb{R}^n$.

For $n=1$, equality (2.27) is trivial because J is an odd function. When $n \geq 2$, we consider a rotation $\Re$ for which $\Re(\mid \zeta \mid e_1) = \zeta$, and we denote by $\Re^T$ its transpose. Hence, by substituting $\widetilde{y}\ \Re^T y$, we obtain

$$\begin{aligned}
J(\zeta) &= \int\limits_{\mathbb{R}^n} \frac{1 - cos((\Re \mid \zeta \mid e_1)).y)}{\mid y \mid^{n+2s}} dy \\
&= \int\limits_{\mathbb{R}^n} \frac{1 - cos((\mid \zeta \mid e_1).(\Re^T y)}{\mid y \mid^{n+2s}} dy \\
&= \int\limits_{\mathbb{R}^n} \frac{1 - cos((\mid \zeta \mid e_1)).\ \widetilde{y})}{|\widetilde{y}|^{n+2s}} d\ \widetilde{y} \\
&= J(\mid \zeta \mid e_1),
\end{aligned}$$

so claim (2.27) is proved.

Then, by (2.27), the substitution $\zeta =\mid \zeta \mid y$ gives that

$$\begin{aligned}
J(\zeta) &= J(\mid \zeta \mid e_1) \\
&= \int\limits_{\mathbb{R}^n} \frac{1 - cos(\mid \zeta \mid y_1)}{\mid y \mid^{n+2s}} dy \\
&= \frac{1}{\mid \zeta \mid^n} \int\limits_{\mathbb{R}^n} \frac{1 - cos\zeta_1}{\mid \zeta / \mid \zeta \mid\mid^{n+2s}} d\zeta \\
&= C(n,s)^{-1} \mid \zeta \mid^{2s},
\end{aligned}$$

thanks to (2.16). In conclusion, relation (2.25) is proved. □

Corollary 2.1. *Let* $s \in (0,1)$ *and let* $C(n,s)$ *be the constant defined in (2.16). Then, for any* $u \in H^s(\mathbb{R}^n)$,

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \int\limits_{\mathbb{R}^n} \mid \zeta \mid^{2s} \mid \mathcal{F}u(\zeta) \mid^2 d\zeta. \tag{2.28}$$

Proof. Let us fix $y \in \mathbb{R}^n$. Using the change of variable $z = x - y$, and applying the Parseval-Plancherel formula, it follows that

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \frac{\mid u(x) - u(y) \mid^2}{\mid x - y \mid^{n+2s}} dx \right) dy &= \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \frac{u(z+y) - u(y)}{\mid z \mid^{n+2s}} dz \right) dy \\
&= \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \left| \frac{u(z+y) - u(y)}{\mid z \mid^{n/2+s}} \right|^2 dy \right) \\
&= \int\limits_{\mathbb{R}^n} \left\| \frac{u(z+\cdot) - u(\cdot)}{\mid z \mid^{n/2+s}} \right\| dz \\
&= \int\limits_{\mathbb{R}^n} \left\| \mathcal{F} \left(\frac{u(z+\cdot) - u(\cdot)}{\mid z \mid^{n/2+s}} \right) \right\|^2_{L^2(\mathbb{R}^n)} dz.
\end{aligned} \tag{2.29}$$

Elementary computations ensure that

$$\int\limits_{\mathbb{R}^n} \left\| \mathcal{F}\left(\frac{u(z+\cdot)-u(\cdot)}{\mid z\mid^{n/2+s}}\right)\right\|^2_{L^2(\mathbb{R}^n)} dz = \int\limits_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{\mid e^{i\zeta.z}-1\mid^2}{\mid z\mid^{n+2s}} \mid \mathcal{F}u(\zeta)\mid^2 d\zeta\right) dz$$
$$= 2 \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1-cos(\zeta.z)}{\mid z\mid^{n+2s}} \mid \mathcal{F}u(\zeta)\mid^2 d\zeta dz$$

so, by (2.25), we can write

$$\int\limits_{\mathbb{R}^n} \left\| \mathcal{F}\left(\frac{u(z+.)-u(.)}{\mid z\mid^{n/2+s}}\right)\right\|^2_{L^2(\mathbb{R}^n)} dz = 2C(n,s)^{-1} \int\limits_{\mathbb{R}^n} \mid \zeta\mid^{2s} \mathcal{F}u(\zeta)^2 d\zeta. \tag{2.30}$$

Hence, relation (2.28) follows by (2.29) and (2.30). □

2.4.3 The fractional Laplacian via Fourier transform

Here we prove that the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo differential operator of symbol $\mid \xi\mid^{2s}$.

Proposition 2.8. *[9]*
Let $s \in (0,1)$. Then, for any $u \in S$,

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(\mid \xi\mid^{2s} (\mathcal{F}(u))(\xi))(x), \quad x \in \mathbb{R}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Proof.
Denote by

$$Lu(x) := -\frac{1}{2}C(n,s) \int\limits_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{\mid y\mid^{n+2s}} dy \quad x \in \mathbb{R}^n, \tag{2.31}$$

where $C(n,s)$ is the constant given in (2.16). We look for a function $S : \mathbb{R}^n \to \mathbb{R}$ such that

$$Lu = \mathcal{F}^{-1}(S(Fu)). \tag{2.32}$$

We will prove that for every $\xi \in \mathbb{R}^n$,

$$S(\xi) = \mid \xi\mid^{2s} \tag{2.33}$$

since

$$\frac{u(x+y)+u(x-y)-2u(x)}{\mid y\mid^{n+2s}} \in L^1(\mathbb{R}^n \times \mathbb{R}^n),$$

by Fubini-Tonelli's result we can exchange the integral in y with the Fourier transform in x. Applying the Fourier transform in the variable x (2.32), we obtain

$$\begin{aligned} S(\xi)(Fu)(\xi) &= \mathcal{F}(Lu) \\ &= -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{F(u(x+y)+u(x-y)-2u(x))}{\mid y \mid^{n+2s}}dy \\ &= -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{e^{i\xi.y}+e^{-i\xi.y}-2}{\mid y \mid^{n+2s}}dy(Fu)(\xi) \\ &= -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{1-cos(\xi.y)}{\mid y \mid^{n+2s}}dy(Fu)(\xi). \end{aligned}$$

Necessarily the function S has the form given in (2.33), and this ends the proof. ☐

2.4.4 A generalization of $(-\Delta)^s$

[9] Here we introduce a general integro differential operator of nonlocal type that generalizes $(-\Delta)^s$. For any fixed $s \in (0,1)$, the operator L_k is given by

$$L_k u(x) := \int\limits_{\mathbb{R}^n} (u(x+y)+u(x-y)-2u(x))K(y)dy,$$

for every $x \in \mathbb{R}^n$, where the kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function with the properties that

$$mK \in L^1(\mathbb{R}^n), \quad where \quad m(x) = min\{\mid x \mid^2, 1\};$$

there exists $\theta > 0$ such that $K(x) \geq \theta \mid x \mid^{-(n+2s)}$, for any $x \in \mathbb{R}^n \setminus \{0\}$. Of course, as a model for K, we can take the function $\theta > 0$ such that

$$K(x) = \mid x \mid^{-(n+2s)}, \; x \in \mathbb{R}^n \setminus \{0\}.$$

In this case, up to some normalization constant, $L_k = -(-\Delta)^s$

Chaptre 3

Nonexistence solutions for evolution equations and systems with spatio-temporal fractional derivatives

we are concerned with finding sufficient conditions and necessary conditions for the solvability of evolution equations and systems with temporal and spatial fractional derivatives. In the first part, attention is paid to the evolution problem :

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t}(u) + (-\Delta)^{\beta/2}(u) = h(x,t) \mid u \mid^{1+\tilde{p}} & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ =: Q, \\ u(x,0) = u_0(x) \geq 0 & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Where $\mathbf{D}^{\alpha}_{0|t}$ order α in the sense of Caputo, $(-\Delta)^{\beta/2}, \beta \in (1,2)$, is the $(\beta/2)$ fractional power of the Laplacian $-\Delta_x$ in the x variable; the function $h(x,t)$ will be specified later. The exponent $\tilde{p}$ is strictly positive. In the case $\alpha = 1, \beta = 2$, the first equation in (3.1) reduces to the usual heat equation which is well documented.
In fact, in his pioneering article [5], Fujita considered the Cauchy problem:

$$\begin{cases} u_t = \Delta u + \mid u \mid^{1+\tilde{p}} & \text{in } Q, \\ u(x,0) = a(x) \geq 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{3.2}$$

Where $0 < \tilde{p}$. If $p_c := \frac{2}{n}$ (c for critical), he proved that:

1 If $0 < \tilde{p} < p_c$ and $a(x_0) > 0$ for some x_0, then any solution to (3.2) blows up in a finite time.

2 If $p > p_c$, then there exist solutions on Q as well as solutions which exist on $\mathbb{R}^n \times (0, T)$ for some finite T but not on Q. (For this p, not all solutions are global; indeed, if $\frac{1}{2}\int\limits_{\mathbb{R}^n} \mid \nabla u_0 \mid^2 dx - (\frac{1}{(p+1)})\int\limits_{\mathbb{R}^n} u_0^p dx < 0$).

3.1 Nonexistence results

Let $\mathbf{D}^\alpha$ the Caputo derivative and D^α the Riemann-Liouville derivative.

3.1.1 The case of sengle equation

Definition 3.1.

Let $p = \tilde{p} + 1$. *A function* $u \in L^1_{loc}\ (Q_T)(Q_T := \mathbb{R}^n \times (0, T))$ *is a local weak solution to (3.1) defined on* Q_T, *if* $uh^{1/p} \in L^p_{loc}$ *and is such that*

$$\int\limits_{Q_T} u_0(x) D^\alpha_{t|T}\varphi(x,t)dxdt + \int\limits_{Q_T} h \mid u \mid^p \varphi dxdt = \int\limits_{Q_T} u(-\Delta)^{\beta/2}\varphi dxdt + \int\limits_{Q_T} u D^\alpha_{t|T}\varphi dxdt \tag{3.3}$$

for any test function $\varphi \in C^{2,1}_{x,t}(Q_T)$, *such that* $\varphi(x, T) = 0$.

Remark 3.1. *The integrals in the above definition are supposed to be convergent. If in the definition* $T = +\infty$, *the solution is called global. Concerning the function* $h(x,t)$ *we require the condition :*

$$h(x,t) \geqslant C_h \mid x \mid^\sigma t^\rho \text{ for } x \in \mathbb{R}^n, \rho \text{ and } \sigma \text{ are constant } t > 0, C_h > 0. \tag{3.4}$$

Theorem 3.1.

Let $N \geqslant 1$ *and* $p > 1$. *Assume that (3.4) is satisfied. If*

$$1 < p \leqslant p_c = \frac{1 + \alpha(\beta + \sigma) + \beta\rho}{\alpha n + \beta(1 - \alpha)},$$

then the problem (3.1) admits no global weak nonnegative solutions other than the trivial one.

Proof. The proof by contradiction. Suppose that u is a nontrivial nonnegative solution which exists globally in time. That is u exists in $(0, T^*)$ for any arbitrary $T^* > 0$. Let T and R be two positive real numbers such that $0 < TR^{\beta\alpha} < T^*$. For later use, let Φ be a smooth non-increasing function such that

$$\Phi(z) = \begin{cases} 1 & if\ z \leqslant 1, \\ 0 & if\ z \geqslant 2, \end{cases}$$

and $0 \leqslant \Phi \leqslant 1$.

The test function φ is chosen so that

$$\int_{Q_T} \mid (-\Delta)^{\beta/2}\varphi \mid^{p'} (h\varphi)^{-p'/p} < \infty, \quad \int_{Q_T} \mid D^{\alpha}_{t|T}\varphi \mid^{p'} (h\varphi)^{-p'/p} < \infty. \tag{3.5}$$

To estimate the right-hand side of (3.3) on $Q_{TR^{2/\theta}}$, we write

$$\int_{Q_{TR^{2/\theta}}} u(-\Delta)^{\beta/2}(\varphi) = \int_{Q_{TR^{2/\theta}}} u(h\varphi)^{1/p}(-\Delta)^{\beta/2}(\varphi)(h\varphi)^{-1/p}.$$

Using the ε-Young inequality

$$XY \leqslant \varepsilon X^p + C(\varepsilon)Y^{p'},\ p + p' = pp',\ X \geqslant 0, Y \geqslant 0,$$

we have the estimate

$$\int_{Q_{TR^{2/\theta}}} u(-\Delta)^{\beta/2}\varphi \leqslant \varepsilon \int_{Q_{TR^{2/\theta}}} \mid u \mid^p h\varphi + C(\varepsilon) \int_{Q_{TR^{2/\theta}}} \mid (-\Delta)^{\beta/2}\varphi \mid^{p'} (h\varphi)^{-p'/p}.$$

Similarly,

$$\int_{Q_{TR^{2/\theta}}} uD^{\alpha}_{t|TR^{2/\theta}}\varphi \leqslant \varepsilon \int_{Q_{TR^{2/\theta}}} \mid u \mid^p h\varphi + C(\varepsilon) \int_{Q_{TR^{2/\theta}}} \mid D^{\alpha}_{t|TR^{2/\theta}}\varphi \mid^{p'} (h\varphi)^{-p'/p}.$$

Now, taking ε small enough, we obtain the estimate

$$\int_{Q_{TR^{2/\theta}}} h \mid u \mid^p \varphi \leqslant C(\varepsilon) \int_{Q_{TR^{2/\theta}}} \left\{ \left|(-\Delta)^{\beta/2}\varphi\right|^{p'} + \left| D^{\alpha}_{t|TR^{2/\theta}}\varphi^{p'} \right| \right\} (h\varphi)^{-p'/p}. \tag{3.6}$$

At this stage, we set

$$\varphi(x,t) := \Phi\left(\frac{\mid x \mid^2 + t^\theta}{R^2}\right),$$

where R and θ are positive real numbers.

Let us perform the change of variables

$$\tau = t/R^{2/\theta}, \quad y = x/R,$$

and set

$$\Omega := \left\{(y,\tau)\in\mathbb{R}^n\times\mathbb{R}^+, \mid y\mid^2+\tau^\theta<2\right\}, \quad \mu(y,\tau):=\tau^\theta+\mid y\mid^2.$$

Now, we choose θ such that the right-hand sides of

$$\begin{aligned}&\int\limits_{Q_{TR^{2/\theta}}} \mid(-\Delta)^{\beta/2}\varphi\mid^{p'}(h\varphi)^{-p'/p}\\ &\leqslant R^{-\beta p'+n+\frac{2}{\theta}-\frac{p'}{p}(\sigma+\frac{2\rho}{\theta})}\int\limits_{\Omega}\mid(-\Delta)^{\beta/2}\Phi\circ\mu\mid^{p'}(C_h\mid y\mid^\sigma\tau^\rho\Phi\circ\mu)^{-p'/p}dyd\tau\end{aligned}$$

and

$$\begin{aligned}&\int\limits_{Q_{TR^{2/\theta}}} \mid D^\alpha_{t|TR^{2/\theta}}\varphi\mid^{p'}(h\varphi)^{-p'/p}\\ &\leqslant R^{-\frac{2}{\theta}\alpha p'+n+\frac{2}{\theta}-\frac{p'}{p}(\sigma+\frac{2\rho}{\theta})}\int\limits_{\Omega}\mid D^\alpha_{\tau|T}\Phi\circ\mu\mid^{p'}(C_h\mid y\mid^\sigma\tau^\rho\Phi\circ\mu)^{-p'/p}dyd\tau\end{aligned}$$

are of the same order in R. In doing so, we find $\theta=\frac{2\alpha}{\beta}$.
We then have the estimate

$$\int\limits_{Q_{TR^{\beta/\alpha}}} h\mid u\mid^p\varphi\leqslant CR^\gamma, \tag{3.7}$$

where

$$\gamma=-\beta p'+n+\frac{\beta}{\alpha}-\left(\sigma+\frac{\rho\beta}{\alpha}\right)\frac{p'}{p}$$

and

$$C=C(\varepsilon)\int\limits_{\Omega}\left(\mid(-\Delta)^{\beta/2}\Phi\circ\mu\mid^{p'}+D^\alpha_{\tau|T}\Phi\circ\mu\mid^{p'}\right)(C_h\mid y\mid^\sigma\tau^\rho\Phi\circ\mu)^{-p'/p}dyd\tau.$$

Now, if we choose $\gamma<0$ (that is $p<p_c$) and let $R\to\infty$ in (3.7), we obtain

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^+} h\mid u\mid^p\leqslant 0. \tag{3.8}$$

This implies that $u=0$ a.e., which is a contradiction. In case $\gamma=0(i.e., p=p_c)$, observe that (because of the convergence of the integral in (3.7)) if

$$C_R=\left\{(x,t)\mathbb{R}^n\times\mathbb{R}^+ : R^2<\mid x\mid^2+t^\theta\leqslant 2R^2\right\},$$

then

$$\lim_{R\longrightarrow\infty}\int\limits_{C_R}\mid u\mid^p h\varphi dxdt=0. \tag{3.9}$$

If instead of using the ε-Young inequality, we rather use the Hölder inequality, then instead of estimate (3.6), we find

$$\int_{Q_{TR^{\beta/\alpha}}} h \mid u \mid^p \varphi dxdt \leqslant L \left(\int_{C_R} \mid u \mid^p h\varphi dxdt \right)^{1/p}, \tag{3.10}$$

where

$$L := \left(\int_{\Omega_1} \mid D^{\alpha}_{\tau|T} \Phi \circ \mu \mid^{p'} (C_h \mid y \mid^{\sigma} \tau^{\rho} \Phi \circ \mu)^{-p'/p} dyd\tau \right)^{1/p'}$$
$$\left(\int_{\Omega_1} \mid (-\Delta^{\beta/2}) \Phi \circ \mu \mid^{p'} (C_h \mid y \mid^{\sigma} \tau^{\rho} \Phi \circ \mu)^{-p'/p} dyd\tau \right)^{1/p'}$$

and

$$\Omega_1 = \left\{ (y,\tau) \in \mathbb{R}^n \times \mathbb{R}^+ : 1 \leqslant \mid y \mid^2 + \tau^{\theta} \leqslant 2 \right\}.$$

Using (3.10), we obtain via (3.9), after passing to the limit as $R \to \infty$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^+} \mid u \mid^p \; h \, dxdt = 0.$$

This leads to $u = 0$ a.e. and completes the proof. □

Remark 3.2.

The requirement $\gamma \leqslant 0$, i.e.,

$$p \leqslant 1 + \frac{\alpha(\beta+\sigma) + \beta\rho}{\alpha n + \beta(1-\alpha)}$$

provides us with a critical exponent which coincides with the well-known Fujita exponent in case $\sigma = \rho = 0, \alpha = 1$ and $\beta = 2$.

Remark 3.3.

The analysis could be performed for more general highly nonlinear equations such as

$$D^{\alpha}_{0|t}(u - u_0) + (-\Delta)^{\beta/2}(\mid u \mid^{m-1} u) + a(x) \cdot \nabla \left(\mid u \mid^{q-1} u \right) = h(x,t) \mid u \mid^p .$$

It works also for other more general problems.

3.1.2 The case Systems of two equations

We show how the method of proof used for the case of one equation can be carried out for the system of reaction-diffusion equations :

$$\begin{cases} D^{\alpha}_{0|t}(u - u_0) + (-\Delta)^{\beta/2} u = \mid v \mid^p & \text{in } Q, \\ D^{\delta}_{0|t}(v - v_0) + (-\Delta)^{\gamma/2} v = \mid u \mid^q & \text{in } Q, \end{cases} \tag{3.11}$$

subject to the initial conditions

$$u(x,0)=u_0(x)\geqslant 0,\quad v(x,0)=v_0(x)\geqslant 0,\quad x\in\mathbb{R}^n,$$

where $0<\alpha,\delta<1\leqslant\gamma,\beta\leqslant 2$.

For simplicity, in system (3.11) the reaction terms are taken equal to $\mid v\mid^p$ and $\mid u\mid^q$. Our analysis holds good for reaction terms of the form $f(t,x)\mid v\mid^p$ and $g(t,x)\mid u\mid^q$ the functions f and g are assumed to satisfy the conditions

$$f(t,x)\geqslant C_1t^{\omega_1}\mid x\mid^{d_1},\quad g(t,x)\geqslant C_2t^{\omega_2}\mid x\mid^{d_2}$$

for $t>0, x\gg 1, \omega_1\geqslant 0, \omega_2\geqslant 0, d_1\geqslant 0, d_2\geqslant 0$.

For the system (3.11), we have

Theorem 3.2.

Let $p>1, q>1$. Assume that

$$n\leqslant max\left\{\frac{\frac{\delta}{q}+\alpha-(1-\frac{1}{pq})}{\frac{\delta}{\gamma qp'}+\frac{\alpha}{\beta q'}},\frac{\frac{\alpha}{p}+\delta-(1-\frac{1}{pq})}{\frac{\alpha}{\beta qp'}+\frac{\delta}{\gamma p'}}\right\}.$$

Then, the system (3.11) (with the initial data) does not admit nontrivial global weak nonnegative solutions.

Proof.

Here again the proof proceeds by contradiction. Therefore, let

$$\xi_j(x,t)=\varPhi\left(\frac{t^2+\mid x\mid^{2\theta_j}}{R^2}\right), j=1,2,$$

where $R>0, \theta_1=\beta/\alpha$ and $\theta_2=\gamma/\delta$. The weak formulation of solutions to system (3.11) reads as

$$\int\limits_{Q_{TR}}\mid v\mid^p\xi_1+\int\limits_{Q_{TR}}u_0(x)D^{\alpha}_{t|TR}\xi_1=\int\limits_{Q_{TR}}uD^{\alpha}_{t|TR}\xi_1+\int\limits_{Q_{TR}}u(-\Delta)^{\beta/2}\xi_1$$

and

$$\int\limits_{Q_{TR}}\mid u\mid^q\xi_2+\int\limits_{Q_{TR}}v_0(x)D^{\delta}_{t|TR}\xi_2=\int\limits_{Q_{TR}}vD^{\delta}_{t|TR}\xi_2+\int\limits_{Q_{TR}}v(-\Delta)^{\gamma/2}\xi_2.$$

Using the Hölder inequality, we may write

$$\int\limits_{Q_{TR}}u\mid D^{\alpha}_{t|TR}\xi_1\mid\leqslant\left(\int\limits_{Q_{TR}}\mid u\mid^q\xi_2\right)^{1/q}\cdot\left(\int\limits_{Q_{TR}}\mid D^{\alpha}_{t|TR}\xi_1\mid^{q'}\xi_2^{-q'/q}\right)^{1/q'}$$

and

$$\int\limits_{Q_{TR^2}} u \mid (-\Delta)^{\beta/2}\xi_1 \mid \leqslant \left(\int_{Q_{TR^2}} \mid u \mid^q \xi_2\right)^{1/q} . \left(\int\limits_{Q_{TR^2}} \mid (-\Delta)^{\beta/2}\xi_1 \mid^{q'} \xi_2^{-q'/p}\right)^{1/q'} ;$$

consequently

$$\int\limits_{Q_{TR^2}} \mid v \mid^p \xi_1 \leqslant \left(\int\limits_{Q_{TR^2}} \mid u \mid^q \xi_2\right)^{1/q} .A \tag{3.12}$$

with

$$A = \left(\int\limits_{Q_{TR^2}} \mid D^{\alpha}_{t|TR}\xi_1 \mid^{q'} \xi_2^{-q'/p}\right)^{1/q'} + \left(\int_{Q_{TR^2}} \mid (-\Delta)^{\beta/2}\xi_1 \mid^{q'} \xi_2^{-q'/p}\right)^{1/q'} .$$

Similarly, we obtain the estimate

$$\int\limits_{Q_{TR^2}} \mid u \mid^q \xi_2 \leqslant \left(\int\limits_{Q_{TR^2}} \mid v \mid^p \xi_1\right)^{1/P} .B \tag{3.13}$$

with

$$B := \left(\int\limits_{Q_{TR^2}} \mid D^{\delta}_{t|TR}\xi_1 \mid^{p'} \xi_1^{-p'/p}\right)^{1/p'} + \left(\int_{Q_{TR^2}} \mid (-\Delta)^{\gamma/2}\xi_2 \mid^{p'} \xi_2^{-p'/p}\right)^{1/p'} .$$

Using inequalities (3.12) and (3.13), we may

$$\left(\int\limits_{Q_{TR^2}} \mid v \mid^p \xi_1\right)^{1-\frac{1}{pq}} \leqslant B^{1/p}.A \tag{3.14}$$

$$\left(\int\limits_{Q_{TR^2}} \mid u \mid^q \xi_1\right)^{1-\frac{1}{pq}} \leqslant B.A^{1/p}. \tag{3.15}$$

Now, in A, we use the variables (τ, y) defined by

$t = R\tau$ and $x = R^{\alpha/\beta}y$,

while in B, we use the variables (τ, y) defined by $t = R\tau$ and $x = R^{\delta/\gamma}y$. We then have the estimate

$$\left(\int\limits_{Q_{TR^2}} \mid v \mid^p \xi_1\right)^{1-\frac{1}{pq}} \leqslant CR^{-l_1{}^{1/q}}R^{l_2}, \tag{3.16}$$

where

$$l_1 = \delta - \frac{1}{p'}\left(n\frac{\delta}{\gamma}+1\right), \quad l_2 = \alpha - \frac{1}{q'}\left(n\frac{\alpha}{\beta}+1\right).$$

That is,

$$\left(\int\limits_{Q_{TR^2}} \mid v \mid^p \xi_1\right)^{1-\frac{1}{pq}} \leqslant CR^{(-\ell_1/q+\ell_2)}. \tag{3.17}$$

Next, we argue as in the case of a single equation (see the argument below formula (3.7) till the end of the proof) in case$-l_1/q+l_2 \geqslant 0$. Note the requirement $-l_1/q+l_2 \geqslant 0$ is equivalent to

$$n \leqslant \frac{\frac{\delta}{q}+\alpha-(1-\frac{1}{pq})}{\frac{\delta}{\beta pq'}+\frac{\alpha}{\gamma p'}}. \tag{3.18}$$

Using (3.15), we obtain, in a similar manner, the estimate

$$n \leqslant \frac{\frac{\alpha}{q}+\delta-(1-\frac{1}{pq})}{\frac{\alpha}{\beta pq'}+\frac{\delta}{\gamma p'}}. \tag{3.19}$$

Observe that either (3.18) or (3.19) is needed to obtain a contradiction, so it suffices to assume

$$1 \leqslant n \leqslant \max\left\{\frac{\frac{\delta}{q}+\alpha-(1-\frac{1}{pq})}{\frac{\delta}{\beta pq'}+\frac{\alpha}{\gamma p'}}, \frac{\frac{\alpha}{q}+\delta-(1-\frac{1}{pq})}{\frac{\alpha}{\beta pq'}+\frac{\delta}{\gamma p'}}\right\}.$$

□

The case where f and g satisfy the above hypotheses may be proved easily along the lines above and the case of a single equation as in the proof of Theorem 3.1.

Remark 3.4.

When $\alpha=\delta=1, \beta=\gamma=2$,we recover the case studied by Escobedo and Herrero, however we have to impose the constraint $p>1, q>1$ while Escobedo and Herrero require$pq>1$.

Remark 3.5.

It is clear that the more general system

$$\begin{cases} D^{\alpha}_{0|t}(u-u_0)+(-\Delta)^{\beta/2}(\mid u \mid^{m-1} u) = h(x,t) \mid v \mid^p + g(x,t) \mid u \mid^r & \textit{in } Q, \\ D^{\delta}_{0|t}(v-v_0)+(-\Delta)^{\gamma/2}(\mid v \mid^{m-1} v) = k(x,t) \mid u \mid^q + l(x,t) \mid v \mid^s & \textit{in } Q, \end{cases}$$

could be analyzed with the same method.

The analysis, here performed, can be used to study systems of convective equations as those, for example, considered by Ames and Straughan. Here, we preferred less general situations to render the ideas as clear as possible.

3.2 Necessary conditions for local and global existence

This part is concerned with the establishment of necessary conditions for the existence of local (as well as global) solutions to problems (3.1) and (3.11). It turns out that these conditions depend on the behavior of the initial data and on the function $h(x,t)$ ($f(x,t)$ and $g(x,t)$ in case of (3.11)) for large x. Previous results concerning the problem

$$\begin{cases} u_t = \Delta u + \tilde{h}(x) \mid u \mid^p & \text{in } Q, \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } \mathbb{R}^n \end{cases} \tag{3.20}$$

In particular, it is showed in that no local weak nonnegative solution to (3.20) exists if the initial data u_0 satisfies

$$\lim_{|x|\longrightarrow\infty} u_0^{p-1}\tilde{h}(x) = +\infty,$$

and any possible local weak nonnegative solution blows up at a finite time if

$$\lim_{|x|\longrightarrow\infty} u_0^{p-1}\tilde{h}(x) \mid x \mid^2 = +\infty.$$

The method developed there is adapted below to the problem (3.1) with, for simplicity $h(x,t) \equiv h(x)$; it will be clear that it can be used for the reaction-diffusion system (3.11). We shall treat the case of a single equation.

Theorem 3.3.
Let u be a local solution to problem (3.1) where $T < +\infty$. Then we have the estimate

$$\liminf_{|x|\longrightarrow\infty} \left[u_0(x)(h(x))^{p'/p}\right] \leqslant CT^{\alpha(1-p')}$$

for some positive constant C.

Proof.
Let us consider the following test function:

$$\varphi(x,t) = \Phi\left(\frac{x}{R}\right) \begin{cases} (1-\dfrac{t}{T})^l, & 0 < t \leqslant T, \\ 0, & t > T, \end{cases}$$

where $\Phi \in W^{1,\infty}(\mathbb{R}^n)$ is nonnegative with $\operatorname{supp}\Phi \subset \{1 <\mid x \mid< 2\}$ (supp stands for support) and satisfy

$$\left((-\Delta)^{\beta/2}\Phi\right)_+ \leqslant k\Phi \text{ for some constant } k > 0.$$

The exponent l is any positive real number if $p \geqslant 1/(1-\alpha)$ and $l > \alpha p'-1$ if $p < 1/(1-\alpha)$. We have

$$D^{\alpha}_{t|T}\left(1-\frac{t}{T}\right)^{l} = \Lambda T^{-\alpha}\left(1-\frac{t}{T}\right)^{l-\alpha},$$

where $\Lambda := \Gamma(1+l)/\Gamma(1+l-\alpha)$.

Using the formulation (3.3) and a similar argument to the one which lead us to (3.6) but keeping the first term in the left-hand side of (3.3), we obtain

$$\int\limits_{Q_T} u_0 D^{\alpha}_{t|T}\varphi(x,t) \leqslant C\int\limits_{Q_T}\left\{(D^{\alpha}_{t|T}\varphi)^{p'}_{+} + \left((-\Delta)^{\beta/2}\varphi\right)^{p'}_{+}\right\}(h\varphi)^{1-p'} \tag{3.21}$$

for some positive constant C. Taking into account the hypotheses on l and the fact that

$$D^{\alpha}_{t|T}\varphi(x,t) = \Lambda\Phi(x)T^{-\alpha}\left(1-\frac{t}{T}\right)^{1-\alpha},$$

if we put $t = T\tau$ and $x = Ry$ in (3.21), we obtain

$$\begin{aligned} &T^{1-\alpha}\int\limits_{\mathbb{R}^n} u_0(Ry)\Phi(y) \\ &\quad\leqslant CT^{1-\alpha p'}\int\limits_{\mathbb{R}^n}\Phi(y)h^{1-p'}(Ry) + CT^{-\beta p'}\int\limits_{\mathbb{R}^N}\Phi(y)h^{1-p'}(Ry). \end{aligned} \tag{3.22}$$

Using the estimate

$$\inf_{|y|>1}\left(u_0(Ry)h(Ry)^{p'-1}\right)\int\limits_{\mathbb{R}^n}\Phi(y)h(Ry)^{1-p'} \leqslant \int\limits_{\mathbb{R}^n} u_0(Ry)\Phi(y).$$

in inequality (3.22) and dividing by the term $\int\limits_{\mathbb{R}^n} u_0(Ry)\Phi(y)$. we obtain

$$\inf_{|y|>1}\left(u_0(Ry)h(Ry)^{p'-1}\right) \leqslant C(T^{-\alpha(p'-1)} + T^{\alpha}R^{-\beta p'}). \tag{3.23}$$

Passing to the limit as $R \to +\infty$, we get

$$\liminf_{|x|\longrightarrow\infty}\left(u_0(x)h(x)^{p'-1}\right) \leqslant CT^{-\alpha(p'-1)}. \tag{3.24}$$

□

Corollary 3.1.

Assume that problem (3.1) has a nontrivial global nonnegative weak solution. Then

$$\liminf_{|x|\longrightarrow\infty}\left(u_0(x)h(x)^{p'-1}\right) = 0.$$

Corollary 3.2.

If $\liminf\limits_{|x|\longrightarrow\infty}\left(u_0(x)h(x)^{p'-1}\right) = +\infty$, *then problem (3.1) can not have any local nontrivial nonnegative weak solution.*

Corollary 3.3.

If $A := \liminf\limits_{|x|\longrightarrow\infty}\left(u_0(x)h(x)^{p'-1}\right) > 0$, *then* $T^{\alpha(p'-1)} \leqslant C/A$, *where* C *is the constant found in Theorem 3.3.*

Theorem 3.4.

Suppose that problem (3.1) has a nontrivial global nonnegative weak solution. Then, there is a positive constant K *such that*

$$\liminf_{|x|\longrightarrow\infty}\left(u_0(x)\mid x\mid^{\alpha-(p'-1)} h(x)^{p'-1}\right) \leqslant K.$$

Proof.

In the relation

$$T^{1-\alpha}\int\limits_{\mathbb{R}^n} u_0(Ry)\Phi(y) \leqslant C\left(T^{-\alpha(p'-1)} + T^{\alpha}R^{-\beta p'}\right)\int\limits_{\mathbb{R}^n}\Phi(y)h^{1-p'}(Ry)$$

found in the proof of Theorem 3.1, we multiply by the expression

$$h^{p'-1}(Ry)\mid Ry\mid^{\alpha(p'-1)} .h^{1-p'}(Ry)\mid Ry\mid^{\alpha(1-p')}$$

inside the integral in the left-hand side and by $\mid Ry\mid^{\alpha(p'-1)} .h^{1-p'}(Ry)$ inside the integral in the right-hand side. We obtain for Φ with $supp\Phi \subset \{x : R <\mid x\mid< 2R\}$,

$$\inf_{|x|>R}\left(u_0(x)\mid x\mid^{\alpha(p'-1)} h(x)^{p'-1}\right)\int\limits_{\mathbb{R}^n}\Phi(y)\mid(Ry)\mid^{\alpha(1-p')} h^{1-p'}(Ry)$$
$$\leqslant C(T^{-\alpha(p'-1)} + T^{\alpha}R^{-\beta p'})(2R)^{\alpha(p'-1)}\int\limits_{\mathbb{R}^n}\Phi(y)\mid(Ry)\mid^{\alpha(1-p')} h^{1-p'}(Ry).$$

Finally, dividing by

$$\int\limits_{\mathbb{R}^n}\Phi(y)\mid Ry\mid^{\alpha(1-p')} h^{1-p'}(Ry)$$

and taking $T = R$, we end up with

$$\inf_{|x|>R}\left(u_0(x)\mid x\mid^{\alpha(p'-1)} h(x)^{p'-1}\right) \leqslant C(1 + R^{(\alpha-\beta)p'}).$$

The conclusion follows by passing to the limit and noticing that $\alpha < \beta$. □

Combining the argument in the proof of Theorem 3.2 with those in the previous two theorems, we obtain similar results (necessary conditions for local existence and for global existence) as those in the previous two theorems and their corollaries for the case of system (3.11).

Chaptre 4

The problem for a nonlocal nonlinear parabolic equation

In this chaptre, we investigate the Cauchy problem for a nonlocal nonlinear parabolic equation

The form Cauchy problem:

$$\begin{cases} u_t + (-\Delta)^{\beta/2}(\mid u \mid^p) = \mid u \mid^q & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $u_0 \in L^1_{loc}(\mathbb{R}^n), n \geq 1, 0 < \beta \leq 2, p > 0, q > 1$, and the nonlocal operator $(-\Delta)^{\beta/2}$ is defined by

$$(-\Delta)^{\beta/2} v(x) := \mathcal{F}^{-1}(\mid \xi \mid^\beta \mathcal{F}(v)(\xi))(x)$$

for every $v \in D((-\Delta)^{\beta/2}) = H^\beta(\mathbb{R}^n)$, where $H^\beta(\mathbb{R}^n)$ is the homogeneous Sobolev space of order β defined by

$$H^\beta(\mathbb{R}^n) = \{u \in S; (-\Delta)^{\beta/2})u \in L^2(\mathbb{R}^n)\} \;\; if \;\; \beta \notin \mathbb{N},$$

$$H^\beta(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); (-\Delta)^{\beta/2})u \in L^2(\mathbb{R}^n)\} \;\; if \;\; \beta \in \mathbb{N},$$

where S is the space of Schwartz distributions; $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for its inverse, Γ is the Euler gamma function.

Lemma 4.1. *In dimension $n \geq 1$ where $\beta \in [0,2]$ and $q \geq 1$, for all nonnegative Schwartz functions φ (in the general case)*

$$(-\Delta)^{\beta/2}\varphi^q \leq q\varphi^{q-1}(-\Delta)^{\beta/2}\varphi.$$

Proof. We give a proof of Ju's inequality. The cases $\beta = 0$ and $\beta = 2$ are obvious, as well as $q = 1$. If $\beta \in (0,2)$ and $q > 1$, we have

$$(-\Delta)^{\beta/2}\varphi(x) = -c_n(\beta/2)\int\limits_{\mathbb{R}^n} \frac{\varphi(x+z)-\varphi}{\mid z \mid^{n+\beta}}dz, forall x \in \mathbb{R}^n,$$

where $c_n = (\dfrac{2^\beta\Gamma(\frac{(n+\beta)}{2})}{(\pi^{\frac{n}{2}}\Gamma(1-\beta/2)})$.Then

$$(\varphi(x))^{q-1}(-\Delta)^{\beta/2}\varphi(x) = -c_n(\beta)\int\limits_{\mathbb{R}^n} \frac{(\varphi(x))^{q-1}\varphi(x+z)-(\varphi(x))^q}{\mid z \mid^{n+\beta}}dz.$$

By Young's inequality we have

$$(\varphi(x))^{q-1}\varphi(x+z) \leq \frac{q-1}{q}(\varphi(x))^q + \frac{1}{q}(\varphi(x+z))^q.$$

Therefore,

$$(\varphi(x))^{q-1}(-\Delta)^{\beta/2}\varphi(x) \geq \frac{^{-}C_n(\beta)}{q}\int\limits_{\mathbb{R}^n} \frac{(\varphi(x+z))^q-(\varphi)^q}{\mid z \mid^{n+\beta}}dz = \frac{1}{q}(-\Delta)^{\beta/2}(\varphi(x))^q.$$

□

Definition 4.1. *(Weak solution).*
Let $u_0 \in L^1_{loc}(\mathbb{R}^n), 0 < \beta \leq 2$ and $T > 0$. We say that u is a weak solution of the problem (4.1) if $u \in L^p((0,T), L^{2p}(\mathbb{R}^n)) \cap L^q((0,T), L^q_{loc}(\mathbb{R}^n))$ and satisfies the equation

$$\begin{aligned}\int\limits_{\mathbb{R}^n} u_0(x)\varphi(x,0)dx + \int\limits_0^T\int\limits_{\mathbb{R}^n} \mid u \mid^q \varphi(x,t)dxdt &= \int\limits_0^T\int\limits_{\mathbb{R}^n} \mid u \mid^p (-\Delta)^{\beta/2}\varphi dxdt \\ &- \int\limits_0^T\int\limits_{\mathbb{R}^n} u\varphi_t(x,t)dxdt,\end{aligned} \tag{4.2}$$

for all compactly supported $\varphi \in C([0,T], H^\beta(\mathbb{R}^n)) \cap C^1([0,T], C(\mathbb{R}^n))$ such that $\varphi(.,T) = 0$.

4.1 The first result

Theorem 4.1.

For $u_0 \in L^1_{loc}(\mathbb{R}^n), u_0 \geq 0$, *if*

$$p < q \leq p + \frac{\beta}{n},$$

then problem (4.1) has no nontrivial global weak solutions.

Proof. Theorem 4.1

The proof is by contradiction. Suppose that u is a global weak solution to (4.1), then, for all $T \gg 1$, we have

$$\begin{aligned}\int\limits_{\mathbb{R}^n} u_0(x)\varphi(x,0)dx + \int\limits_0^T\int\limits_{\mathbb{R}^n} \mid u \mid^q \varphi(x,t)dxdt \ &= \int\limits_0^T\int\limits_{\mathbb{R}^n} \mid u \mid^p (-\Delta)^{\beta/2}\varphi dxdt \\ &- \int\limits_0^T\int\limits_{\mathbb{R}^n} u\varphi_t(x,t)dxdt,\end{aligned}$$

for all test function $\varphi \in C([0,T], H^\beta(\mathbb{R}^n)) \cap C^1([0,T], C(\mathbb{R}^n))$ such that *supp*φ is compact with $\varphi(.,T) = 0$.

Now we take $\varphi(x,t) := \varphi_1^\ell(x)\varphi_2^\eta(t)$ *with* $\varphi_1(x) := \Phi\left(\frac{\mid x \mid}{T^\alpha}\right), \varphi_2(t) := \Phi\left(\frac{t}{T}\right)$, where $\alpha = \frac{q-p}{\beta(q-1)}, \ell, \eta \gg 1$ and Φ a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq \ r \leq \dfrac{1}{2}, \\ 0 & \text{if } r \geq 1, \end{cases}$$

$0 \leq \Phi \leq 1, \mid \Phi'(r) \mid \leq \frac{C_1}{r}$, for all $r > 0$. We have

$$\begin{aligned}\int\limits_{\Omega} u_0(x)\varphi_1^\ell(x)dx + \int\limits_0^T\int\limits_{\Omega} \mid u \mid^q \varphi(x,t)dxdt &= \int\limits_0^T\int\limits_{\mathbb{R}^n} \mid u \mid^p \varphi_2^\eta(t)(-\Delta)^{\beta/2}(\varphi_1^\ell(x))dxdt \\ &- \int\limits_0^T\int\limits_{\Omega} u\varphi_1^\ell(x)\frac{d}{dt}\varphi_2^\eta(t)dxdt,\end{aligned}$$

where

$$\Omega := \{x \in \mathbb{R}^n; \mid x \mid \leq T^\alpha\}.$$

Using Ju's inequality $(-\Delta)^{\beta/2}(\varphi_1^\ell) \leq \ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}(\varphi_1)$ (see 4.1), and the fact that $u_0 \geq 0$, we obtain

$$\begin{aligned}
&\int\limits_0^T \int\limits_\Omega \mid u \mid^q \varphi(x,t)dxdt \\
&\leq C \int\limits_0^T \int\limits_\Omega \mid u \mid^p \varphi_2^\eta(t)\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x))dxdt \\
&+ C \int\limits_0^T \int\limits_\Omega \mid u \mid \varphi_1^\ell(x)\varphi_2^{\eta-1}(t)\left|\frac{d}{dt}\varphi_2(t)\right| dxdt \\
&\leq C \int\limits_0^T \int\limits_\Omega \mid u \mid^p \varphi^{\frac{p}{q}}\varphi^{\frac{-p}{q}}(t)\varphi_2^\eta(x)\varphi_1^{\ell-1}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid dxdt \\
&+ C \int\limits_0^T \int\limits_\Omega \mid u \mid \varphi^{\frac{1}{q}}\varphi^{\frac{-1}{q}}(x)\varphi_1^\ell(x)\left|\frac{d}{dt}\varphi_2(t)\right| dxdt. \qquad (4.3)
\end{aligned}$$

Therefore, using Young's inequality

$$ab \leq \frac{1}{4}a^{\frac{q}{p}} + Cb^{\frac{q}{q-p}} \qquad (4.4)$$

with

$$\begin{cases} a = \mid u \mid^p \varphi^{\frac{p}{q}} \\ b = C\varphi^{\frac{-p}{q}}\varphi_2^\eta(t)\varphi_1^{\ell-1}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid \end{cases}.$$

in the first integral of the right-hand side of (4.3), and the following Young's inequality

$$ab \leq \frac{1}{4}a^q + Cb^{\tilde{q}} \qquad (4.5)$$

$$textwhere\ \tilde{q} = \frac{q}{q-1}, \qquad (4.6)$$

with

$$\begin{cases} a = \mid u \mid^{\frac{1}{q}} \\ b = C\varphi^{\frac{-1}{q}}\varphi_1^\ell(x)\varphi_2^{\eta-1}(t) \mid \frac{d}{dt}\varphi_2(t) \mid \end{cases}.$$

in the second integral of the right-hand side of (4.3), we get

$$\begin{aligned}
&\frac{1}{2}\int\limits_0^T \int\limits_\Omega \mid u \mid^q \varphi(x,t)dxdt \\
&\leq C \int\limits_0^T \int\limits_\Omega \varphi_2^\eta(t)\varphi_1^{\ell-\frac{q}{q-p}}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid \frac{q}{q-p}dxdt \\
&+ C \int\limits_0^T \int\limits_\Omega \varphi_1^\ell(x)\varphi^{\eta-\tilde{q}}(t).\frac{d}{dt}\varphi_2(t)^{\tilde{q}}dxdt. \qquad (4.7)
\end{aligned}$$

At this stage, we introduce the scaled variables:$\tau = T^{-1}t, \xi = T^{-\alpha}x$, in the right-hand side of (4.7), we conclude that

$$\int_0^T \int_\Omega | u |^q \varphi(x,t) dx dt \leq C T^{-\delta}, \tag{4.8}$$

where $\delta := \frac{q}{q-1} - n\alpha = $. Now, noting that, as

$$q \leq q^* := p + \frac{\beta}{n} \Longleftrightarrow \delta \geq 0,$$

we have to distinguish two cases:

- **Case 1:** $q < q^*$: We pass to the limit in (4.8), as T goes to ∞, we get

  $$\lim_{T\to\infty} \int_0^T \int_{|x|\leq T^\alpha} | u |^q \varphi(x,t) dx dt = 0.$$

  Using the Lebesgue dominated convergence theorem and the fact that $\varphi(x,t) \to 1$ as $T \to \infty$, we conclude that

  $$\int_0^\infty \int_{\mathbb{R}^n} | u |^q (x,t) dx dt = 0,$$

  and the by the continuity in time and space of u we infer that $u \equiv 0$.

- **Case 2:** $q = q^*$: Using inequality (4.8) with $T \to \infty$ and taking into account the fact that $q = q^*$, we have

  $$u \in L^q((0,\infty), L^q(\mathbb{R}^n));$$

  which implies that

  $$\begin{aligned} &\lim_{T\to\infty} \int_{\frac{T}{2}}^T \int_{|x|\leq (BT)^\alpha} | u |^q \varphi(x,t) dx dt \\ &= \lim_{T\to\infty} \int_0^T \int_{|x|\leq (BT)^\alpha} | u |^q \varphi(x,t) dx dt - \lim_{T\to\infty} \int_0^{\frac{T}{2}} \int_{|x|\leq (BT)^\alpha} | u |^q \varphi(x,t) dx dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} | u |^q \varphi(x,t) dx dt - \int_0^\infty \int_{\mathbb{R}^n} | u |^q \varphi(x,t) dx dt = 0. \end{aligned} \tag{4.9}$$

On the other hand, repeating the same calculation as above by taking this time $\varphi_1(x) := \Phi\left(\dfrac{\mid x \mid}{B^\alpha T^\alpha}\right)$, where $1 \leq B < T$ is large enough such that when $T \to \infty$ we don't have $B \to \infty$ at the same time, and applying Hölder's inequality

$$\int ab \leq \left(\int a^q\right)^{\frac{1}{q}} \left(\int b^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}},$$

with

$$\begin{cases} a =\mid u \mid^{\frac{1}{q}} \\ b = C\varphi^{\frac{-1}{q}}\varphi_1^{\ell}(x)\varphi_2^{\eta-1}(t) \mid \dfrac{d}{dt}\varphi_2(t) \mid \end{cases}.$$

in the second integral of the right-hand side of (4.3) instead of Young's inequality, we arrive at

$$\begin{aligned}
&\int_0^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt \\
&\leq \frac{1}{4}\int_0^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt + C\int_0^T\int_{\Omega_1} \varphi_2^{\eta}(t)\varphi_1^{\ell-\frac{q}{q-p}}(x) \mid (-\Delta)^s(\varphi_1(x)) \mid^{\frac{q}{q-p}} dxdt \\
&+ C\left(\int_{\frac{T}{2}}^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt\right)^{\frac{1}{q}} \left(\int_0^T\int_{\Omega_1} \varphi_1^{\ell}(x)\varphi_2^{\eta-\tilde{q}}(t)\left|\frac{d}{dt}\varphi_2(t)\right|^{\tilde{q}} dxdt\right)^{\frac{1}{\tilde{q}}},
\end{aligned}$$

where

$$\Omega_1 := \{x \in \mathbb{R}^n; \mid x \mid\leq (BT)^\alpha\};$$

therefore

$$\begin{aligned}
&\int_0^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt \\
&+ C\int_0^T\int_{\Omega_1} \varphi_2^{\eta}(t)\varphi_1^{\ell-\frac{q}{q-p}}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid \frac{q}{q-p}dxdt \\
&+ C\left(\int_{\frac{T}{2}}^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt\right)^{\frac{1}{q}} \left(\int_0^T\int_{\Omega_1} \varphi_1^{\ell}(x)\varphi_2^{\eta-\tilde{q}}(t)\left|\frac{d}{dt}\varphi_2(t)\right| |^{\tilde{q}}dxdt\right)^{\frac{1}{\tilde{q}}}.
\end{aligned}$$

Thanks to the following rescaling:$\tau = T^{-1}t, \xi = (TB)^{-\alpha}x$, taking into account the fact that $q = q^*$, we can easily conclude that

$$\int_0^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt \leq CB^{-1} + C(B^{n\alpha})^{\frac{1}{\tilde{q}}}\left(\int_{\frac{T}{2}}^T\int_{\Omega_1} \mid u \mid^q \varphi(x,t)dxdt\right)^{\frac{1}{q}}.$$

Thus, taking the limits when $T \to \infty$ and then $B \to \infty$, using (4.9), we get :

$$\int_0^\infty \int_{\mathbb{R}^n} | u |^q (x,t) dx dt = 0,$$

i.e.$u \equiv 0$. This completes the proof. $\square$

4.2 The second result

Theorem 4.2.

For $u_0 \in L^1_{loc}(\mathbb{R}^n), u_0 \geq 0$, *assume that there exists a constant* $\varepsilon > 0$ *such that, for every* $0 < \gamma < n$, *the initial datum verifies the following sign assumption:*

$$u_0(x) \geq \varepsilon (1 + | x |^2)^{-\frac{\gamma}{2}},$$

if

$$p < q < p + \frac{\beta}{\gamma},$$

then problem (4.1) has no global weak solutions.

Proof. Theorem 4.2

The proof is also by contradiction. Suppose that u is a global weak solution to (4.1), then, for all $T \gg 1$, we have

$$\int_{\mathbb{R}^n} u_0(x)\varphi(x,0)dx + \int_0^T \int_{\mathbb{R}^n} | u |^q \varphi(x,t) dx dt = \int_0^T \int_{\mathbb{R}^n} | u |^p (-\Delta)^{\beta/2} \varphi dx dt - \int_0^T \int_{\mathbb{R}^n} u \varphi_t(x,t) dx dt,$$

for all test function $\varphi \in C([0,T], H^\beta(\mathbb{R}^n)) \cap C^1([0,T], C(\mathbb{R}^n))$ such that *supp*φ is compact with $\varphi(\,\cdot\,, T) = 0$.

Now we take $\varphi(x,t) := \varphi_1^\ell(x)\varphi_2^\eta(t)$ *with* $\varphi_1(x) := \Phi\left(\frac{| x |}{T^\alpha}\right), \varphi_2(t) := \Phi\left(\frac{t}{T}\right)$, where $\alpha = \dfrac{q-p}{\beta(q-1)}, \ell, \eta \gg 1$ and Φ a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & if\ 0 \leq r \leq \dfrac{1}{2}, \\ 0 & if\ r \geq 1, \end{cases}$$

$0 \leq \Phi \leq 1, \mid \Phi'(r) \mid \leq \dfrac{C_1}{r}$, for all $r > 0$. We have

$$\int\limits_{\Omega} u_0(x)\varphi_1^{\ell}(x)dx + \int\limits_0^T \int\limits_{\Omega} \mid u \mid^q \varphi(x,t)dxdt = \int\limits_0^T \int\limits_{\mathbb{R}^n} \mid u \mid^p \varphi_2^{\eta}(t)(-\Delta)^{\beta/2}(\varphi_1^{\ell}(x))dxdt$$
$$- \int\limits_0^T \int\limits_{\Omega} u\varphi_1^{\ell}(x)\frac{d}{dt}\varphi_2^{\eta}(t)dxdt,$$

where

$$\Omega := \{x \in \mathbb{R}^n; \mid x \mid \leq T^{\alpha}\}.$$

Using Ju's inequality $(-\Delta)^{\beta/2}(\varphi_1^{\ell}(x) \leq \ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}(\varphi^1)$ (see 4.1), we obtain

$$\begin{aligned}
&\int\limits_0^T \int\limits_{\Omega} \mid u \mid^q \varphi(x,t)dxdt + \int\limits_{\Omega} u_0(x)\varphi_1^{\ell}(x)dx \\
&\leq C \int\limits_0^T \int\limits_{\Omega} \mid u \mid^p \varphi_2^{\eta}(t)\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x))dxdt \\
&+ C \int\limits_0^T \int\limits_{\Omega} \mid u \mid \varphi_1^{\ell}(x)\varphi_2^{\eta-1}(t)\left|\frac{d}{dt}\varphi_2(t)\right| dxdt \\
&\leq C \int\limits_0^T \int\limits_{\Omega} \mid u \mid^p \varphi^{\frac{p}{q}}\varphi^{\frac{-p}{q}}(t)\varphi_2^{\eta}(x)\varphi_1^{\ell-1}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid dxdt \\
&+ C \int\limits_0^T \int\limits_{\Omega} \mid u \mid \varphi^{\frac{1}{q}}\varphi^{\frac{-1}{q}}(x)\varphi_1^{\ell}(x)\left|\frac{d}{dt}\varphi_2(t)\right| dxdt. \qquad (4.10)
\end{aligned}$$

As $u_0(x) \geq \varepsilon(1+ \mid x \mid^2)^{\frac{-\gamma}{2}}$, then

$$\int\limits_{\Omega} u_0(x)\varphi_1^{\ell}(x)dx \geq \int\limits_{|x| \leq \frac{T^{\alpha}}{2}} u_0(x) \geq \varepsilon \int\limits_{|x| \leq \frac{T^{\alpha}}{2}} (1+ \mid x \mid^2)^{\frac{-\gamma}{2}}dx \geq C\varepsilon T^{\alpha(n-\gamma)}.$$

Therefore, using Young's inequality

$$ab \leq \frac{1}{4}a^{\frac{q}{p}} + Cb^{\frac{q}{q-p}}, \qquad (4.11)$$

with

$$\begin{cases} a = \mid u \mid^p \varphi^{\frac{p}{q}}, \\ b = C\varphi^{\frac{-p}{q}}\varphi_2^{\eta}(t)\varphi_1^{\ell-1}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid . \end{cases}$$

In the first integral of the right-hand side of (4.10), and the following Young's inequality

$$ab \leq \frac{1}{4}a^q + Cb^{\tilde{q}} \quad \text{where} \quad \tilde{q} = \frac{q}{q-1}, \qquad (4.12)$$

with

$$\begin{cases} a =\mid u \mid^{\frac{1}{q}}, \\ b = C\varphi^{\frac{-1}{q}}\varphi_1^{\ell}(x)\varphi_2^{\eta-1}(t) \mid \frac{d}{dt}\varphi_2(t) \mid . \end{cases}$$

In the second integral of the right-hand side of (4.10), we get

$$\begin{aligned} & C\varepsilon T^{\alpha(n-\gamma)} + \frac{1}{2}\int\limits_0^T\int\limits_\Omega \mid u \mid^q \varphi(x,t)dxdt \\ & \leq C\int\limits_0^T\int\limits_\Omega \varphi_2^{\eta}(t)\varphi_1^{\ell-\frac{q}{q-p}}(x) \mid (-\Delta)^{\beta/2}(\varphi_1(x)) \mid^{\frac{q}{q-p}} dxdt \\ & + C\int\limits_0^T\int\limits_\Omega \varphi_1^{\ell}(x)\varphi^{\eta-\tilde{q}}(t) \mid \frac{d}{dt}\varphi_2(t) \mid^{\tilde{q}} dxdt. \end{aligned} \tag{4.13}$$

At this stage, we introduce the scaled variables : $\tau = T^{-1}t, \xi = T^{-\alpha}x$, in the right-hand side of (4.13), we conclude that

$$C\varepsilon T^{\alpha(n-\gamma)} \leq CT^{-\frac{q}{q-1}+n\alpha+1},$$

that is

$$C\varepsilon \leq CT^{-\delta^*}. \tag{4.14}$$

Where $\delta^* := \dfrac{q}{q-1} - n\alpha - 1 + \alpha(n-\gamma)$. Now, noting that, as

$$q < q^{**} := p + \frac{\beta}{\gamma} \iff \delta^* > 0,$$

then, by passing to the limit in (4.14), as T goes to ∞, we get a contradiction. □

Remark 4.1. *By applying the same calculation to the corresponding nonlocal elliptic equation*

$$(-\Delta)^{\beta/2}(\mid u \mid^p) =\mid u \mid^q, x \in \mathbb{R}^n, \tag{4.15}$$

where $n \geq 1, 0 < \beta \leq 2, p > 0,$ *and* $q > 1$. *We can obtain the following result: if*

$$p < q < \frac{np}{(n-\beta)_+},$$

then problem (4.15) has no nontrivial weak solutions.

In the end, we notice that the second result is better than the first result.

Conclusion

- In this memoir, we studied the problem of non-existence of non-trivial solutions to fractional partial differential equations and systems.

- The problem of non-existence of non-trivial solutions for fractional partial differential equations and systems fractionial order remains among the importants problems that necessitate much more research.

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Résumé

Ce travail est consacrée à l'étude des problèmes de non-existence des solutions non-trivial pour certaines EDPs fractionnaires et systèmes des EDPs fractionnaires, l'idée de base des preuves repose sur la méthode des fonctions tests.

Mots clés : Dérivée fractionnaire - Laplacen fractionnaire - Exposant critique - Méthode des fonctions tests - Equation d'évolution - Non-existence des solutions.

Abstract

This work is devoted to the study of the non-existence problems of non-trivial solutions for some fractional PDEs and fractional partial differential systems, the basic idea of proofs is based on the method of the test functions.

Key words: Fractional derivative - Fractional Laplacen - Critical exponent - Test function method -Evolution equation - non-existence solutions.

الملخص

هذا العمل مخصص لدراسة مسألة عدم وجود الحلول غير التافهة لبعض الأنماط للمعادلات التفاضلية الجزئية ذات رتب كسرية وذلك بإستعمال طريقة الدوال الإختبارية.

كلمات مفتاحية: مشتقات كسرية-لابلاس الكسري- الأس الحرجة - طريقة التوابع الاختبارية -المعادلة التطورية - عدم وجود الحلول .