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Theme

Numerical approximation of the Caputo-Fabrizio fractional derivative and its application in solving fractional differential equations.

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Dedication

قال تعالى : ﴿ وَآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ ﴾

الحمدلله عند البدء وعند الختام، فما تناهى دربٌ، ولا خُتم جهد، ولا تم سعي إلا بفضله.

وُجد الإنسان على وجه البسيطة، ولم يعش بمعزل عن باقي البشر وفي جميع مراحل الحياة، يُوجد أناس يستحقُّون منَّا الشُّكر وأولى الناس بالشُّكر هما الأبوان؛ لما لهما من الفضل ما يبلغ عنان السماء, فوجودهما سبب للنجاة والفلاح في الدينا والآخرة.

إلى أصدقائي الذين أشهد لهم بأنهم نعم الرُّفقاء في جميع الأمور..

ولا ينبغي أن أنسى أساتذتي ممن كان لهم الدور الأكبر في مُساندتي ومدِّي بالمعلومات القيِّمة...

أُهدي لكم بحث تخرُّجي هذا

داعيًا المولى - عزَّ وجلَّ - أن يُطيل في أعماركم، ويرزقكم بالخيرات.

اللَّهُمَّ انْفَعْنِي بِمَا عَلَّمْتَنِي، وَعَلِّمْنِي مَا يَنْفَعُنِي، وَزِدْنِي عِلْمًا .

Thanks and appreciation

مِصداقِا لقولهِ تعالى :- * وَإِذْ تَأَذَّنَ رَبُّكُمْ لَئِن شَكَرْتُمْ لَأَزِيدَنَّكُمْ ۖ وَلَئِن كَفَرْتُمْ إِنَّ عَذَابِي لَشَدِيدٌ *

سورة إبراهيم الأية 07.

نحمد الله عزّ وجل أن وفقنا لإتمام هذا العمل المتواضع

وعملًا بقوله صلّى الله عليه وسلم :

(مَنْ لمْ يشْكُر النَّاسَ لَمْ يشْكُر الله) رواه الترمذي .

نتقدم بالشكر والجزيل والعرفان بالجميل :

إلى كُل من مدّ يد العون لإثارة دربنا , إلى كل من علمنا علماً نافعاً بدّءا من معلمي الإبتدائي وصولاً بأساتذة الجامعة .

تحية عطرة وشكر خاص للأستاذ المشرف "بشير د حده " الذي أفادنا بنصحه

وتوجيهاته لنا طيلة فترة إنجاز هذه المذكرة .

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ولم يبخل علينا بمعلوماته وإمكانياته لإعداد هذه المذكرة .

Contents

Notations

$\mathbb{N}$	Set of complex numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{C}$	Set of complex numbers.
$\mathfrak{Re}$	Real part.
Γ	Gamma function.
β	Beta function.
$\mathcal{L}$	Laplace Transforms.
$\mathcal{L}^{-1}$	inverse Laplace Transforms.
s	Parameter in the Laplace transformation.
${}_{a}^{RL}D_x^{(\alpha)}$	Riemann-Liouville fractional derivative.
${}_{a}^{C}D_x^{(\alpha)}$	Caputo fractional derivative.
${}_{a}^{CL}D_x^{(\alpha)}$	Caputo-Fabrizio fractional derivative.
${}_{a}^{\alpha}I$	Riemann-Liouville fractional integral.
${}^{CF}I^{\alpha}$	Caputo-Fabrizio fractional integral.
$\mathscr{C}[0,T]$	Space of all continuous functions defined on the interval $[0,T]$.
$\binom{n}{k}$	$C_n^k = \dfrac{n!}{k!(n-k)!}$.
R_i^{j+1}	The remainder term of the approximation for using the Crank-Nicolson scheme.
${}_{t}^{\alpha}\partial u(t)$	The time-fractional ODE model.

Introduction

Fractional differential calculus has gained much interest by the many researcher in the last decades and it has strong mathematical background and many papers are attributed to the development of it.Among them, we can cite some for example,[5, 39]. Fractional calculus has been also used for modeling physical phenomena including control systems, mechanics and viscoelasticity.

Most fractional differential equations describing real-world (physical) problems are highly complicated and cannot sometimes be solved analytically. A lot of numerical approaches in connection with derivatives of fractional order describing these real-world problems alter essentially in the many in which the derivative of fractional order is tailored, see, for instance, [40] and references therein. Numerical approximation of a derivative of fractional order has a highly complicated formula compared to those of integer order due to their non-local nature, and therefore the calculation at a particular point requires knowledge of the function further out of the region close to that point. Accordingly, finite difference approximations of derivatives of fractional order engage a quantity of points that alters according to how faraway we are from the borderline.

Several researchers have proposed new definitions of the concept of derivative with fractional order. These definitions go from Riemann-Liouville to the newly proposed one by Caputo and Fabrizio.

where it was shown that the new-fangled derivative contains additional encouraging properties in comparison with the older version. For example,they have shown that it can represent substance heterogeneities and configurations with different scales, which clearly cannot be overseeing with the prominent local theories and also the known fractional derivative. Another application is in the investigation of the macroscopic behaviours of some materials that are associated with non-local communications between atoms, which are recognized to be important of the properties of material.

The aim of this work is ***numerical approximate of Caputo-Fabrizio fractional derivative and apply it to solving fractional differential equations.***

This work is divided into three chapter:

• In the first chapter, we will mention the fractional analysis of derivatives and integration, using fractional derivatives, the fractional Riemann-Léouville derivative, the fractional Caputo derivative, and the fractional Caputo-Fabrizio derivative and some of their properties. , etc...[5, 3, 8]

and we will studying the existence and uniqueness of solutions of linear fractional differential equations by applying the Laplace transform definition.They results in this chapter are taken from they articles. [6, 9]

As we will apply some fixed point theorems (Banach's principle of contraction theorem, Krasnoselskii's fixed point theorem) to nonlinear Caputo-Fabrizio fractional differential equations.

• In the second chapter, we will study Numerical approximation of the space-time Caputo-Fabrizio fractional derivative and application to groundwater pollution equation. [13, 15, 18]

• In the last chapter, we will present some results of application of to Ordinary Fractional Differential Equations .[22, 31, 32]

Chapter 1

A reminder about fractional analysis of derivative and integration

1.1 Riemann-Liouville and Caputo fractional derivatives

1.1.1 Riemann-Liouville fractional integrals

The definition of fractional integral in the Riemann-Liouville sense is a generalization, of the Cauchy formula (1789 - 1857), which is obtained as follows: [1]
Let $f : [a; b] \to \mathbb{R}$ be a continuous function, we denote

$$I_a^1 f(x) = \int_a^x f(t)dt$$

Double integration

$$I_a^2 f(x) = \int_a^x \int_a^t f(\mu)d\mu dt = \int_a^x (x-t)f(t)dt.$$

By repeating the process (n - 1) times, we obtain the following relation:

$$I_a^n f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt.$$

For all $n \in \mathbb{N}$ where the generalization of the factorial by the Gamma function:

$$(n-1)! = \Gamma(n)$$

.

Definition 1.1.1. *[1] The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$, for a function $f \in \mathscr{C}$ ([a , b]) is defined by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

Proposition 1.1.1.

1. $I_a^\alpha \left(I_a^\beta\right)(x) = I_a^\beta \left(I_a^\alpha f\right)(x) = I_a^{\alpha+\beta} f(x).$

2. *Let A,B are fixed elements of the body $\mathbb{R}$ or $\mathbb{C}$,*

$$I_a^\alpha \left[Af(x) + Bg(x)\right] = AI_a^\alpha f(x) + BI_a^\alpha g(x).$$

1.1.2 Riemann-Liouville fractional derivative

Definition 1.1.2. *[2] The popular definition of fractional derivative is this one:*

$$^{RL}D_a^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t) dt.$$

$$(n-1 \leqslant [\alpha] < n), \textit{and } x > a.$$

Proposition 1.1.2. *Riemann-Liouville operator has the following important properties:*

1. ${}_a^{RL}D_t^{(\alpha)} \; ({}_a^{RL}D_t^{(\beta)}) f(x) = {}_a^{RL}D_t^{(\alpha+\beta)} f(x).$

2. *For $\alpha = m \in \mathbb{N}$ we have:*

$$^{RL}D_a^{(0)} f(x) = \frac{1}{\Gamma(1)} \left(\frac{d}{dx}\right) \int_a^x f(t) dt = f(x),$$

$$^{RL}D_a^{(m)} f(x) = \frac{1}{\Gamma(1)} \left(\frac{d^{m+1}}{dx^{m+1}}\right) \int_a^x f(t) dt = \frac{d^m}{dx^m} f(x),$$

consequently the fractional derivative in the sense of Riemann-Liouville coincides with the derivative classic by $\alpha \in \mathbb{N}$.

Remarque 1.1.1.

$$^{RL}D_a^{(0)} f(x) = \left(\frac{d}{dx}\right)^n (I_a^{n-\alpha} f)(x)$$

such that : $n = [\alpha] + 1, x > a$.

1.1.3 Caputo fractional derivative

Although fractional derivation in the sense of Riemann-Liouville played an important role in the development of fractional calculus, because of its applications in pure and applied mathematics. However, given that the derivative in the sense of Riemann-Liouville of a constant is not zero and that the initial conditions of the Cauchy problem are expressed by fractional order derivatives, Caputo offers another approach where the derivative of the constant is zero and the initial conditions are expressed as in the classical case by whole order derivatives.

Definition 1.1.3. *[2] Let* $0 < n - 1 < \alpha < n$ *and* f *be a function of class* $C^n([a, b])$*. The fractional derivative of order* α *in the sense of Caputo of the function* f *is defined by :*

$$^{C}D^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x f^n(t)(x-t)^{n-\alpha-1} dt.$$

• *Under natural conditions on the function f(x), for* $\alpha \to n$ *the Caputo derivative becomes a conventional* n *- th derivative of the function f(x).*

Indeed let us assume that $0 \leqslant n - 1 < \alpha < n$ *and that the function f(x) has n + 1 continuous bounded derivatives in [a , T] for every* $T > a$ *then [3]*

$$\lim_{\alpha \to n} {}_a^C D_x^{(\alpha)} f(x) = \lim_{\alpha \to n} \left(\frac{1}{\Gamma(n-\alpha)} \int_a^x f^n(t)(x-t)^{n-\alpha-1} dt \right),$$

from integration by parts we have

$$\begin{aligned}\lim_{\alpha\to n} {}_a^C D_x^{(\alpha)} f(x) &= \lim_{\alpha\to n}\left(\frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)}\int_a^x (x-t)^{n-\alpha} f^{(n+1)}(t)dt\right)\\ &= f^{(n)}(a)\int_a^x f^{(n+1)}(t)dt\\ &= f^{(n)}(x), n=1,2,...\end{aligned}$$

- *Non-commutation [3]*

$$ {}_a^C D_x^{(\alpha)}\left({}_a^C D_x^{(m)} f(x)\right) \neq {}_a^C D_x^{(m)}\left({}_a^C D_x^{(\alpha)} f(x)\right), (m=0,1,2,...,n-1<\alpha<n).$$

The interchange of the differentiation operators in formulas is allowed under different conditions:

$$ {}_a^C D_x^{(\alpha)}\left({}_a^C D_x^{(m)} f(x)\right) = {}_a^C D_x^{(m)}\left({}_a^C D_x^{(\alpha)} f(x)\right) = {}_a^C D_x^{(\alpha+m)}$$

$$f^s(0)=0, s=n,n+1,...,m,\ (m=0,1,2,...,n-1<\alpha<n).$$

Exemple 1.1.1. *Consider the function:*

$$f(x)=x^\beta$$

for $0<n-1<\alpha<n$ *, we have :*

$$ {}^C D^{(\alpha)} f(x) = I^{n-\alpha}\left(D^n x^\beta\right)$$

or

$$D^n x^\beta = \left(\frac{\Gamma(\beta+1)}{\beta+1-n} x^{\beta-n}\right)$$

As a result:

$$I^{n-\alpha}\left(\frac{\Gamma(\beta+1)}{\beta+1-n} x^{\beta-n}\right) = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)\Gamma(n-\alpha)}\right)\int_0^x (x-t)^{(n-\alpha-1)} t^{\beta-n} dt$$

by performing the change of variable $t = yx$ so $dt = xdy$, we obtain:

$$\begin{aligned}\int_0^x (x-t)^{n-\alpha-1} t^{\beta-n} dt &= \int_0^x (x-t)^{n-\alpha-1} t^{\beta-n} dt \\ &= \int_0^1 x^{n-\alpha-1}(1-y)^{n-\alpha-1} y^{\beta-n} x^{\beta-n+1} dy \\ &= \int_0^1 x^{\beta-\alpha}(1-y)^{n-\alpha-1} y^{\beta-n} dy \\ &= x^{\beta-\alpha} \int_0^1 (1-y)^{n-\alpha-1} y^{\beta-n} dy \\ &= x^{\beta-\alpha} \left(\frac{\Gamma(n-\alpha)\Gamma(\beta-n-1)}{\Gamma(\beta-n-1)} \right).\end{aligned}$$

So

$$I^{n-\alpha}\left(\frac{\Gamma(\beta+1)}{\beta+1-n} x^{\beta-n}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} \frac{1}{\Gamma(n-\alpha)} \frac{\Gamma(n-\alpha)\Gamma(\beta+1-n)}{\Gamma(\beta+1-n)} x^{\beta-\alpha},$$

finally, we obtain

$$^{C}D^{(\alpha)} x^0 = \left(\frac{\Gamma(\beta+1)}{\beta+1-n} x^{\beta-n}\right).$$

unlike the Riemann-Liouville derivative, the fractional order derivative in the Caputo sense of a constant is zero.

1.1.4 Some properties of fractional derivatives

Theorem 1.1.1. *(Linearity) [3] Similarly to integer-order differentiation, fractional differentiation is a linear operator:*

$$D^{(\alpha)}(\lambda f(x) + \mu g(x)) = \lambda D^{(\alpha)} f(x) + \mu D^{(\alpha)} g(x)$$

Proof. *For example, if $D(\alpha)$ is the Caputo operator (where $n-1 \leq \alpha < n$ and $n = 1$) , by definition, we have:*

$$\begin{aligned}
D^{(\alpha)}(\lambda f(x)+\mu g(x)) &= \frac{1}{\Gamma(1-\alpha)}\int_a^x (\lambda f(t)+\mu g(t))'(x-t)^{-\alpha}dt \\
&= \frac{1}{\Gamma(1-\alpha)}\int_a^x (\lambda f'(t)+\mu g'(t))'(x-t)^{-\alpha}dt \\
&= \frac{\lambda}{\Gamma(1-\alpha)}\int_a^x f'(t)(x-t)^{-\alpha}dt + \frac{\mu}{\Gamma(1-\alpha)}\int_a^x g'(t)(x-t)^{-\alpha}dt \\
&= \lambda_a^C D_x^\alpha f(x) + \mu_a^C D_x^\alpha g(x).
\end{aligned}$$

Theorem 1.1.2. *.**(The Leibniz Rule)** [3]*

For all $n \in \mathbb{N}$ we have

$$\frac{d^n}{dt^n}(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x).$$

The generalization of this formula gives us

$$D^{(\alpha)}(f(x)g(x)) = \sum_{k=0}^{n} \binom{\alpha}{k} f^{(k)}(x)D^{(\alpha)}g^{(\alpha-k)}(x) + R_n^\alpha(x).$$

or $n \geqslant \alpha+1$ and

$$R_n^\alpha(x) = \frac{1}{n!\Gamma(-\alpha)}\int_\alpha^t (x-s)^{-\alpha-1}g(s)ds\int_s^t f^{(n+1)}(\xi)d\xi,$$

with

$$\lim_{n\to\infty} R_n^\alpha(x) = 0$$

If f and g are continuous in [a,t] as well as all their derivatives, the formula becomes:

$$D^{(\alpha)}(f(x)g(x)) = \sum_{k=0}^{n} \tbinom{k}{\alpha} f^{(k)}(x)D^{(\alpha)}g^{(\alpha-k)}(x).$$

where $D^{(\alpha)}$ is the fractional derivative in the sense of Riemann-Liouville.

1.2 Caputo-Fabrizio fractional derivative (CFFD)

Because of the singularity in the kernel of the Caputo fractional derivative [4],[3] at the end point of the interval of integration, the Caputo fractional derivative is not always a suitable kernel to accurately describe the memory effect in a real system. Caputo and Fabrizio [5] has recently proposed a new fractional derivative without any singularity in its kernel. The kernel of the new fractional derivative has the form of an exponential function. More recently, Losada and Nieto [6] derived the fractional integral associated with the new fractional Caputo-Fabrizio fractional derivative.

This section is devoted to studying the basic definitions and results about the Caputo-Fabrizio fractional derivative.

Let us recall the usual Caputo fractional time derivative (CFD) of order α*, is given by :*

$$ {}^{C}D_t^{(\alpha)} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f'(\tau)(t-\tau)^{-\alpha} d\tau. \tag{1.1}$$

Definition 1.2.1. *[5] Let* $f \in H^1(a,b), b > a, \alpha \in [0,1]$ *the Caputo-Fabrizio fractional derivative is defined as*

$$ {}^{CF}D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau)\ exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau, \tag{1.2}$$

where $M(\alpha)$ *is a normalization function such that* $M(0) = M(1) = 1$.

If the function does not belong to $H^1(a,b)$ *then, the derivative can be reformulated as*

$$ {}^{CF}D_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(\tau))\ exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau.$$

The definition of the CFFD was improved by Losada and Nieto to become [6]

$$ {}^{CF}D_t^{(\alpha)} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_a^t f'(\tau)\ exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau.$$

Now, it is worth to observe that if we put [5]

$$\sigma = \frac{1-\alpha}{\alpha} \in\ [0,\infty],\ \alpha = \frac{1}{1+\sigma} \in [0,1].$$

the definition (1.2.1) of CFFD assumes the form

$$\tilde{D}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma}\int_a^t f'(\tau) exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau.$$

where $\sigma \in [0,\infty]$ *and* $N(\sigma)$ *is the corresponding normalization term of* $M(\alpha)$*, such that*

N(0) = N(∞) = 1. Moreover because

$$\lim_{\sigma\to 0}\frac{1}{\sigma} exp[-\frac{(t-\tau)}{\sigma}] = \delta(t-\tau),$$

and for $\alpha \to 1$*, we have* $\sigma \to 0$

$$\begin{aligned}\lim_{\alpha\to 1} {}^{CF}D_t^{(\alpha)} f(t) &= \lim_{\alpha\to 1}\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\ exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\ &= \lim_{\sigma\to 0}\frac{N(\sigma)}{\sigma}\int_a^t f'(\tau)\ exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = f'(t).\end{aligned}$$

Otherwise, when $\alpha \to 1$*, then* $\sigma \to +\infty$*. Hence,*

$$\begin{aligned}\lim_{\alpha\to 0} {}^{CF}D_t^{(\alpha)} f(t) &= \lim_{\alpha\to 1}\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\ exp\ [-\frac{\alpha(t-\tau)}{1-\alpha}\ ]d\tau \\ &= \lim_{\sigma\to 0}\frac{N(\sigma)}{\sigma}\int_a^t f'(\tau)\ exp\ [-\frac{(t-\tau)}{\sigma}\ ]d\tau = f(t) - f(\alpha).\end{aligned}$$

Let us consider the (CFFD) of a particular function, as f(t) = sin ωt,

for α = 0.66 , a = - 8 and $\omega = 1$

$${}^{CF}D_t^{0.66}\ sin\ \omega t = \frac{M(0.66)}{0.33}\int_a^t cos\ \tau\ exp(-2(t-\tau))d\tau$$

The simulation of this derivative produces the following pictures.

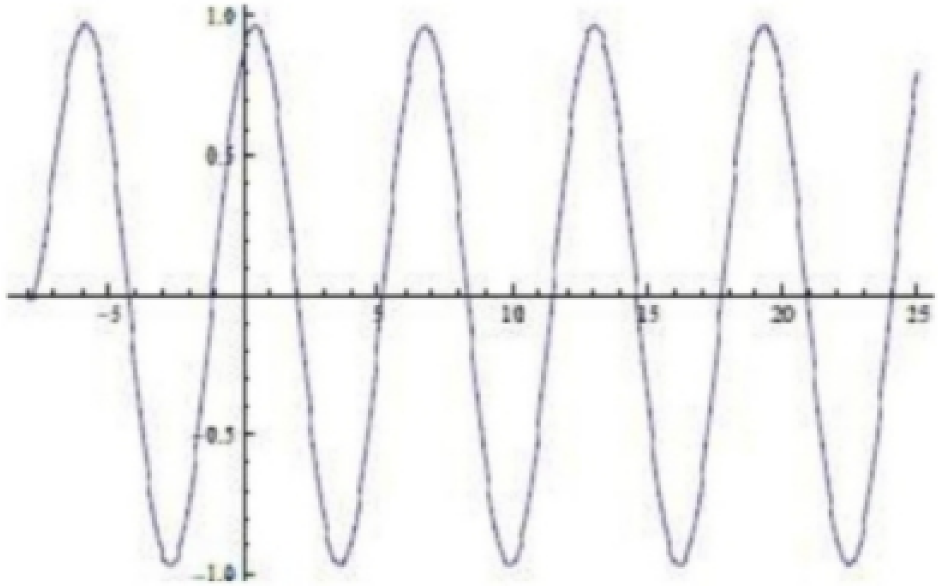


Figure 1.1: *Simulation of (CFFD), with* $\alpha = 0.66$

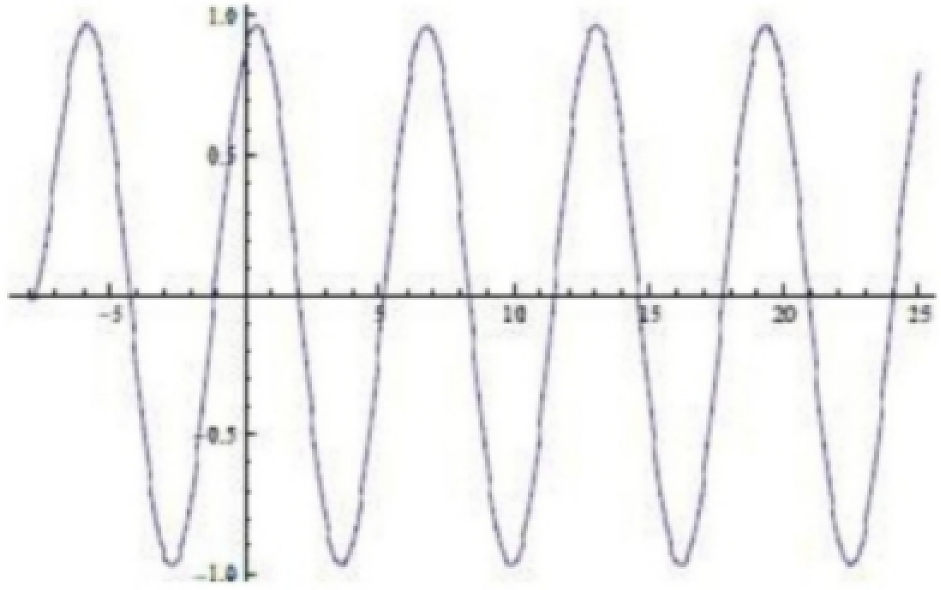


Figure 1.2: *Simulation of (CFD), with* $\alpha = 0.66$

From these two simulations with (CFFD) $\alpha = 0.66$*, it appears as the classical is very similar to the (CFD). Otherwise, when we study models with* α *close to 0, we see a different behavior.*

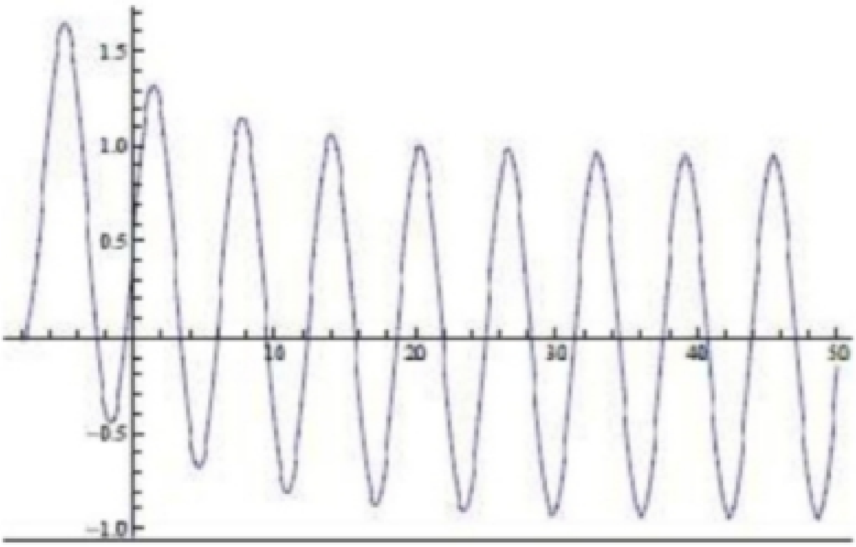


Figure 1.3: *Simulation of (CFFD), with* $\alpha = 0.1$

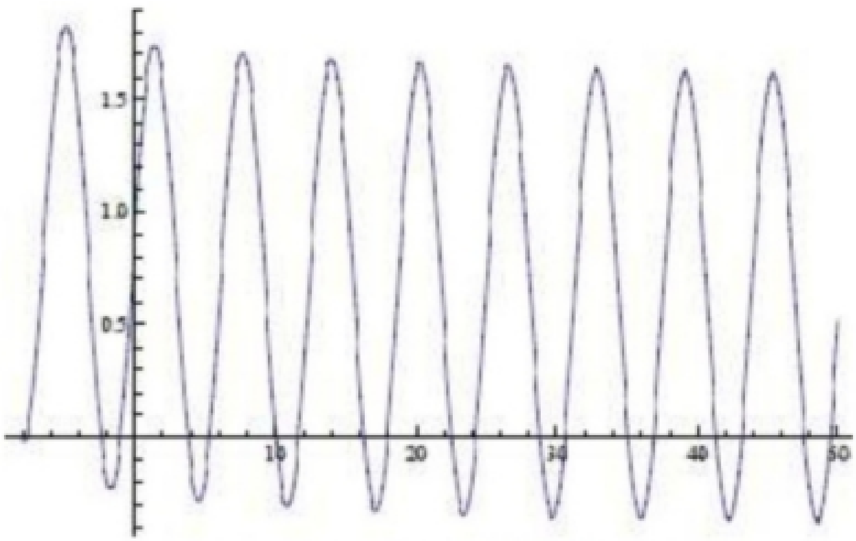


Figure 1.4: *Simulation of (CFD), with* $\alpha = 0.1$

So that, for $\alpha = 0.1$ *in Figure 1.3 and Figure 1.4 we observe different actions between (CFFD) and (CFD) simulations. In particular the classical (CFD) is more affected by past, compared with the (CFFD) which show a rapid stabilization.*
If $n \geqslant 1$, *and* $\alpha \in [0,1]$ *the fractional time derivative* ${}^{CF}D_t^{(\alpha+n)}f(t)$ *of order* $(n+\alpha)$ *is defined by:*

$$ {}^{CF}D_t^{(\alpha+n)}f(t) := {}^{CF}D_t^{(\alpha)}\left({}^{CF}D_t^{(n)}f(t)\right) \tag{1.3}$$

Theorem 1.2.1. *[5]*

For (CFFD), if the function f (t) is such that

$$f^{(s)}(0) = 0, \;\; s = 1, 2, 3, ..n$$

then, we have

$$^{CF}D_t^{(\alpha)}\left(^{CF}D_t^{(n)}f(t)\right) :=^{CF} D_t^{(n)}\left(^{CF}D_t^{(\alpha)}f(t)\right)$$

Proposition 1.2.1. ***(Linearity)***

Let $^{CF}D^{(\alpha)}$ *Caputo-Fabrizio operator satisfy*

$$^{CF}D^{(\alpha)}(\lambda f(x) + \mu g(x)) = \lambda^{CF}D^{(\alpha)}f(x) + \mu^{CF}D^{(\alpha)}g(x).$$

Proof. *According to the definition(1.2.1) , we have*

$$\begin{aligned}
{}^{CF}{}_aD_t^{(\alpha)}(\lambda f(t) + \mu g(t)) &= (\frac{M(\alpha)}{1-\alpha}\int_a^t (\lambda f(\tau) + \mu g(\tau))' \; exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\
&= \frac{M(\alpha)}{1-\alpha}\left(\lambda f'(\tau) + \mu g'(\tau)\right) \; exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\
&= \frac{\lambda M(\alpha)}{1-\alpha}\int_a^t f'(\tau) \; exp\left[-\frac{\alpha(t-\tau)}{1-\alpha} d\tau\right] \\
&+ \frac{\mu M(\alpha)}{1-\alpha}\int_a^t g'(\tau) \; exp\left[-\frac{\alpha(t-\tau)}{1-\alpha} d\tau\right] \\
&= \lambda_a^{CF}D_t^{(\alpha)}f(t) + \mu_a^{CF}D_t^{(\alpha)}g(t).
\end{aligned}$$

1.2.1 Laplace transform of the CFFD

Definition 1.2.2. *[5]*

It is well known that Laplace Transform plays an important role in the study of ordinary differential equations. In the case of this new fractional definition (CFFD) with a = 0 the Laplace Transform becomes like this, for $0 < \alpha < 1$

$$\mathcal{L}[^{CF}D_t^{(\alpha)}f(t)](s) = \frac{s\mathcal{L}[f(t)](s) - f(0)}{2(s + \alpha(1-s))}, \;\; s > 0 \tag{1.4}$$

Lemme 1.2.1. *[5]*

The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0\ ,\ 1)$ and $n \in \mathbb{N}$ is given by :

$$\mathcal{L}\left[{}^{CF}D_t^{(\sigma)} f(t)\right](s) = \frac{s^{n+1}\mathcal{L}[f(t)](s) - s^n f(0) - s^{n-1}f'(0) - ... - f^n(0)}{s + \alpha(1-s))} \tag{1.5}$$

1.2.2 The Inverse Laplace Transform

Definition 1.2.3. *[7]*

If G(s) = $\mathcal{L}[g(t)](s)$; then the inverse transform of G(s), is defined as:

$$\mathcal{L}^{-1}G(s) = g(t).$$

properties of the inverse Laplace transform

1. $\mathcal{L}^{-1}[aG_1(s) + bG_2(s)] = ag_1(t) + bg_2(t)$.
2. $\mathcal{L}^{-1}G(s-a) = e^{at}g(t)$.
3. $\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] = \int_0^t g(t)dt$.

1.2.3 The fractional integral associated to the CFFD

After the notion of fractional derivative of order $0 < \alpha < 1$, that of fractional integral of order $0 < \alpha < 1$ becomes a natural requirement. In this section we obtain the fractional integral associated to the Caputo-Fabrizio fractional derivative previously introduced.

Proposition 1.2.2. *[6]*

Let $0 < \alpha < 1$: The fractional integral of order α of a function f is defined by:

$$^{CF}I^{(\alpha)}f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds,\ t \geqslant 0 \tag{1.6}$$

1.2.4 Composition of CFFD Operators

Here we present a theoretical property related to the Caputo-Fabrizio fractional derivative.

Theorem 1.2.2. *[8] Let be $n \in \mathbb{N} - \{0\}$, a, $b \in \mathbb{R}$ $(a < b)$ and $u \in C^n([a,b])$. Then the equality*

$$\frac{d^n}{dt^n}({}^{CF}D_{at}^{(\alpha)}u(t)) = \sum_{i=1}^{n}(-1)^{n-i}\frac{\alpha^{n-i}}{(1-\alpha)^{n+1-i}}u^{(i)}(t) + (-1)^n\left(\frac{\alpha}{1-\alpha}\right)^n \, {}^{CF}D_{at}^{(\alpha)}u(t).$$

is true. Proof. see [8]

Corollaire 1.2.1. *Let be $a, b \in \mathbb{R}$ $(a < b)$ and $u \in C^1([a,b])$. Then the equality*

$$\int_a^b ({}^{CF}D_{at}^{(\alpha)}u(t))dt = \frac{1}{\alpha}(u(b) - u(a)) - \frac{1-\alpha}{\alpha}({}^{CF}D_{ab}^{(\alpha)}u(b)).$$

is true.

In this section, we give some theoretical properties concerning the composition of Caputo-Fabrizio fractional operators.

Theorem 1.2.3. *[8] Let be $a, \alpha, \beta \in \mathbb{R}$ such that $0 < \alpha, \beta < 1 (\alpha \neq \beta)$. Then the equality*

$$ {}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = \frac{1}{\beta - \alpha}\left(\beta . {}^{CF}D_{at}^{(\beta)}u(t) - \alpha . {}^{CF}D_{at}^{(\alpha)}u(t)\right). \qquad (1.7)$$

is true.

Theorem 1.2.4. *[8] Let be $a, \alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the equality*

$$ {}^{CF}D_{at}^{(\alpha)}\left({}^{CF}I_{at}^{(\alpha)}u(t)\right) = u(t) - exp\left(-\frac{a}{1-a}(t-a)\right)u(a). \qquad (1.8)$$

is true.

Theorem 1.2.5. *[8] Let $a, \alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the equality*

$$ {}^{CF}I_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\alpha)}u(t)\right) = u(t) - u(a).$$

is true.

Theorem 1.2.6. *[8] Let $0 < \alpha < 1, a \in \mathbb{R}$. Then the equality*

$$ {}^{CF}I_{at}^{(\alpha)}\left(I_{at}^{\alpha}u(t)\right) = (1-\alpha)I_{at}^{(\alpha)}u(t) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}I_{at}^{(\alpha+1)}u(t).$$

1.3 Some Ordinary linear Caputo-Fabrizio fractional differential equations

1.3.1 Some Results of Existence and Uniqueness of the Solution

Lemme 1.3.1. *[6] Let $0 < \alpha < 1$. and f be a solution of the following fractional differential equation,*

$$^{CF}D^{(\alpha)}f(x) = 0,\ t \geq 0. \tag{1.9}$$

Then, f is a constant function. The converse, as indicated in the introduction, is also true.

Proposition 1.3.1. *[6] Let $0 < \alpha < 1$. Then, the unique solution of the following initial value problem*

$$\begin{cases} ^{CF}D^{\alpha}f(t) = \sigma(t),\ t \geq 0 \\ f(0) = f_0 \in R \end{cases} \tag{1.10}$$

is given by :

$$f(t) = f_0 + a_\alpha(\sigma(t) - \sigma(0)) + b_\alpha \int_0^t \sigma(t),\ t \geq 0 \tag{1.11}$$

where

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}, b_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)}. \tag{1.12}$$

Theorem 1.3.1. *[9] For $\sigma = \alpha + 1$, $\alpha \in (0,1)$, and $g : [0,\infty) \to \mathbb{R}$ with $g \in L_1(0,\infty)$, the following boundary value problem of Caputo-Fabrizio fractional differential equation*

$$\begin{cases} ^{CF}D^{\sigma}f(x) = g(x), x \geq 0, \\ f(0) = f_0\ f(1) = f_1, \end{cases} \tag{1.13}$$

has the unique solution given by :

$$\begin{aligned} f(x) &= f_0 + (f_1 - f_0)x + (1-\alpha)(1-x)\int_0^x g(t)dt \\ &+ \alpha(x-1)\int_0^x tg(t)dt - (1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt. \end{aligned}$$

Remarque 1.3.1. *In Theorem (1.3.1), if we let $h(x) = g(x) - g(0)$, then $h(0) = 0$, so that the initial value problem*

$$\begin{cases} {}^{CF}D^{\sigma} f(x) = h(x), \ x \geq 0 \\ f(0) = A, f'(0) = B \end{cases}$$

has the unique solution of much simpler form given by

$$f(x) = A + Bx + (1 - \alpha) \int_0^x h(t)dt + \alpha \int_0^x (x - t)h(t)dt.$$

Theorem 1.3.2. *[9] If $\sigma \in (1, 2)$ and $g \in L^1(0, \infty) \cap C^1[0, \infty)$, then the following linear boundary value problem of Caputo-Fabrizio fractional differential equation has the unique solution for all $\eta \in \mathbb{R}$.*

$$\begin{cases} {}^{CF}D(\sigma)f(x) = \eta f(x) + g(x), \eta \neq 0, \ x \geq 0 \\ f(0) = f_0, f(1) = f_1 \end{cases} \tag{1.14}$$

Exercice : *[9] Consider the initial value problem*

$$\begin{cases} {}^{CF}D^{\sigma} f(x) + f(x) = 0, \\ f(0) = 1, f'(0) = 0 \end{cases}$$

where $\sigma = \alpha + 1$ with $\alpha \in (0, 1)$.
Applying the Laplace transformation leads to have

$$F(s)(s^2 + s + \alpha(1 - s)) = s.$$

Now, the inverse Laplace transformation gives the exact solution as follows

$$\begin{aligned} f(x) &= exp(x(\alpha/2 - 1/2))(cosh(x(\alpha^2/4 - 3\alpha/2 + 1/4)^{1/2}) \\ &\quad + sinh(x(\alpha^2/4 - 3\alpha/2 + 1/4)^{1/2})(\alpha/2 - 1/2))/(\alpha^2/4 - 3\alpha/2 + 1/4)^{1/2}). \end{aligned}$$

1.4 Some Ordinary Nonlinear Caputo-Fabrizio fractional differential equations

1.4.1 Results of applying the Banach's Principle of Contraction theorem to boundary value problems

We prove the existence and uniquness of the nonlinear boundary value problems of the Caputo-Fabrizio differential equations by the help of the Banach contraction principle. [9]

Let $\mathbb{C}(I)$ *be the Banach space of continuous functions on* $I = [0, 1]$ *with supremum norm*

$$\|x\| = \sup_{s\in[0,1]} |x(s)|, \ x \in \mathscr{C}(I).$$

We now state the existence and uniquness of the solution in the next theorem.

Theorem 1.4.1. *[9] If* $\sigma = \alpha + 1, \alpha \in [0, 1] and F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function with the property that*

$$|F(x, u_1) - F(x, u_2)| \leq q|u_1 - u_2| u_1, u_2 \in \mathbb{R}, q > 0.$$

then the boundary value problem

$$\begin{cases} {}^{CF}D^{\sigma}u(x) = F(x, u(x)), \ x \geq 0, \\ u(0) = u_0, u(1) = u_1, \end{cases} \tag{1.15}$$

has a unique solution in $\mathscr{C}(I)$ *provided* $q < 1$.

1.4.2 Results of applying the Banach's Principle of Contraction theorem to initial value problems

Theorem 1.4.2. *[6] Let $0 < \alpha < 1, T > 0$ and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ a continuous function such that there exits $L > 0$ satisfying*

$$|F(x,t_1) - F(x,t_2)| \leq L|t_1 - t_2| \ \forall \ t_1, t_2 \in \mathbb{R}.$$

If $(a_\alpha + b_\alpha T)L < 1$, then the initial value problem

$$\begin{cases} {}^{CF}D^\alpha u(x) = F(x,u(x)), \ x \in [0,T] \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{1.16}$$

has a unique solution on $\mathscr{C}[0,T]$.

1.4.3 Results of applying the Krasnoselskii's fixed point theorem to Caputo-Fabrizio fractional differential equations

In this section, we use Krasnoselskii's fixed point theorem to obtain some results for the existence and uniqueness of a solution to caputo-fabrizio fractional differential equations. Hence, the equation reads as [10]

$$\begin{cases} {}_0^{CF}D_x^\alpha u(x) = f\left(x, u(x), {}_0^{CF} D_x^\alpha u(x)\right), \ x \in [0,T] = I \\ u(0) = u_0, \ u_0 \in \mathbb{R}, \end{cases} \tag{1.17}$$

Lemme 1.4.1. *[10] Let $v \in C[0,T]$, then the solution of fractional differential equations*

$$\begin{cases} {}_0^{CF}D_x^\alpha u(x) = v(x), \ x \in [0,T], \ 0 < \alpha \leq 1, \\ u(0) = u_0, u_0 \in \mathbb{R}, \end{cases} \tag{1.18}$$

is given as

$$u(x) = u_0 + B_\alpha[v(x) - v_0] + \bar{B}_\alpha \int_0^t v(\tau)d\tau \tag{1.19}$$

where

$$B_\alpha = \frac{1-\alpha)}{M(\alpha)}, \bar{B}_\alpha = \frac{\alpha}{M(\alpha)}.$$

Proof: Using the definition of ${}_0^{CF}I_x^{(\alpha)}$ *, (1.18) implies that*

$$u(x) = c + B_\alpha v(x) + \bar{B}_\alpha \int_0^x v(\tau)d\tau, \ c \in \mathbb{R} \tag{1.20}$$

Using the initial condition $u(0) = u_0$ *and* $v(0) = v_0 \in \mathbb{R}$*, from (1.20), we get* $c = u_0 - B_\alpha v_0$*. Hence by plugging the value of* c *in (1.20), we get (1.19).*

Remarque 1.4.1. *Henceforth, for simplicity, we use* ${}_0^{CF}D_x^{(\alpha)}u(x) = g_u(x)$ *for the implicit term in our analysis. Further, for simplicity, we use* $f(0, u(0), g_{\bar{u}}(0)) = f_0$*.*

Lemme 1.4.2. *[10] Under the conditions of Lemma (1.15), the solution of (1.16) is given by*

$$u(x) = u0 + B_{(\alpha)}[f(x, u(x), g_u(x)) - f_0] + \bar{B}_{(\alpha)} \int_0^t f(\tau, u(\tau), g_u(\tau))d\tau. \tag{1.21}$$

To proceed further, we assume that (H_1) *There exist* $L_f > 0$ *and* $0 < M_f < 1$ *such that*

$$|f(x, u, g_u) - f(x, \bar{u}, g_{\bar{u}})| \leq L_f|u - \bar{u}| + M_f \ |g_u - g_{\bar{u}}| \ \forall u, \bar{u}, g_u, g_{\bar{u}} \in \mathbb{R}.$$

Let $X = C(I)$ *be a Banach space with norm* $\|x\| = \sup_{x \in I} |u(x)|$*.*

Theorem 1.4.3. *[10] Under the assumption* (H_1)*, if the condition* $(B_\alpha + \bar{B}_\alpha T)\dfrac{L_f}{1 - M_f} < 1$ *holds, then the considered problem (1.16) has a unique solution.*

Theorem 1.4.4. *[10] Let the given assumption hold:* (H_2) *There exist constants* $a_f, b_f, c_f > 0$ *with* $0 < c_f < 1$ *such that*

$$|f(x, u, v)| \leq a_f + b_f|u| + c_f|v|.$$

Under the assumption (H_2)*, if* $0 < B_\alpha \dfrac{L_f}{1 - M_f} < 1$ *holds, then the considered problem (1.16) has at least one solution.*

Chapter 2

Numerical approximation of the space-time Caputo-Fabrizio fractional derivative and application to groundwater pollution equation

2.1 Introduction

In the last decade,many physical problems have been modeled using the concept of noninteger-order derivative.These derivatives of fractional order range from Riemann-Liouville via Caputo to Caputo-Fabrizio [12, 13]. We can find in the literature many analytical approaches to deal with differential equations with fractional equations [15, 14]. Most of these techniques are dealing with linear fractional differential equations. However, most fractional differential equations describing real-world problems are highly complicated and cannot sometime be handled via analytical methods.In order to solve these problems in many cases, researchers rely on the use of numerical methods because these problems have initial conditions, boundary condition, and source terms that turn hard to find an analytical solution.
Several numerical approaches in connection with derivatives of fractional order describing real-world problems alter essentially in the many in which the derivative of fractional order is tailored [11, 16]. Approximation representation of a derivative of fractional order has a highly complicated formula compared to those of integer order because fractional derivatives are nonlocal, and therefore the calculation at a particular point requires knowl-edge

of the function further out of the region close to that point. Accordingly, finite difference approximations of derivatives of fractional order engage a quantity of points that alters according to how faraway we are from the border line [17, 19]. The most recent derivative of fractional order was proposed by Caputo and Fabrizio [12],who demonstrated that the new-fangled derivative encompasses extra encouraging properties in comparison with the old version. They demonstrated, for example, that it can depict substance heterogeneities and configurations with different scales, which obviously cannot be overseeing with the prominent local theories and also the well-known fractional derivative. An additional application is in the investigation of the macroscopic behaviors of some materials, associated with nonlocal communications between atoms, which are recognized to be important of the properties of material. We present the definition of the Caputo fractional derivative.

Definition 2.1.1. *The Caputo derivative of fractional order old editor of a function f is given as*

$$ {}_0^C D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n}(f(x))dt, n-1 < \alpha \leq n. \tag{2.1} $$

Definition 2.1.2. *Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$. Then the new Caputo fractional derivative is defined as*

$$ {}_0^{CF} D_t^\alpha(f(t)) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(x) exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \tag{2.2} $$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$ [12]. However, if the function does not belong to $H^1(a,b)$ then, the derivative can be redefined as

$$ {}_0^{CF} D_t^\alpha(f(t)) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(x)) exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx \tag{2.3} $$

The aim of this paper is to propose a numerical approximation of the space and time Caputo-Fabrizio derivative of fractional order that will be used by researchers in the field of numerical analysis.

2.2 Caputo-Fabrizio approximations

In this section, we derive a numerical approximation based upon the definition of newly proposed derivative of fractional order [18],

$$ {}_{0}^{CF}D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{1-\alpha}\int_0^t f'(x)exp\left[-\alpha\frac{t-x}{1-\alpha}\right]dx \tag{2.4}$$

For some positive integer N, the grid size in time for finite difference technique is defined by $k = \frac{1}{N}$.

The grid points in the time interval $[0,T]$ *are labeled* $t_n = nk, n = 0,1,2,...,TN$. *The value of the function f at the grid point is* $f_i = f(t_i)$.

A discrete approximation to the Caputo-Fabrizio derivative of fractional order can be obtained by simple quadrature formula as follows:

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_n)) = \frac{M(\alpha)}{1-\alpha}\int_0^{t_n} f'(x)exp\left[-\alpha\frac{t_n-x}{1-\alpha}\right]dx. \tag{2.5}$$

This equation can be modified using the first-order approximation to

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_j)) = \frac{M(\alpha)}{1-\alpha}\sum_{j=1}^{n}\int_{(j-1)k}^{j_k}\left(\frac{f^{k+1}-f^k}{\Delta t}+O(\Delta t)\right)exp\left[-\alpha\frac{t_j-x}{1-\alpha}\right]dx \tag{2.6}$$

Before integration we obtain the following expression

$$ \frac{M(\alpha)}{1-\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^j}{\Delta t}+O(\Delta t)\right)\int_{(j-1)k}^{jk} exp\left[-\alpha\frac{t_n-x}{1-\alpha}\right]dx, \tag{2.7}$$

$${}_{0}^{CF}D_t^{\alpha}(f(t_j)) = \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^j}{\Delta t}+O(\Delta t)\right)d_{j,k},$$

where

$$ d_{j,k} = exp\left[-\alpha\frac{k}{1-\alpha}(n-j+1)\right] - exp\left[-\alpha\frac{k}{1-\alpha}(n-j)\right]. \tag{2.8}$$

We finally have that

$$ {}_{0}^{CF}D_t^{\alpha}(f(t_n)) = \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^j}{\Delta t}+\right)d_{j,k} + \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}d_{j,k}O(\Delta t). \tag{2.9}$$

Theorem 2.2.1. *Let $f(x)$ be a function in $C^2[a,b]$, and let the order of the fractional derivative be $0 < \alpha \leq 1$. Then the first-order approximation of the Caputo-Fabrizio derivative at a point t_n is*

$$ {}_0^{CF}D_t^{\alpha}(f(t_n)) = \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^{j}}{\Delta t} + O((\Delta t)^2)\right)d_{j,k}. \tag{2.10} $$

Proof *From equation (2.8) we have*

$$ \begin{aligned} {}_0^{CF}D_t^{\alpha}(f(t_n)) &= \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^{j}}{\Delta t}\right)d_{j,k} \\ &+ \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(exp\left[-\alpha\frac{k}{1-\alpha}(n-j+1)\right] - exp\left[-\alpha\frac{k}{1-\alpha}(n-j)\right]\right)O((\Delta t)^2) \end{aligned} $$

However,

$$ \sum_{j=1}^{n}\left(exp\left[-\alpha\frac{k}{1-\alpha}(n-j+1)\right] - exp\left[-\alpha\frac{k}{1-\alpha}(n-j)\right]\right) = exp\left[-\alpha\frac{k}{1-\alpha}(n)\right] - 1. \tag{2.11} $$

Now the approximation of the exponential function can be obtained as

$$ exp\left[-\alpha\frac{k}{1-\alpha}(n)\right] \approx 1 - \alpha\frac{k}{1-\alpha}(n). \tag{2.12} $$

Then replacing the above in equation (2.11), we obtain

$$ \sum_{j=1}^{n}\left(exp\left[-\alpha\frac{k}{1-\alpha}(n-j+1)\right] - exp\left[-\alpha\frac{k}{1-\alpha}(n-j)\right]\right) \approx -\alpha\frac{k}{1-\alpha}(n). \tag{2.13} $$

Then equation (2.11) becomes

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^{j}}{\Delta t}\right)d_{j,k} + \frac{M(\alpha)k}{1-\alpha}(n)O(\Delta t). \tag{2.14} $$

We therefore obtain the requested result

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^{j}}{\Delta t}d_{j,k}\right) + O((\Delta t^2)) \tag{2.15} $$

This completes the proof.

We now conclude that the first-order approximation method for the computation of the Caputo-Fabrizio derivative of time fractional order is given as

$$ {}_0^{CF}D_t^{\alpha}(f(t)) = \frac{M(\alpha)}{\alpha}\sum_{j=1}^{n}\left(\frac{f^{j+1}-f^{j}}{\Delta t}\right)d_{j,k}. \tag{2.16} $$

We next propose the first order for the space fractional order.

For some positive integer N, the grid sizes in time for finite difference technique is defined by $i = \frac{1}{M}$

The grid points in the time interval [0,X] are labeled $x_i = mi$, *m =0,1,2,...,XM.*

The value of the function f at the grid point is $f_i^k = f(x_i, t_k)$. *We have*

$$ {}_0^{CF}D_t^{\alpha}(f(x_m,t_i)) = \frac{M(\alpha)}{\sqrt{\pi}(1-\alpha)}\int_0^{x_m}\frac{\partial}{\partial y}f'(y,t_i)exp\left[-\alpha^2\frac{(x_m-y)^2}{(1-\alpha)^2}\right]dy. \tag{2.17} $$

Now employing the Crank-Nicolson approximation for the first-order derivative, the above equation is converted to

$$ \begin{aligned} {}_0^{CF}D_x^{\alpha}(f(x_m,t_k)) =& \frac{M(\alpha)}{\sqrt{\pi}(1-\alpha)}\int_0^{x_m}\left(\frac{(f_{i+1}^{k+1}-f_{i-1}^{k+1})-(f_{i+1}^{k}-f_{i-1}^{k})}{4\Delta x}+O(\Delta x)\right) \\ & \times exp\left[-\alpha^2\frac{(x_m-y)^2}{(1-\alpha)^2}\right]dy. \end{aligned} \tag{2.18} $$

The latter equation can be converted to

$$ \begin{aligned} {}_0^{CF}D_x^{\alpha}(f(x_m,t_i)) =& \frac{M(\alpha)}{\sqrt{\pi}(1-\alpha)}\sum_{l=1}^{m}\left\{\frac{(f_{l+1}^{k+1}-f_{l-1}^{k+1})-(f_{l+1}^{k}-f_{l-1}^{k})}{4\Delta x}+O(i)\right\} \\ & \times\int_{(l-1)i}^{li}exp\left[-\alpha^2\frac{(im-y)^2}{(1-\alpha)^2}\right]dy. \end{aligned} \tag{2.19} $$

where the integral part is given as

$$\begin{aligned}&\int_{(l-1)i}^{li} exp\left[-\alpha^2\frac{(im-y)^2}{(1-\alpha)^2}\right]dy = \\ &\quad\frac{(1-\alpha)\sqrt{\pi}}{2\alpha}\{erf\left[(mi-li)\frac{\alpha}{1-\alpha}\right]-erf\left[(mi-li+i)\frac{\alpha}{1-\alpha}\right]\},\end{aligned} \tag{2.20}$$

so that equation (2.15) *becomes*

$$\begin{aligned}{}_0^{CF}D_x^{\alpha}(f(x_m,t_k)) = \frac{M(\alpha)}{1-\alpha}\sum_{l=1}^{m}\left\{\frac{(f_{l+1}^{k+1}-f_{l-1}^{k+1})-(f_{l+1}^{k}-f_{l-1}^{k})}{4\Delta x}+O(i)\right\}\\ \times\frac{(1-\alpha)}{2\alpha}\left\{erf\left[(m-l)\frac{\alpha i}{1-\alpha}\right]-erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\right\},\end{aligned} \tag{2.21}$$

From the above we obtain

$$\begin{aligned}&{}_0^{CF}D_x^{\alpha}(f(x_m,t_k)) = \\ &\quad\frac{M(\alpha)}{1-\alpha}\sum_{l=1}^{m}\{\frac{(f_{l+1}^{k+1}-f_{l-1}^{k+1})-(f_{l+1}^{k}-f_{l-1}^{k})}{4\Delta x}\frac{(1-\alpha)\sqrt{\pi}}{2\alpha}\\ &\quad\times\{erf\left[(m-1)\frac{\alpha i}{1-\alpha}\right]-erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\}\}\\ &\quad+O(i)\frac{(1-\alpha}{2\alpha}\sum_{l=1}^{m}\{erf\left[(m-1)\frac{\alpha i}{1-\alpha}\right]-erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\}\end{aligned} \tag{2.22}$$

Theorem 2.2.2. *Let f (x,t) be a function in* $C^2([a,b]\times[0,T])$*, and let the order of the fractional derivative be* $0<\alpha\leq 1$*. Then the first-order approximation of the Caputo-Fabrizio derivative at a point* (x_m,t_n) *is*

$${}_0^{CF}D_x^{\alpha}(f(x_m,t_k)) = \frac{M(\alpha)}{2\alpha}\sum_{l=1}^{m}\left\{\frac{(f_{l+1}^{k+1}-f_{l-1}^{k+1})-(f_{l+1}^{k}-f_{l-1}^{k})}{4\Delta x}\right\}d_{i,l}+R(\alpha,i,l) \tag{2.23}$$

where

$$d_{i,l}=\left\{erf\left[(m-1)\frac{\alpha i}{1-\alpha}-erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\right\}\right\},\ \parallel R(\alpha,i,l)\parallel < M.$$

Proof *From equation (2.23) we have that*

$$
\begin{aligned}
&{}_0^{CF}D_x^{\alpha}(f(x_m,t_k)) = \\
&\quad \frac{M(\alpha)}{2\alpha}\sum_{l=1}^{m}\{\frac{(f_{l+1}^{k+1}-f_{l-1}^{k+1})-(f_{l+1}^{k}-f_{l-1}^{k})}{4\Delta x}\{erf\left[(m-1)\frac{\alpha i}{1-\alpha}\right] \\
&\quad - erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\}\} \\
&\quad + O(i)\frac{M(\alpha)}{2\alpha}\sum_{l=1}^{m}\{erf\left[(m-1)\frac{\alpha i}{1-\alpha}\right] - erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\}
\end{aligned}
$$

We put

$$
R(\alpha,i,l) = O(i)\frac{M(\alpha)}{2\alpha}\sum_{l=1}^{m}\left\{erf\left[(m-1)\frac{\alpha i}{1-\alpha} - erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\right\}\right\}. \tag{2.24}
$$

Then taking the norm to both sides, we have

$$
\begin{aligned}
&\parallel R(\alpha,i,l) \parallel = \parallel O(i)\frac{M(\alpha)}{2\alpha}\sum_{l=1}^{m}\left\{erf\left[(m-l)\frac{-\alpha i}{1-\alpha}\right] - erf\left[(m-l+1)\frac{-\alpha i}{1-\alpha}\right]\right\} \parallel \\
&\parallel R(\alpha,i,l) \parallel = \parallel O(i)\frac{M(\alpha)}{2\alpha}\left(erf\left[m\frac{-\alpha i}{1-\alpha}\right]\right) \parallel
\end{aligned}
\tag{2.25}
$$

This completes the proof. Then, the first-order approximation method for the computation of Caputo-Fabrizio derivative of space fractional order is given as

$$
\begin{aligned}
{}_0^{CF}D_x^{\alpha}(f(x_m,t_i)) = &\frac{M(\alpha)}{2\alpha}\sum_{l=1}^{m}\{\frac{(f_{l+1}^{k+1}-f_{l-1}^{k+1})-(f_{l+1}^{k}-f_{l-1}^{k})}{4\Delta x} \\
&\times\{erf\left[(m-1)\frac{\alpha i}{1-\alpha}\right] - erf\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\}\}
\end{aligned}
\tag{2.26}
$$

2.3 Application to some well-known equations

In this section, we present a numerical solution of the time fractional advection diffusion equation in heterogeneous medium. The fractional derivative used here is of the Caputo-Fabrizio type. The reason of this modification is that the fractional derivatives are recollection operational which recurrently distinguish indulgence of force or damage in the passable *as in the case of inelastic media or reconsideration of the porosity in the thinning out in permeable media and supplementary in comprehensive they are in traditional values throughout the subsequent theory of hydrology. They are accredited not merely for the motivation that they match appropriately a variety of noticeable actuality, nevertheless, additionally for the motive that they own the well-designed alongside with scrupulous property that although the order of differentiation is integer, they match by means of the traditional derivative of that order. On the other hand, this chattel is not pertinent to the effect they characterize in the physical observable fact and conjectures if using other differential operators, probably simpler nevertheless devoid of this property, one may get similar responds of fractional order derivative. Therefore, in order to well replicate the flow of the particles via porous media in different scale in the medium, we replace the ordinary derivative in time with the scale time derivative proposed by Caputo and Fabrizio. The equation under consideration here is*

$$ {}_{0}^{CF}D_t^{\alpha}(P(x,t)) + \frac{\varphi u}{c_p}\frac{\partial p(x,t)}{\partial x} - \frac{\lambda}{c_p}\frac{\partial^2 P(x,t)}{\partial x^2} = \frac{Q(x,t)}{c_p}. \tag{2.27} $$

In equation (2.27), four terms represents transient, advection, and source terms, respectively,P(x,t) is the particle, heat, pollution,or other physical quantities; c is the specific of heat, particles, or other physical quantities; φ *is the porosity that is the ratio of the liquid volume to the total volume of the medium via which the flow is taken place,*ρ, λ *are the mass density and thermal conductivity, respectively, and, finally,Q(x,t) is the source term. Now substituting equation (2.26) into (2.27), we obtain*

$$ \begin{aligned} &\frac{M(\alpha)}{2\alpha}\sum_{k=1}^{j+1}\frac{P_i^k - P_i^{k-1}}{\tau}\left(erf\left((j-k)\frac{\alpha k}{1-\alpha}\right) - erf\left(((j-k+1)\frac{\alpha k}{1-\alpha}\right)\right) = \\ &\frac{\lambda}{2h^2 c_p}\left\{(P_{i+1}^{j+1} - 2P_i^{j+1} + P_{i-1}^{j+1}) + (P_{i+1}^{j} - 2P_i^{j} + P_{i-1}^{j})\right\} \\ &- \frac{\varphi u}{c_p}\left\{(P_{i+1}^{j+1} - P_{i-1}^{j+1}) + (P_{i+1}^{j} - P_{i-1}^{j})\right\} + \frac{Q_i^{j+1} + Q_i^j}{2c_p}. \end{aligned} \tag{2.28} $$

The above equation can be converted to

$$\begin{aligned}
&\frac{M(\alpha)}{2\alpha}\left(\frac{P_i^{j+1}-P_i^j}{\tau}+\sum_{k=1}^{j+1}\frac{P_i^{j+1-k}-P_i^{j-k}}{\tau}\right)d_{k,j}^{\alpha}\\
&=\frac{\lambda}{2h^2c_p}(P_{i+1}^{j+1}-2P_i^{j+1}+P_{i-1}^{j+1})+(P_{i+1}^{j}-2P_i^{j}+P_{i-1}^{j})\\
&-\frac{\varphi u}{4hc_p}(P_{i+1}^{j+1}-P_{i-1}^{j+1})+(P_{i+1}^{j}-P_{i-1}^{j})+\frac{Q_i^{j+1}+Q_i^j}{2c_p}.
\end{aligned}\tag{2.29}$$

For simplicity, let us put
$a=\dfrac{M(\alpha)}{2\alpha\tau\tau}, b=\dfrac{\lambda}{2h^2c_p}, c=\dfrac{\varphi u}{4hc_p}$

Rearranging, we obtain the following recursive formula:

$$\begin{aligned}
(ad_{k,j}^{\alpha}+2b)P_i^{j+1}&=(ad_{k,j}^{\alpha}-2b)P_i^{j}+\alpha\sum_{k=1}^{j}(P_i^{j+1-k}-P_i^{j-k})d_{k,j}^{\alpha}\\
&+b(P_{i+1}^{j+1}+P_{i-1}^{j+1})+(P_{i+1}^{j}+P_{i-1}^{j})-c(P_{i+1}^{j+1}-P_{i-1}^{j+1})+(P_{i+1}^{j}-P_{i-1}^{j})\\
&+\frac{Q_i^{j+1}+Q_i^j}{2c_\rho}.
\end{aligned}\tag{2.30}$$

2.4 Stability analysis of the numerical scheme

We present in this section the stability analysis of the Crank-Nicolson scheme for time fractional advection diffusion equation. For this, we let $e_i^j=P_i^j-p_i^j$ *with* p_i^j *the approximate solution at the point* $(x_i,t_j)(i=1,2,...,N;j=1,2,...,M)$ *and, as usual,* $e^j=[e_1^j,...,e_N^j]^T$. *The error committed while solving the time fractional advection diffusion equation with the Crank-Nicolson scheme is*

$$\begin{aligned}
(ad_{k,j}^{\alpha}+2b)e_i^{j+1}&=(ad_{k,j}^{\alpha}-2b)e_i^{j}+a\sum_{k=1}^{j}(\exp_i^{j+1-k}-\exp_i^{j-k})d_{k,j}^{\alpha}\\
&+b(\exp_{i+1}^{j+1}+\exp_{i-1}^{j+1})+(\exp_{i+1}^{j}+\exp_{i-1}^{j})-c(\exp_{i+1}^{j+1}-\exp_{i-1}^{j+1})+(\exp_{i+1}^{j}-\exp_{i-1}^{j})\\
&+\frac{Q_i^{j+1}+Q_i^j}{2c_\rho}.
\end{aligned}\tag{2.31}$$

Here, we assume that

$$\exp_i^j = f(j)exp(\tau\sigma_{ij}), \tag{2.32}$$

where σ *is the real spatial wave number [18] . However, substituting equation (2.32) into equation (2.31), we obtain, for* $j = 0$,

$$\left(ad_{k,0}^{\alpha} + 4b\sin^2\left(\frac{\sigma i}{2}\right)\right) f(1) = \left(ad_{k,0}^{\alpha} - 4b\sin^2\left(\frac{\sigma i}{2}\right)\right) f(0), \tag{2.33}$$

and for $j > 0$, *we have*

$$\left(ad_{k,j}^{\alpha} + 4b\sin^2\left(\frac{\sigma i}{2}\right)\right) f(j) = \left(ad_{k,j}^{\alpha} - 4b\sin^2\left(\frac{\sigma i}{2}\right)\right) f(j-1) - a\sum_{l=1}^{j-1} f(j-l)d_{k,j}^{\alpha} + f(j+1)d_{k,0}^{\alpha}. \tag{2.34}$$

Theorem 2.4.1. *Assume that* $f(k)$ *satisfies equations (2.33) and (2.34). Then, for all* $k > 0$,

$$|f(j)| \leq |f(0)|. \tag{2.35}$$

Proof *We shall prove this theorem by employing the recursive method on the natural number* j. *Then, when* $j = 0$, *we have equation (2.33), and we reformulate it as follows:*

$$\left|\frac{f(1)}{f(0)}\right| = \left|\frac{(ad_{k,0}^{\alpha} - 4b\sin^2(\sigma i))}{(ad_{k,0}^{\alpha} + 4b\sin^2(\sigma i))}\right| \leq 1. \tag{2.36}$$

This implies

$$|f(j)| \leq |f(0)|.$$

The property is verified for $j = 0$. *Let us assume that this property is also satisfied for any* $j \geq 1$. *We shall verify that the property holds also for* $j + 1$:

$$\left(ad_{k,j}^{\alpha} + 4b\sin^2\left(\frac{\sigma i}{2}\right)\right) f(j+1) = \left(ad_{k,j}^{\alpha} - 4b\sin^2\left(\frac{\sigma i}{2}\right)\right) f(j) - a\sum_{l=1}^{j} f(j-l)d_{k,l}^{\alpha}. \tag{2.37}$$

Now taking the norms of both sides of equation (2.37), we obtain

$$\left|ad_{k,j}^{\alpha}+4b\sin^2\left(\frac{\sigma i}{2}\right)\right||f(j+1)|\leq\left|\left(ad_{k,j}^{\alpha}-4b\sin^2\left(\frac{\sigma i}{2}\right)\right)\right||f(j)|+\sum_{l=1}^{j}|f(j-l)|d_{k,l}^{\alpha}. \quad (2.38)$$

Nonetheless, we recall that the property holds up to j .Thus we transform the above equation into

$$\left|ad_{k,j}^{\alpha}+4b\sin^2\left(\frac{\sigma i}{2}\right)\right||f(j+1)|\leq\left|\left(ad_{k,j}^{\alpha}-4b\sin^2\left(\frac{\sigma i}{2}\right)\right)\right||f(0)|+\sum_{l=1}^{j}|f(0)|d_{k,l}^{\alpha}.$$

Rearranging, we obtain

$$\begin{aligned}&\left|ad_{k,j}^{\alpha}+4b\sin^2\left(\frac{\sigma i}{2}\right)\right||f(j+1)|\\&\leq\left\{\left|\left(ad_{k,j}^{\alpha}-4b\sin^2\left(\frac{\sigma i}{2}\right)\right)\right|+\left\{erf\left[\frac{m\alpha i}{1-\alpha}\right]-erf\left[\frac{\alpha i}{1-\alpha}\right]\right\}\right\}|f(0)|.\end{aligned} \quad (2.39)$$

It is important to recall that

$$|erf[x]\ |\leq1,\qquad erf\left[\frac{m\alpha i}{1-\alpha}\right]-erf\left[\frac{\alpha i}{1-\alpha}\right]\leq0. \quad (2.40)$$

Therefore,

$$\frac{|f(j+1)|}{|f(0)|}\leq|\{|\left(ad_{k,j}^{\alpha}-4b\sin^2\left(\frac{\sigma i}{2}\right)\right)|+\{erf\left[\frac{m\alpha i}{1-\alpha}\right]-erf\left[\frac{\alpha i}{1-\alpha}\right]\{|\leq\ 1. \quad (2.41)$$

Then,

$$\frac{|f(j+1)|}{|f(0)|}\ \leq\ 1. \quad (2.42)$$

The property also holds for $j+1$. *According to the inductive technique, the property is satisfied for any natural number. This completes the proof of Theorem* (2.4.1). *Theorem* (2.4.1) *shows that the Crank-Nicolson scheme is stable for the advection diffusion equation with the time fractional Caputo-Fabrizio derivative.*

2.5 Convergence analysis of the numerical solution

Let us suppose that, at the point (x_i, t_j)*, the exact solution of our considered equation is* $P(x_i, t_j)(i = 1, 2, 3, ..., N; j = 1, 2, 3, 4, ..., M)$*. We assume that the difference between the exact solution and the approximate solution at that particular point is provided by* $\delta_i^j = P(x_i, t_j) - P_i^j$*.The transpose matrix associated with the matrix* $\delta_i^j = P(x_i, t_j) - P_i^j(i =$ $(i = 1, 2, 3, ..., N; j = 1, 2, 3, 4, ..., M)$ *is* $(\delta_1^j, \delta_2^j, ..., \delta_N^j)^T$*.However, the row* δ^0 *is zero because it represents the initial condition. The recursive relation in connection with the Crank-Nicolson scheme for the time fractional advection diffusion equation is given as*

$$\begin{aligned}
&(ad_{k,0}^{\alpha} + 2b)\delta_i^1 + (c - b)(\delta_{i+1}^1 + \delta_{i-1}^1 - \frac{Q_i^1 - Q_i^0}{2c_\rho} = R_i^1, \ for j = 0. \\
&(ad_{i,j}^{\alpha} + 2b)\delta_i^{j+1} - (ad_{i,j}^{\alpha} + 2b)\delta_i^j + (c - b)(\delta_{i+1}^{j+1} + \delta_{i-1}^{j+1} + (c - b)(\delta_{i+1}^j + \delta_{i-1}^j - \frac{Q_i^{j+1} - Q_i^j}{2c_\rho} \\
&= -a\sum_{l=1}^{j-1} \delta_i^{j-l} d_{i,j}^{\alpha} + R_i^{j+1}, \ for j > 0.
\end{aligned} \tag{2.43}$$

The remainder term of the approximation for using the Crank-Nicolson scheme to solve the modified advection diffusion equation is given in this case as
$R_i^{j+1} \leqslant D(2k + h^2)$.

Theorem 2.5.1. *The Crank-Nicolson scheme for the advection diffusion equation with time fractional Caputo-Fabrizio derivative converges, and there exists a positive constant D such that*

$$\parallel P(x_i, t_i) - P_i^j \parallel \leqslant D(2k + h^2). \ for\ all\ (i = 1, 2, 3, ..., M; \ j = 1, 2, 3, 4, ..., N). \tag{2.44}$$

2.6 Numerical simulation for different values of alpha

In this section, using the new numerical scheme, we present the numerical simulation of the advection diffusion equation with the Caputo-Fabrizio derivative of fractional order for different values of alpha.

We choose in this case $Q(x,t) = sin[x + \frac{\pi}{4}]P(x,0) = cos[x], P(0,t) = cos[t], P(x,10) = 0, u = 0.1, c_\rho = 0.9, \varphi = 3, \lambda = 0.75$.

The numerical simulations are depicted in Figures 1, 2, 3, and 4. It is worth noting that each figure represents the flow at scale alpha. It is very important to realize that fractional differentiation is able to control the variabilities of the plume movement within the geological formations. The pollution does not only move within a homogeneous medium but also within heterogeneous one; therefore, the plume paths cannot be predicted by the classical advection dispersion equation. In these figures, we can see that the proportionally of the density of pollution within the geological formation is not the same everywhere due to the heterogeneity, and

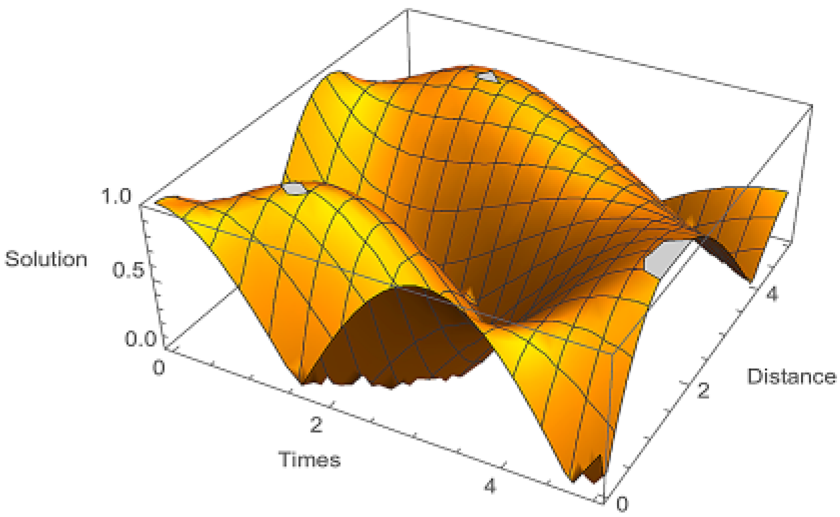


Figure 1 Numerical simulation at scale alpha = 0.15.

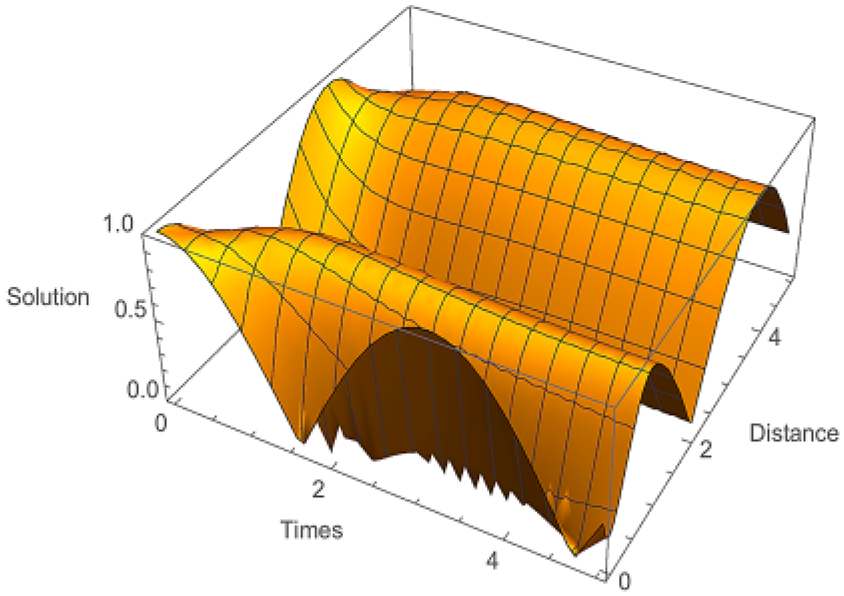


Figure 2 Numerical simulation at scale alpha = 0.55.

this is better described via the concept of fractional differentiation with nonsingular kernel

We have proposed in this work the numerical approximation of the newly proposed derivative of fractional order in order to fit this derivative in the scope of numerical investigations. The new derivative is easy to use even numerically and display important characteristics that cannot be observed in the commonly used fractional derivatives. In order to test the possible application of the new numerical approximation of the new Caputo Fabrizio derivative of fractional order, we presented a model of advection diffusion equation with the time fractional of the new derivative. We solved this equation numerically using the Crank-Nicolson technique. We showed the stability analysis together with some numerical simulations for different values of alpha.

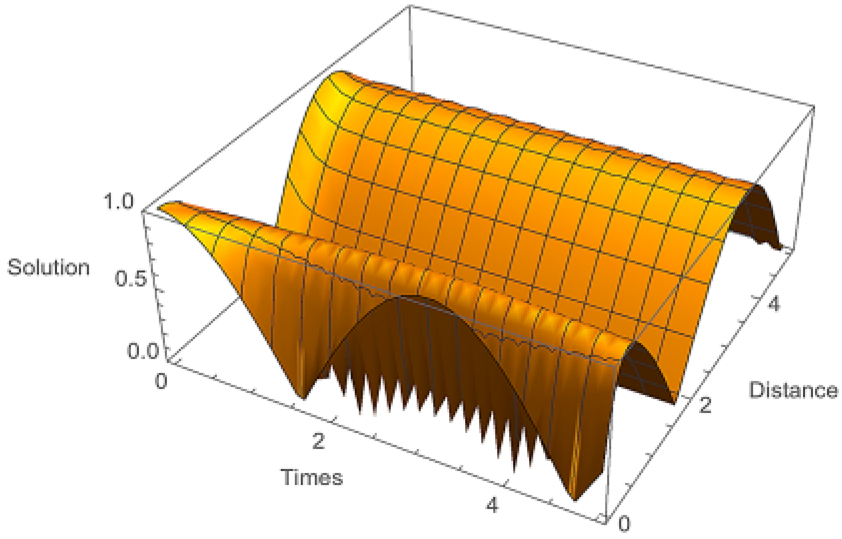


Figure 3 **Numerical simulation at scale alpha = 0.85.**

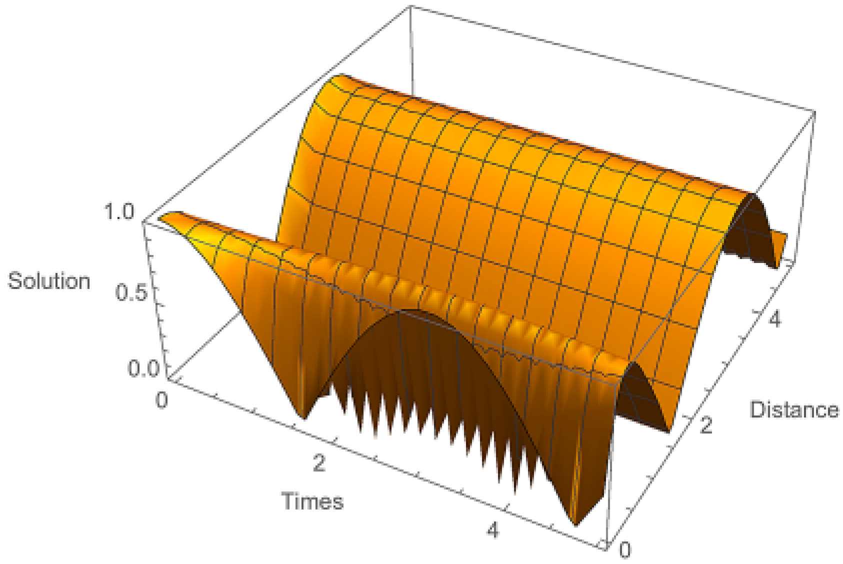


Figure 4 **Numerical simulation at scale alpha = 0.95.**

Chapter 3

Application to Ordinary Fractional Differential Equations

3.1 Numerical Approximation of Fractional Ordinary Differential Equation with the Caputo Derivative

The numerical approximation of classical ordinary differential equations is relatively simple and, being a focus of mathematical studies for the last few decades, has been by now almost completely investigated. However, a fractional case is much less studied and is still poorly understood despite the fact that there has been a growing interest in the research of this area. In short, there has been just handful of research papers and books considering the numerical approximation of time fractional ordinary differential equations.

Langlands and Henry [20] examined the fractional-order time diffusion equation, and introduced an L_1-stable scheme for this equation. Sun and Wu [22] derived a finite difference method with L_1 approximation for the fractional-in-time derivative. Lin and Xu [21] construct and analyse a finite difference scheme for the time discretization of the time-fractional diffusion equation, and showed that the time convergence is of order $2-\alpha$ .Zhao et al.[23] derived two second-order approximation schemes for time-fractional derivatives involved in anomalous diffusion and wave propagation. Numerical technique for a class of fractional *ordinary differential equations was proposed by Kumar and Agrawal[24], in which their approach can be reduced into a Volterra-type integral equation. A general technique for high-order numerical schemes based on this approach is constructed by Cao and Xu[25].*

In this chapter, we use the following numerical approximation of the Caputo derivative, which follows closely the work done in [21, 26]. We assume a uniform time step size k*, and let* $t^n = nk$*,* n = *0,1,2,... . We also assume* U^n*to be the numerical approximation of* $u(t^n)$*. To derive a first-order method, for* t = t *n+1, we assume a uniform partition in time,*

$$0 = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = t.$$

Next, we approximate each standard time derivative by the forward difference in the form

$$\begin{aligned}
\partial_t^\alpha u(t^{n+1}) &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{s=0}^{n} \int_{t_s}^{t_{s+1}} \frac{U^{s+1} - U^s}{k(t_{n+1} - \xi)} d\xi \\
&= \frac{1}{\Gamma(1-\alpha)(1-\alpha)} \sum \frac{(n+1-s)^{1-\alpha} - (n-s)^{1-\alpha}}{k^\alpha} (U^{s+1} - U^s) \\
&= \frac{1}{\Gamma(2-\alpha)k^\alpha} \left(U^{n+1} - \sum_{n=0}^{k} d_s^{n+1} U^s \right) \\
&:= D_t^\alpha U^{n+1}
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
d_s^{n+1} &= 2(n+1-s)^{1-\alpha} - (n+2-s)^{1-\alpha} - (n-s)^{1-\alpha}, \quad s = 1, 2, .., n, \\
d_0^{n+1} &= (n+1)^{1-\alpha} - n^{1-\alpha}.
\end{aligned}$$

Bear in mind that $y = x^{1-\alpha}$ *is a concave and increasing function for which* $x > 0$*, by direct calculation, we obtain*

$$\sum_{n=0}^{n} d_s^{n+1} = 1, \quad d_s^{n+1} > 0, \quad s = 0, 1, 2, ..., n. \tag{3.2}$$

Therefore, while the standard time derivative yields the instantaneous rate of change, from its numerical approximation (3.1), one can interpret the Caputo derivative as the rate of change of a quantity from a convex combination of its history values, and the coefficients d_s^{n+1} *indicate the influence strength due to the memory effect. As memory effect becomes weaker, the influence strength of a history values decreases in time.*

It should be mentioned that, we obtain for a fixed value of order α,

$$d_n^{n+1} = 2 - 2^{1-\alpha} - 0^{1-\alpha} = 2 - 2^{1-\alpha},$$

independent of n. *Further,we denote* d_n^{n+1} *by* $\widehat{d}$ *which plays a vitalrole in the Courant-Friedrichs-Lewy conditions for explicit upwind schemes, as we shall discuss later.*

3.1.1 Numerical Schemes and Stability Analysis

Here, we adopt the numerical approximation of the Caputo derivative which was introduced in the previous chapter to formulate numerical schemes for ordinary differential equations, with conservation laws. We focus on examining the stability condition for each scheme and demonstrate how they differ from the models with classical time derivatives.

Backward Euler Scheme

We consider the time-fractional ODE model

$$\partial_t^\alpha u(t) = \lambda u(t). \tag{3.3}$$

where λ *is complex number with* $\Re(\lambda) \leq 0$, *which is similar to the eigenvalue of an operator.*

The stability analysis of many numerical schemes for ODE in conjunction with the Caputo derivatives has been examined in some previous research materials and books, see, for instance, [27, 25, 28, 29]. By applying the backward Euler method for(3.3)*, we have*

$$D_t^{\alpha} U^{n+1} = \lambda U^{n+1} \tag{3.4}$$

Multiply each side of(3.4) *by* $k^{\alpha}\Gamma(2-\alpha)$*, the above equation becomes*

$$(1 - \lambda\kappa^{\alpha}\Gamma(2-\alpha))U^{n+1} = \sum_{s=0}^{n} d_s^{n+1} U^s.$$

If we let $z = \lambda\kappa^{\alpha}\Gamma(2-\alpha)$ *, the stability polynomial* $\pi(\xi; z)$ *for the above scheme is*

$$\pi(\xi; z) = (1-z)\xi^{n+1} - \sum_{s=0}^{n} d_s^{n+1}\xi^s.$$

In what follows, we discuss various scenarios, that is, when $\lambda \neq 0$ *and when* $\lambda = 0$*.*

For the case $\lambda \neq 0$*, we have* $\mathfrak{R}(z) \leq 0$ *and* $z \neq 0$*, then we finally obtain* $\mid 1 - z \mid > 1$ *. If we assume* ξ_0 *with* $\mid \xi_0 \mid \geqslant 1$ *is a root to* $\pi(\xi; z)$*, then for* $s \leq n$*, we get*

$$|\xi_0^s| \leq |\xi_0|^s \leq |\xi_0^{n+1}|.$$

Then, we obtain

$$\begin{aligned} |(1-z)\xi_0^{n+1}| = |1-z||\xi_0|^{n+1} &= \left| \sum_{s=0}^{n} d_s^{n+1}\xi_0^s \right| \\ &\leqslant \sum_{s=0}^{n} d_s^{n+1}|\xi_0|^s \leqslant \left(\sum_{s=0}^{n} d_s^{n+1} \right) |\xi_0|^{n+1} = |\xi_0|^{n+1}, \end{aligned}$$

which is a contradiction. This implies that the stability polynomial has only roots with modulus less than 1, and hence the method is absolute stable. When $\lambda = 0$*, then* $z = 0$*, and the stability analysis in this case reduces to the zero stability of the time discretization. If* *the modulus of the root of the stability polynomial is unity, that is, 1, we assume that the root is* $\xi_0 = e^{i\Theta}$. *If* $\xi = 0$*, then* $\eta_0 = 1$*, so that*

$$\pi(1;0) = 1^{n+1} - \sum_{s=0}^{n} d_s^{n+1} = 0,$$

and we compute

$$\frac{d\pi(\xi;0)}{d\xi} = (n+1)\xi^n - \sum_{s=0}^{n} d_s^{n+1} s \xi^{s-1}.$$

The coefficients $\{d_s^{n+1}\}_{s=0}^{n}$ *satisfy* (3.2)*, and therefore*

$$\left|\sum_{s=0}^{n} d_s^{n+1} s\right| < \left|\sum_{s=0}^{n} d_s^{n+1} n\right| = n.$$

We can now conclude that

$$\frac{d\pi}{d\xi} = (n+1) - \sum_{s=0}^{n} d_s s > n+1 = 1 \neq 0.$$

Hence, 1 is not a repeated root of the stability polynomial.

If $\theta \neq 0$*, then we obtain the following equation:*

$$e^{i(n+1)\theta} = \sum_{s=0}^{n} d_s^{n+1} e^{is\theta}.$$

We divide each side by $e^{i(n+1)\theta}$ *to have*

$$\sum_{s=0}^{n} d_s^{n+1} e^{i(s-1-n)\theta}.$$

Since $\theta \neq 0$*, at least one* $e^{i(s-1-n)\theta}$ *is not real valued. Then, the right-hand side of the above equation is a convex combination of* $n+1$ *unit complex numbers.*

Hence, $e^{i\theta}$ *with* $\theta \neq 0$ *is not a root to the stability polynomial.*

Explicit Upwind Scheme for the Scalar Conservation Law

The First-Order Method

We consider the one-dimensional conservation law

$$\partial_t^\alpha u + (g(u))_x = 0, \tag{3.5}$$

where the flux function is decomposed as

$$g = g^+ + g^-, \quad (g^+)' \geqslant 0, \quad (g^-)' \leqslant 0.$$

The above assumption is made to simplify our analysis. It should be noted that the flux decomposition is not the major requirement for designing such numerical methods. For general discussion on this issue, we refer our readers to [26, 30] Without loss of generality, the flux function g(u) could either be linear or nonlinear. But obviously, it also involves the linear advection case, when $g = \sigma u$.

Likewise, we assume a uniform time step k, and let $t^n = nk$, $n = 0, 1, 2, \ldots$ *In addition, on the computational interval [a, b], we assume uniform spatial grids* $x_j = \sigma + jh$, *for* $j = 0, 1, 2, \ldots, N$, *with spatial grid size* $h = \dfrac{b-a}{N}$. *Let* U_j^k *be the numerical approximation of* $u(x = x_j, t = t^s)$, *then the first-order upwind scheme for the nonlinear conservation law is given as*

$$D_t^\alpha U_j^{n+1} + \frac{1}{h}\{g^+(U_j^n) - g^+(U_{j-1}^n)\} + \frac{1}{h}\{g^-(U_{j+1}^n) - g^-(U_j^n)\} = 0 \tag{3.6}$$

If we let

$$\lambda_j^{+,n} = \frac{\sigma_j^{+,n} k^\alpha \Gamma(2-\alpha)}{h} \quad \text{and} \quad \lambda_j^{-,n} = \frac{\sigma_j^{-,n} k^\alpha \Gamma(2-\alpha)}{h},$$

where for some ξ_j^n *between* U_{j-1}^n *and* U_j^n *and some* μ_j^n *between* U_j^n *and* U_{j+1}^n, *we obtain*

$$\sigma_j^{+,n} = \frac{g^+(u_j^n) - g^+(u_{j-1}^n)}{U_j^n - U_{j-1}^n} = (f^+)' + (\xi_j^n) \geq 0,$$
$$\sigma_j^{-,n} = \frac{g^-(u_{j+1}^n) - g^-(u_j^n)}{U_{j+1}^n - U_j^n} = (f^-)' + (\mu_j^n) \leq 0,$$

then, we rewrite the numerical scheme as

$$U_j^{n+1} = \left(\overline{d} - \lambda_j^{+,n} + \lambda_j^{-,n}\right) U_j^n + \lambda_j^{+,n} U_{j-1}^n - \lambda_j^{-,n} U_{j+1}^n + \sum_{s=0}^{n-1} d_s^{n+1} U_j^s \tag{3.7}$$

we therefore propose the CFL condition for the first-order upwind scheme as

$$\frac{k^\alpha \Gamma(2-\alpha)}{h} \left\{\max|(g^+)'| + \max|(g^-)'|\right\} \leq \overline{d}. \tag{3.8}$$

The CFL condition is observed to be essentially agreed with the conservation law and standard time derivatives, exception is that the time step k gains an exponent α due to the Caputo derivative. With the CLF condition, we obtain

$$\overline{d} - \lambda_j^{+,n} + \lambda_j^{-,n} \geqslant 0.$$

Therefore, we say that the maximum principle for the upwind scheme, $\forall n \in \mathbb{N}^+$

$$\max_j |U_j^n| \leq \lim_j |U_j^0|.$$

In addition, we can verify that this scheme is total variation diminishing (TVD) if the CFL condition (3.8) holds. In actual sense, we can rewrite (3.7) in the form

$$U_j^{n+1} = \overline{d} U_j^n - \rho g^+(U_j^n) + \rho g^+(U_j^n) + \rho g^+(U_{j-1}^n) - \rho g^+(U_{j+1}^n) + \sum_{s=0}^{n-1} d_s^{n+1} U_j^s. \tag{3.9}$$

where $\rho = \dfrac{k^\alpha \Gamma(2-\alpha)}{h}$.

Next, we consider another solution, say Z, satisfying the same difference equation

$$Z_j^{n+1} = \overline{d} Z_j^n - \rho g^+(Z_j^n) + \rho g^+(Z_j^n) + \rho g^+(Z_{j-1}^n) - \rho g^+(Z_{j+1}^n) + \sum_{s=0}^{n-1} d_s^{n+1} Z_j^s. \tag{3.10}$$

On subtracting (3.10) *from*(3.9), *we have*

$$\begin{aligned} U_j^{n+1} - Z_j^{n+1} = \overline{d}(U_j^n - Z_j^n) - \rho(g^+(U_j^n) - g^+(Z_j^n)) + \rho(g^-(U_j^n) - g^-(Z_j^n)) \\ + \rho(g^+(U_{j-1}^n) - g^+(Z_{j-1}^n)) - \rho(g^-(U_{j+1}^n) - g^-(Z_{j+1}^n)) \\ + \sum_{s=0}^{n-1} d_s^{n+1}(U_j^s - Z_j^s). \end{aligned}$$

Then, by adopting the mean value theorem, we get

$$\begin{aligned} U_j^{n+1} - Z_j^{n+1} = \left[\, \overline{d} - \rho(g^+)'(\xi_j^+) + \rho(g^-)'(\xi_j^-) \,\right] \left(U_j^n - Z_j^n\right) + \rho(g^+)' \left(\xi_{j-1}^+\right) \left(U_{j-1}^n - Z_{j-1}^n\right) \\ - \rho(g^-)' \left(\xi_{j+1}^-\right) \left(U_{j+1}^n - Z_{j+1}^n\right) + \sum_{s=0}^{n-1} d_s^{n+1}(U_j^k - Z_j^k), \end{aligned}$$

where ξ_j^+ *and* ξ_j^- *are the numbers between* U_j^n *and* Z_j^n , *respectively. When the CFL condition in* (3.8) *is held, we get*

$$\overline{d} - \rho(g^+)'(\xi_j^+) + \rho(g^-)'(\xi_j^-) \geqslant 0$$

By triangle inequality, we have

$$\begin{aligned} \left|U_j^{n+1} - Z_j^{n+1}\right| = \left[\, \overline{d} - \rho(g^+)'(\xi_j^+) + \rho(g^-)'(\xi_j^-) \,\right] \left(U_j^n - Z_j^n\right) + \rho(g^+)' \left(\xi_{j-1}^+\right) \left|U_{j-1}^n - Z_{j-1}^n\right| \\ - \rho(g^-)' \left(\xi_{j-1}^-\right) \left|U_{j+1}^n - Z_{j+1}^n\right| + \sum_{s=0}^{n-1} d_s^{n+1} \left|U_j^k - Z_j^k\right|. \end{aligned}$$

Summing the equation over j gives

$$\sum_j \left|U_j^{n+1} - Z_j^{n+1}\right| \leqslant \sum_{s=0}^{n} d_s^{n+1} \sum |U_j^s - Z_j^s|.$$

One observed here that the flux terms have been removed. By induction, we have

$$\| U^n - Z^n \|_{l^1} \leq \| U^0 - Z^0 \|_{l^1},$$

as the desired result for the following theorem.

Theorem 3.1.1. *The first-order upwind scheme(3.6) for the scalar conservative law,*

$$\partial_t^{\alpha} u(x,t) + g(u(x,t))_x = 0,\ x \in \mathbb{R},\ t > 0,$$

is l^1 contracting, when the CFL condition(3.8) holds.

3.2 Modelling of Ebola Hemorrhagic Fever: Fractional Derivative Approach

Known that the filoviruses comes from a virus family called Filoviridae. This virus may cause unembellished hemorrhagic fever in both humans and monkeys [33, 34, 35] . In most research papers and monographs, only two classes of this virus family have been identified so far, they are, Ebola virus and Marburg virus. Also, according to the literature, only five species of Ebola virus have been distinguished including Bundibugyo, Ivory Coast, Sudan, Zaire and Reston. Among these species, Ebola virus is known to be the only family of the Zaire Ebola virus species and the most dangerous and largest recorded epidemic outbreaks [36, 35].

Ebola is an uncommon but dangerous virus that results in bleeding inside and outside of the body [37]. As the virus spreads through the body, it breaks down the immune system and organs. Ultimately, it drops the levels of blood clotting in the cells [38]. This results in severe and uncontrollable bleeding [37, 35]. The biggest challenge in most West African nations is issue of unemployment. Findings have revealed that about 133 million working class people which constitute about 50 % of the population are illiterate and jobless. Many of these young people lack economic or social life skills which completely render them useless in the labour market. As a means to survive and provide minimum basic needs for their family, the only available jobs include fishing, farming and hunting. In the case of hunting and destruction of ecosystem, many of the wild animals are killed and endangered, the practice majority do routinely to survive. Most of the animals killed are either consumed fresh or dry and sold in the market places. Incidentally, it is a common belief that the Ebola virus disease can only occur after an Ebola virus is transmitted to an initial human by contact with an infected animal's body fluid. On contrary, human-to-human transmission when direct contact is made to the bodily fluid or blood of the infected person. Fruit bats are known to be the most likely natural source of the Ebola virus. As the early transmission, the bat *drops incompletely eaten fruits and pulp, and when it lands animals such as monkeys and gorillas feed on fallen fruits. Later, the humans hunt down these animals as food or sell the infected animal bodies to make money. Definitely the affected human will make contact with the rest of his family. So, the chain of transmission continues. For the mathematical formulation, we let $S(t), I(t), R(t)$ and $D(t)$ be the respec-tive susceptible, infected, recovery and the total mortality or death populations. Like-wise, let s, i, r and δ be the susceptibility rate, infection rate, recovery rate and death rate by Ebola.*

The mathematical model describing the rate of change of susceptible individuals is represented as

$$\frac{dS(t)}{dt} = -iS(t)I(t) + sR(t) - \beta N, \tag{3.11}$$

where i describes the rate of infectious class from recovery individuals converted to be vulnerable at rate s, and β denote the number of population that die naturally due to other diseases.

The rate of change of infected population is given by the differential equation

$$\frac{dI(t)}{dt} = -iS(t)I(t) - \delta I(t) - rI(t), \tag{3.12}$$

which suggests that the total number of individuals removed from susceptible class can be expressed mathematically as $iS(t)I(t)$. Obviously, due to medication, a rea-sonable number of populations will be recovered at rate r, while infected individual will die at rate δ. The recovery population and the change in mortality rate can be described by the following respective ordinary differential equations:

$$\frac{dR(t)}{dt} = rI(t) - sR(t), \tag{3.13}$$

and

$$\frac{dD(t)}{dt} = \delta I(t) + \beta N. \tag{3.14}$$

Thus, the mathematical equation governing the Ebola virus disease is given as

$$\begin{aligned}
\frac{dS(t)}{dt} &= -iS(t)I(t) + s(t) - \delta N,\\
\frac{dI(t)}{dt} &= iS(t)I(t) - \delta I(t) - rI(t),\\
\frac{dR(t)}{dt} &= rI(t) - sR(t),\\
\frac{dD(t)}{dt} &= \delta I(t) + \beta N.
\end{aligned} \tag{3.15}$$

The above equation will be formulated in terms of the Caputo-Fabrizio and the Atangana-Baleanu fractional derivatives, respectively, as

$$\begin{aligned}
{}_0^{CF}D_t^{\alpha} S(t) &= -iS(t)I(t) + s(t) - \delta N,\\
{}_0^{CF}D_t^{\alpha} I(t) &= iS(t)I(t) - \delta I(t) - rI(t),\\
{}_0^{CF}D_t^{\alpha} R(t) &= rI(t) - sR(t),\\
{}_0^{CF}D_t^{\alpha} D(t) &= \delta I(t) + \beta N.
\end{aligned} \tag{3.16}$$

***Table .3.T** Parameters based on some reported data*

Parameters	$S(0)$	$I(0)$	$R(0)$	$D(0)$	N	β	r	i	s	δ
Values	*900*	*10*	*0*	*0*	*1000*	*0.01*	*0.4*	*0.01*	*0.02*	*0.6*

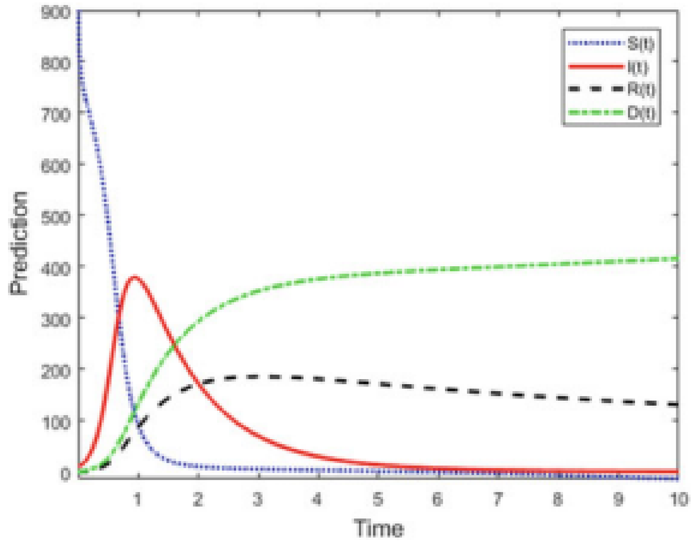


Fig. 3.1 *Prediction of the Caputo-Fabrizio fractional derivative model* (3.16) *for* $\alpha = 0.25$

and

$$\begin{aligned}
{}_{0}^{ABC}D_t^{\alpha}S(t) &= -iS(t)I(t) + s(t) - \delta N, \\
{}_{0}^{ABC}D_t^{\alpha}I(t) &= iS(t)I(t) - \delta I(t) - rI(t), \\
{}_{0}^{ABC}D_t^{\alpha}R(t) &= rI(t) - sR(t), \\
{}_{0}^{ABC}D_t^{\alpha}D(t) &= \delta I(t) + \beta N.
\end{aligned} \tag{3.17}$$

The parameters used for the numerical simulations according to reported data are given in ***Table (3.T)****.* ***Figures 3.1****,* ***3.2****,* ***3.3****,* ***3.4*** *and* ***3.5*** *depict the approximate solution of the Caputo-Fabrizio time-fractional Ebola system* (3.16) *for different values of* α *as shown in the figures' captions.* ***Figures 3.6****,* ***3.7****,* ***3.8****,* ***3.9****,* ***3.10*** *and* ***3.11*** *represent*

Fig. 3.2 Prediction of the Caputo-Fabrizio fractional derivative model (3.16) *for* $\alpha = 0.45$

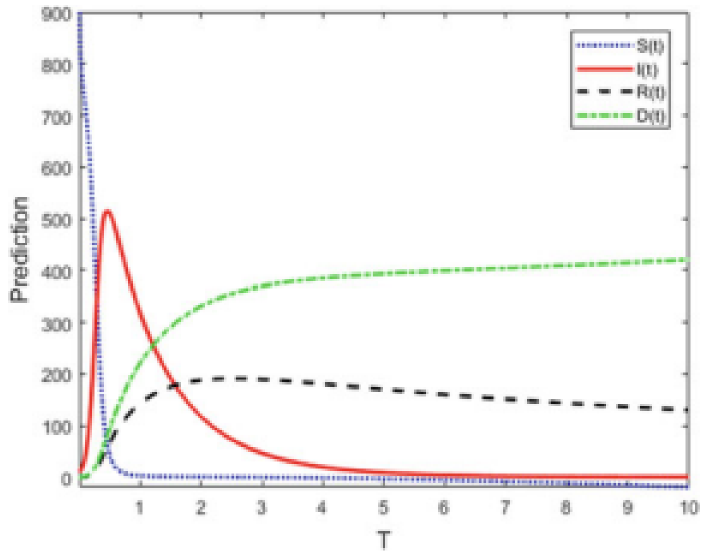


Fig. 3.3 Prediction of the Caputo-Fabrizio fractional derivative model (3.16) *for* $\alpha = 0.67$

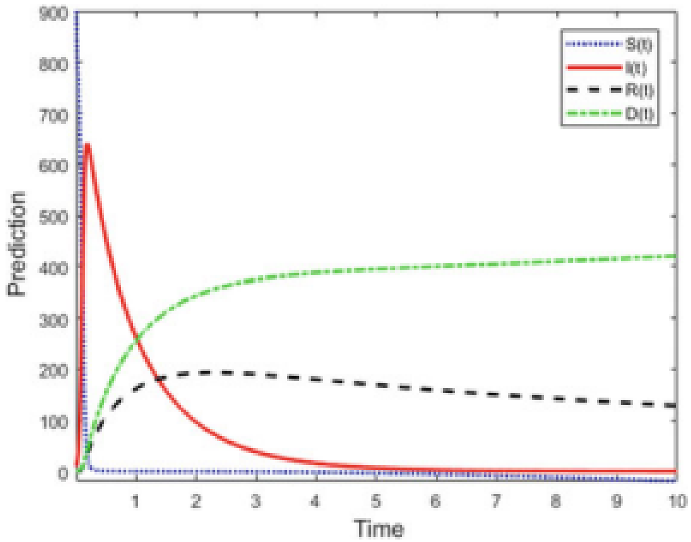
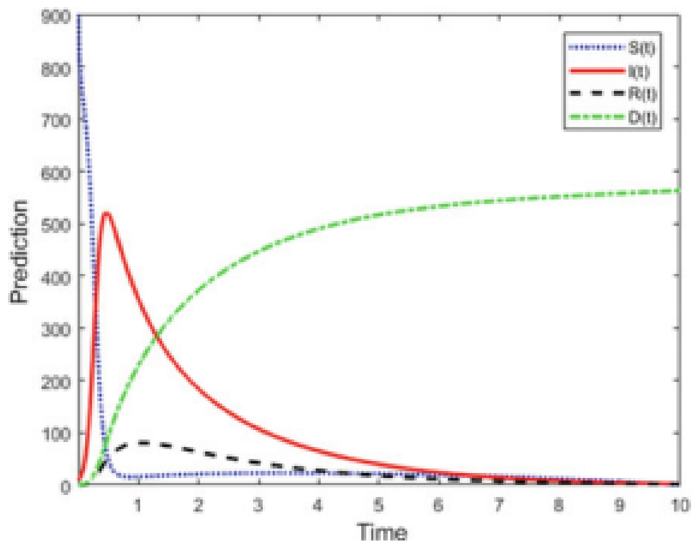


Fig. 3.4 Prediction of the Caputo-Fabrizio fractional derivative model (3.16) *for* $\alpha = 0.83$



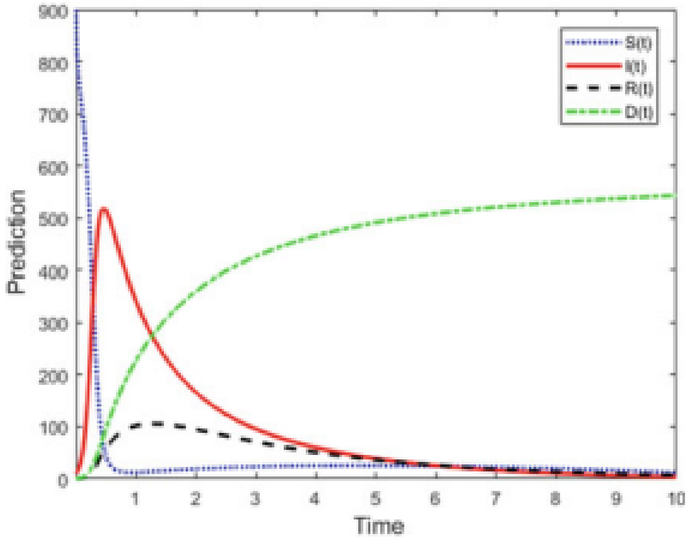


Fig. 3.5 *Prediction of the Caputo-Fabrizio fractional derivative model* (3.16) *for* $\alpha = 0.90$

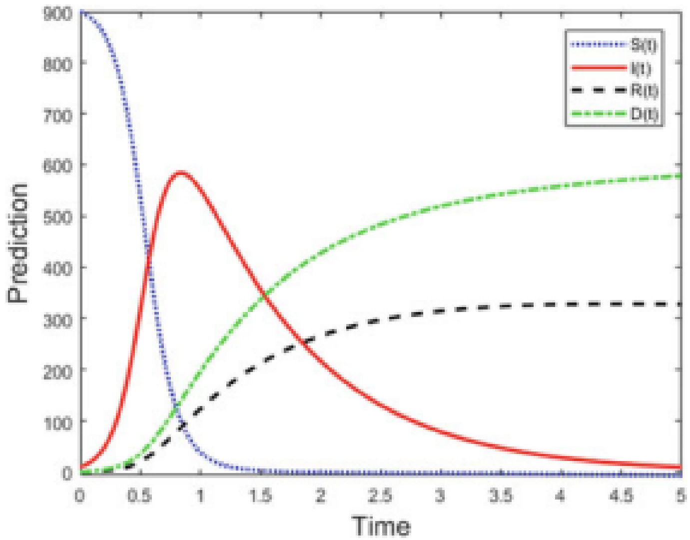


Fig. 3.6 *Prediction using the Atangana-Baleanu fractional derivative model* (3.17) *for* $\alpha = 0.25$

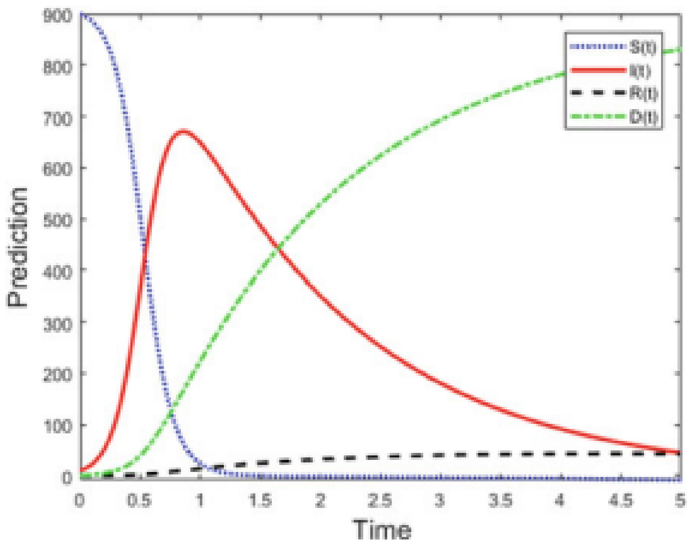


Fig. 3.7 *Prediction using the Atangana-Baleanu fractional derivative model* (3.17) *for* $\alpha = 0.55$

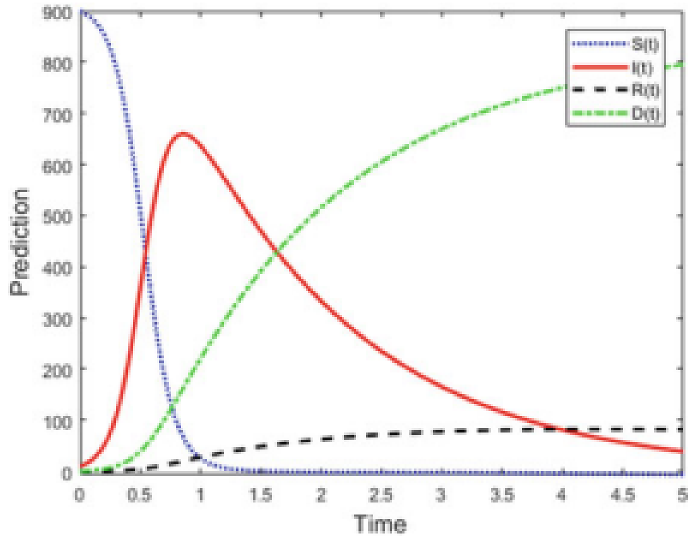


Fig. 3.8 *Prediction using the Atangana-Baleanu fractional derivative model* (3.17) *for* $\alpha = 0.59$

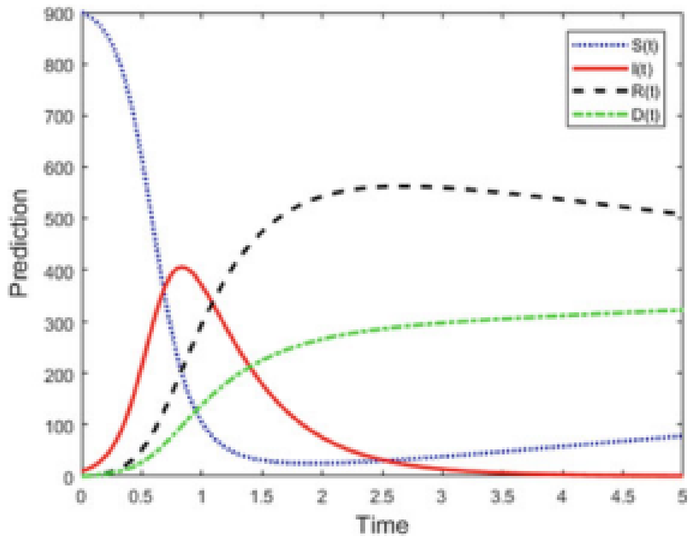


Fig. 3.9 *Prediction using the Atangana-Baleanu fractional derivative model* (3.17) *for* $\alpha = 0.83$

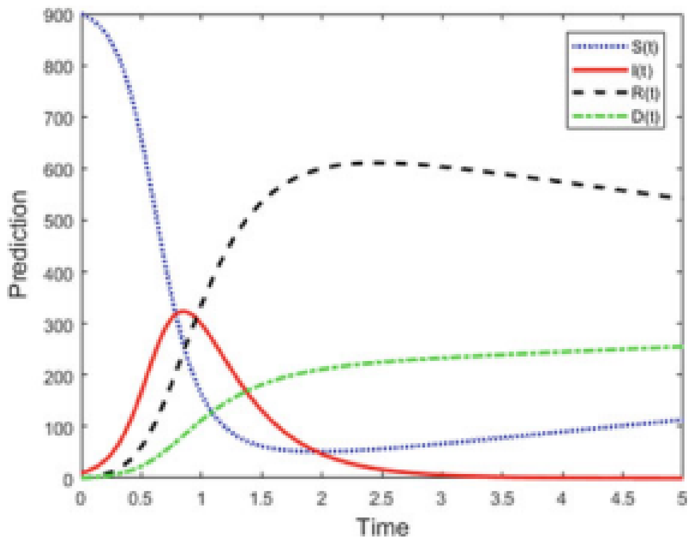


Fig. 3.10 *Prediction using the Atangana-Baleanu fractional derivative model* (3.17) *for* $\alpha = 0.89$

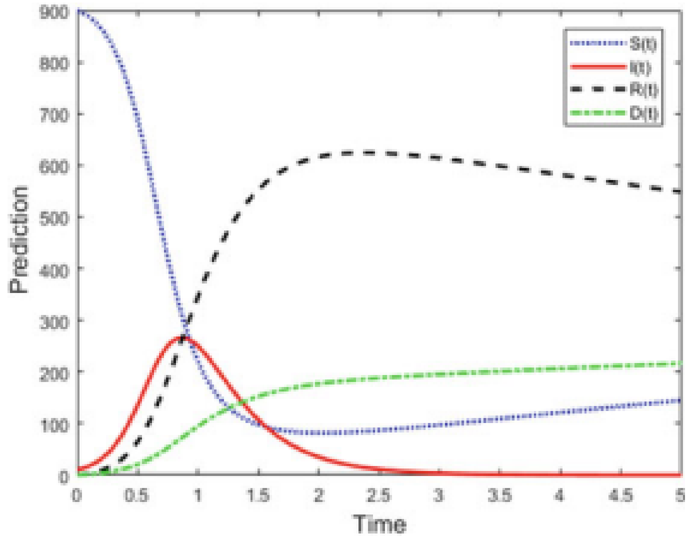


Fig. 3.11 *Prediction using the Atangana-Baleanu fractional derivative model* (3.17) *for* $\alpha = 0.93$

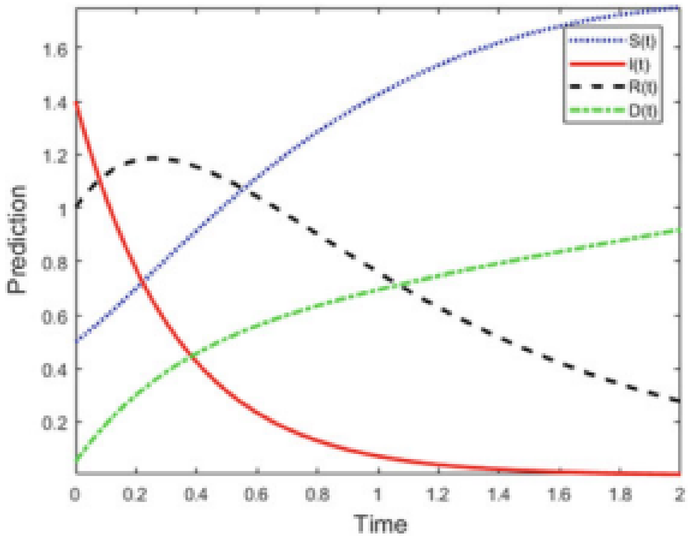


Fig. 3.12 *Prediction for system* (3.17) *at* $t = 2$

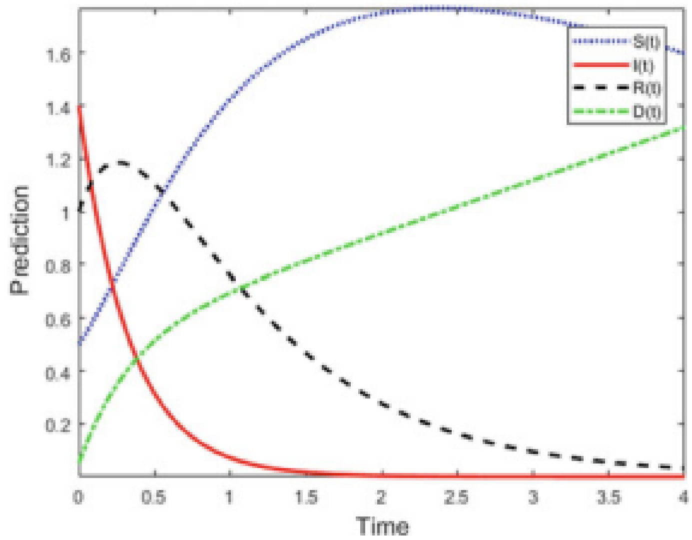


Fig. 3.13 *Prediction for system* (3.17) *at* $t = 4$

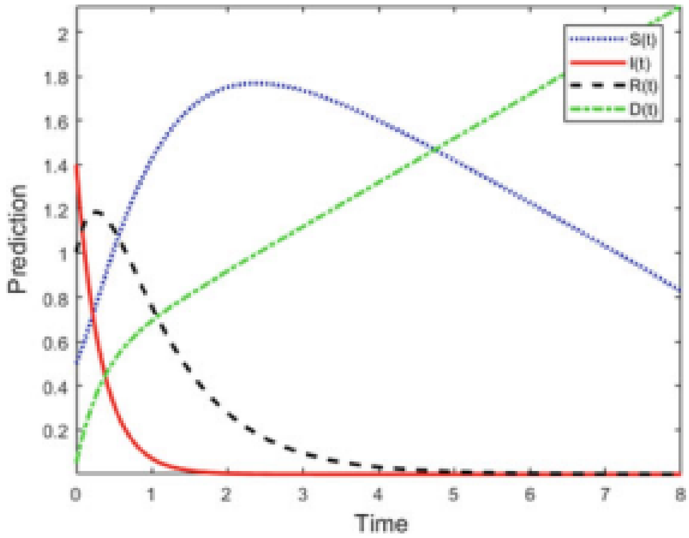


Fig. 3.14 *Prediction for system (3.17) at $t = 8$*

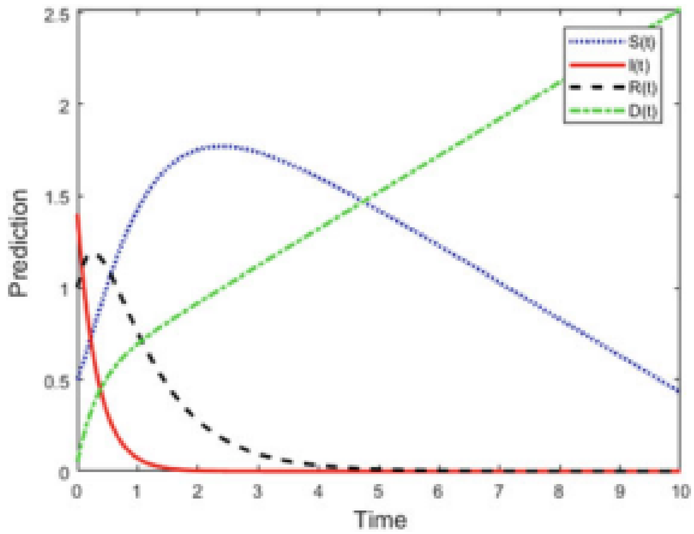


Fig. 3.15 *Prediction for system (3.17) at $t = 10$*

the approximate numerical solution of the Atangana-Baleanu fractional derivative system (3.17) for different instances of α as displayed in the figures' captions. Finally, we fixed $\alpha = 0.50$ with perturbed initial conditions to examine the effect of time as given in ***Figures. 3.12***, ***3.13***, ***3.14*** *and* ***3.15***.

Conclusion

- *Ordinary fractional differential equations represent a powerful mathematical tool that allows for a more accurate and realistic description of physical, biological, and engineering systems. Due to their versatility and ability to handle complex dynamic patterns, these equations provide a flexible framework for analyzing phenomena that cannot be explained using traditional ordinary differential equations.*
- *As demonstrated through various applications, the use of ordinary fractional differential equations can lead to significant improvements in modeling and predicting different systems, whether in the fields of control, thermodynamics, biology, or even financial sciences. With ongoing research in this area, we anticipate that new and diverse applications will continue to emerge, enhancing our understanding of the world around us and contributing to the sustainable development of science and technology.*
- *As a future work, we planning to use the properties presented in this work to generalize this definition to Caputo-Fabrizio fractional differential equations and fractional differential calculus remains among the important problems that necessitate much more research.*
- *Ultimately, the continuous challenge lies in developing and simplifying numerical solutions for these equations, allowing more researchers and engineers to exploit their unique capabilities in various fields.*

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Abstracte

Capito-Fabrizio 's definition of fractional derivation is one of the latest definitions to improve derivation fractional derivation.
In this memory, this new definition has been applied to some Cauchy fractional problems and some boundary values problems.
We treat linear cases with Laplace transform and nonlinear cases by some fixed point theories.

Keywords and phrases: **Capito-Fabrizio fractional derivative, Laplace transform, fixed point, numerical approximation.**

Résumé

La définition de Caputo-Fabrizio de la dérivation fractionnaire est l'une des dernières définitions pour améliorer la dérivation fractionnaire.
Dans cette mémoire, cette nouvelle définition a été appliquée à certains problème fractionnaires de Cauchy et certains problèmes aux limites.
Nous avons traité des cas linéaire par la transformée de Laplace et des cas non linéaires par des certaines théorèmes de point fixe.

Mots clés et expressions: dérivé fractionnaire Caputo-Fabrizio, transformée de Laplace, point fixe, approximation numérique .

الملخص

يُعتبر تعريف كابيتو- فابريزيو لِلإشتقاق الكسري من أحدث التعاريف لتحسين الإشتقاق الكسري.في هذه المذكرة تم تطبيق هذا التعريف على بعض مسائل كوشي الكسرية وبعض مسائل القيم الحدية,حيثُ عالجنا الحالات الخطية بواسطة تحويل لابلاس والحالات الغير خطية بواسطة بعض نظريات النقطة الثابتة.

الكلمات والعبارات الدالة : ***المشتق الكسري لكابيتو- فابريزيو , تحويل لابلاس,النقطة الثابتة,التقريب العددي.***