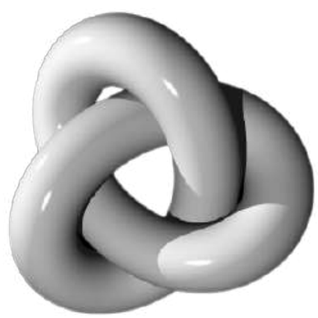
Differential Geometry

A course for Master students

Hacen Zelaci



Mathematical department, University of El Oued.
Current address: El Oued - Algeria
E-mail address: `zelaci-hacen@univ-eloued.dz`

Abstract

A branch of geometry dealing with geometrical forms, mainly with curves and surfaces, by methods of mathematical analysis. In differential geometry the properties of curves and surfaces are usually studied on a small scale, i.e. the study concerns properties of sufficiently small pieces of them. Properties of families of curves and surfaces are also studied.

Differential geometry arose and developed in close connection with mathematical analysis, the latter having grown, to a considerable extent, out of problems in geometry. Many geometrical concepts were defined prior to their analogues in analysis. For instance, the concept of a tangent is older than that of a derivative, and the concepts of area and volume are older than that of the integral.

Differential geometry first appeared in the 18th century and is linked with the names of L. Euler and G. Monge. The first synoptic treatise on the theory of surfaces was written by Monge (Une application d'analyse à la géométrie, 1795). In 1827 a study under the (English) title A general study on curved surfaces was published by C.F. Gauss; this study laid the foundations of the theory of surfaces in its modern form. From that time onwards differential geometry ceased to be a mere application of analysis, and has become an independent branch of mathematics.

The discovery of non-Euclidean geometry by N.I. Lobachevskii in 1826 played a major role in the development of geometry as a whole, including differential geometry. Lobachevskii rejected in fact the a priori concept of space, which was predominating in mathematics and in philosophy. He found that spaces different from Euclidean spaces exist. This idea of Lobachevskii was reflected in numerous mathematical studies. Thus, in 1854 B. Riemann published his course ber die Hypothesen, welche der Geometrie zuGrunde liegen and thus laid the foundations of Riemannian geometry, the application of which to higher-dimensional manifolds is related to the geometry of n-dimensional space similarly as the relation between the interior geometry of a surface and Euclidean geometry on a plane.

Historically, differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces. Mathematical analysis of curves and surfaces had been developed to answer some of the nagging and unanswered questions that appeared in calculus, like the reasons for relationships between complex shapes and curves, series and analytic functions. These unanswered questions

indicated greater, hidden relationships. The general idea of natural equations for obtaining curves from local curvature appears to have been first considered by Leonhard Euler in 1736, and many examples with fairly simple behavior were studied in the 1800s. When curves, surfaces enclosed by curves, and points on curves were found to be quantitatively, and generally, related by mathematical forms, the formal study of the nature of curves and surfaces became a field of study in its own right, with Monge's paper in 1795, and especially, with Gauss's publication of his article, titled "Disquisitiones Generales Circa Superficies Curvas", in Commentationes Societatis Regiae Scientiarum Gottingesis Recentiores in 1827. Initially applied to the Euclidean space, further explorations led to non-Euclidean space, and metric and topological spaces.

This course gives an introduction to this domain, it doesn't requires any background in this mathematical discipline.

Our main references are [**Spi99**], [**RS18**],[**Oan18**], [**Pet**], [**Sch08**], [**Gua18**], [**Csi14**].

Contents

NOTATIONS: Through out this work, we use the following general notations:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the set of nonnegative integers, integers, the fields of rational numbers, real numbers, complex numbers respectively.
- Maps are generally denoted by f, g, h....
- Integers are denoted by k, n, m....
- We say that a function f is C^k if it is $k-$times differentiable and its $k^{th}-$differential is continuous.
- The differential of a function f at p is denoted $d_p f$.
- The Jacobian of a function f at p is denoted $J_p(f)$.

CHAPTER 1

Introduction

Recall the definition of a submanifold of $\mathbb{R}^n$.

Definition 0.1. Let $M \subset \mathbb{R}^n$ be a subset. We say that M is a submanifold of dimension d ($\leqslant n$) and of class C^k if for all x in M, there is a neighborhood U of x in $\mathbb{R}^n$, a neighborhood V of 0 in $\mathbb{R}^n$ and a C^k-diffeomorphism $\phi : U \to V$ such that $\phi(U \cap M) = V \cap (\mathbb{R}^d \times \{0\})$.

Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^m$ be a smooth function. An element $c \in \mathbb{R}^m$ is called a *regular value* of f if, for all $p \in U$, we have

$$f(p) = c \Longrightarrow d_p f : \mathbb{R}^n \to \mathbb{R}^m \text{ is surjective.}$$

We have

Proposition 0.2. *A subset $M \subset \mathbb{R}^n$ is a submanifold iff for any $p \in M$, there exists an open set $U \subset \mathbb{R}^n$ and a smooth map $f : U \to \mathbb{R}^{n-m}$ such that $p \in U$,$U \cap M = f^{-1}(0)$ and 0 is a regular value for f.*

Recall that a vector $v \in \mathbb{R}^n$ is called a *tangent vector* of M at p if there exists a smooth curve $\gamma :]-1, 1[\to M$ such that

$$\gamma(0) = p,\ \gamma'(0) = v.$$

The set of tangent vectors to M at p is called the linear tangent space of M at p and it is denoted T_pM. The affine tangent space of M at p is the translate of T_pM with the vector $\overrightarrow{0p}$.

In the notation of the above proposition, we have $T_pM = \mathrm{Ker}(d_p f)$.

A vector field on a smooth manifold M is a smooth map

$$X : M \to TM,$$

such that for any $p \in M$, $X(p) \in T_pM$. In other words, X is a section for the canonical projection $TM \to M$. We denote by $\Gamma(M)$ the space of vector fields.

Let X be a vector field on M and $I \subset \mathbb{R}$ an interval. A smooth map $\gamma : I \to M$ is called an *integral curve* for X if for any $t \in I$ we have

$$\gamma'(t) = X(\gamma(t)).$$

CHAPTER 2

Differential manifolds

1. Abstract manifolds

Let T be a topological space. A neighborhood of a point $x \in T$ is a subset $U \subset T$ with the property that it contains an open set containing x. A map $f : T \to W$ which is continuous, bijective and has a continuous inverse is called a homeomorphism.

Definition 1.1. A topological space X is said to be Hausdorff if for every pair of distinct points $x, y \in X$ there exist disjoint neighborhoods of x and y.

Let M be a Hausdorff topological space, and let $m \geqslant 0$ be a fixed non negative integer.

Definition 1.2. An $m-$dimensional smooth atlas of M is a collection $(O_i)_{i\in I}$ of open sets $O_i \subset M$ such that $M = \cup_{i\in I} O_i$, together with a collection $(U_i)_{i\in I}$ of open sets in $\mathbb{R}^m$ and a collection of homeomorphisms, called charts, $\phi_i : O_i \to U_i = \phi_i(U_i)$, with the following property of smooth transition on overlaps:
For each pair $i, j \in I$ the map $\phi_j \circ \phi_i^{-1} : \phi_i(O_i \cap O_j) \to \phi_j(O_i \cap O_j)$ is smooth.
Two smooth atlases are compatible if their union is again an atlas.

If $\mathscr{A}$ is an atlas, then so is the collection $\bar{\mathscr{A}}$ of all charts compatible with each member of $\mathscr{A}$. The atlas $\bar{\mathscr{A}}$ is obviously maximal. In other words, every atlas extends uniquely to a maximal atlas.

Example 1.3. The map

$$\varphi(u, v) = (\cos v, \sin v, u), u, v \in \mathbb{R}$$

is smooth and covers the cylinder $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$, but it is not injective. Let

$$U_1 = \{(u, v) \in \mathbb{R}^2 | -\pi < v < \pi\} , U_2 = \{(u, v) \in \mathbb{R}^2 | 0 < v < 2\pi\},$$

and let φ_i denote the restriction of φ to U_i for $i = 1, 2$. Then φ_1 and φ_2 are both injective, φ_1 covers S with the exception of a vertical line on the back where $x = -1$, and φ_2 covers S with the exception of a vertical line on the front where $x = 1$. Together they cover the entire set and thus they constitute an atlas.

Example 1.4. Assume that $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open non-empty sets, and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous map. Denote by M the graph of f, i.e.

$$M = \{(x, y)|\; y = f(x)\}.$$

Let $U = (W \times V) \cap M$ and let $\phi(x, y) = x$ be the projection of U onto W. Then the pair (ϕ, U) is a chart on M. The inverse map is given by $\phi^{-1}(x) = (x, f(x))$.

Definition 1.5. An abstract manifold (or just a manifold) of dimension m, is a Hausdorff topological space M, equipped with a maximal $m-$dimensional smooth atlas.

Remark 1.6. If in the definition, we don't assume that M is a topological space, and that O_i are just subset, then there is a unique topology on M making O_i open and ϕ_i homeomorphisms.

Example 1.7. The stereographic projections endows the spheres S^n with an atlas given as follows; let $U_\pm$ be the subsets of S^n obtained by removing $(\mp 1, 0, \cdots, 0)$. Stereographic projection defines bijections $\phi_\pm : U_\pm \to \mathbb{R}^n$ given by

$$\phi_\pm(x_0, \cdots, x_n) = (\frac{x_1}{1 \pm x_0}, \cdots, \frac{x_n}{1 \pm x_0}).$$

One can check (exercise!) that

$$\phi_- \circ \phi_+^{-1}(x) = \frac{x}{\|x\|^2},$$

hence it is smooth. The same is true for $\phi_+ \circ \phi_+^{-1}$. Hence $\{(\phi_+, U_+), (\phi_-, U_-)\}$ is an atlas on S^n.

Definition 1.8. Let $f : M \to N$ be a map between abstract manifolds. Then f is called smooth if for each $p \in M$ there exists a chart $\phi : O \to U$ around p, and a chart $\psi : O' \to V$ around $f(p)$, such that $f(O) \subset O'$ and such that the coordinate expression $\psi \circ f \circ \phi^{-1}$ is smooth. A bijective map $f : M \to N$, is called a diffeomorphism if f and f^{-1} are both smooth.

Notice that a smooth map $f : M \to N$ is continuous. This follows immediately from the definition above, by writing $f = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi$ in a neighborhood of each point. Also, every chart $\phi : O \to U$

is a smooth diffeomorphism. And every diffeomorphism ϕ of a non-empty open subset $V \subset \mathbb{R}^m$ onto an open subset in M is a chart on M.

Now, an atlas of an abstract manifold is said to be *countable* if the set of charts in the atlas is countable

LEMMA 1.9. *Let M be an abstract manifold. Then M has a countable atlas if and only if there exists a countable base for the topology in M. In particular, a submanifold of $\mathbb{R}^n$ has a countable atlas.*

PROOF. Assume M has a countable atlas. Let $\phi : O \to U \subset \mathbb{R}^m$ be a chart, then there is a countable base for the topology of U. This induces a countable base for the topology on O, because ϕ is a homeomorphism. The collection of these bases for all the charts in the atlas gives a base for the topology of M.
Conversely, suppose that there exists a countable base $(V_k)_k$ of the topology of M. For each k, choose a chart $\phi : O \to U$, such that $V_k \subset O$ (if it exists!). This gives a countable atlas $\mathscr{A}$ in M. Indeed, it is clearly countable. Let now $x \in M$. Then there exists a chart $\psi : O' \to U'$ such that $x \in O'$, and there exists k such that $x \in V_k \subset O'$. Hence there exists a chart $\phi : O \to U$ in $\mathcal{A}$ such that $x \in O$. So $\mathscr{A}$ is an atlas. □

THEOREM 1.10 (Whitney theorem). *Let M be an abstract smooth manifold of dimension d, and assume that there exists a countable atlas for M. Then there exists a diffeomorphism of M onto a submanifold in $\mathbb{R}^{2d}$.*

EXAMPLE 1.11 (Projective space). Let V be a finite dimensional vector space, The *projective space* of V is the set of lines (through the origin) in V . In other words,

$$P(V) = \{l \subset V | l \text{ is a 1-dimensional } \mathbb{R}\text{-linear subspace}\}.$$

It can be seen a $P(V) = V^*/\sim$, where the equivalence relation is given by

$$u \sim v \Leftrightarrow u = \lambda v \text{ for some } \lambda \in \mathbb{R}.$$

If $V = \mathbb{R}^n$, then $P(V)$ is denoted simply $\mathbb{P}^n$. The projective space $\mathbb{P}^n$ can be given the structure of an abstract $n-$dimensional manifold as follows. Let $\pi : x \to [x]$ denote the natural map $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, and let $S^n \subset \mathbb{R}^n$ denote the unit sphere. A set $U \subset \mathbb{P}^n$ is declared to be open if and only if its preimage $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$ (or equivalently, if $\pi^{-1}(U) \cap S^n$ is open in S^n). This makes $\mathbb{P}^n$ a Hausdorff topological space (it is actually the quotient topology, that's a map $f : \mathbb{P}^n \to Y$ is continuous if and only if $f \circ \pi$ is continuous).

For $i = 1, \dots, n+1$, let $O_i = \{[x] \in \mathbb{P}^n | x_i \neq 0\}$. They are clearly open and cover $\mathbb{P}^n$. Define the maps $\phi_i : O_i \to \mathbb{R}^n$ given by

$$\phi_i([x_1 : \cdots : x_{n+1}]) = (x_1/x_i, \cdots, x_{n+1}/x_i),$$

where the i^{th} component is omitted. Clearly ϕ_i is a homeomorphism from O_i to $\mathbb{R}^n$. Moreover, the collection $\{(O_i, \phi_i)\}$ is a smooth atlas (exercise).
Note also that $\mathbb{P}^1$ is homeomorphic to the circle S^1. It is clearly isomorphic to $S^1/\sim$, where $\sim$ identifies x and $-x$ in S^1. Then the map $z \to z^2$ is a homeomorphism $S^1/\sim \to S^1$.

Definition 1.12. Let M be an abstract manifold. A subset $N \subset M$ is called an *abstract submanifold* of M if N is an abstract manifold on its own such that the topology of N is induced from M and the inclusion map $i : N \to M$ is a smooth immersion.

2. Tangent space

Let M be an $m-$dimensional manifold. A *curve* γ on M is a smooth map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is open. This means (see Definition 1.8) that $\phi \circ \gamma$ is smooth for all chart ϕ on M. The expression $\phi \circ \gamma$ is called *coordinate expression* of γ with respect to ϕ. If $p \in M$ is a point, a parametrized curve on M through p, is a parametrized curve on M together with a point $t_0 \in I$ for which $p = \gamma(t_0)$.

Let $\gamma_i : I_i \to M$, for $i = 1, 2$, be parametrized curves on M with $p = \gamma_1(t_1) = \gamma_2(t_2)$, and let ϕ be a chart around p. We say that γ_1 and γ_2 are tangential at p, if the coordinate expressions satisfy

$$(\phi \circ \gamma_1)'(t_1) = (\phi \circ \gamma_2)'(t_2).$$

Lemma 2.1. *Being tangential at p is an equivalence relation on curves through p. It is independent of the chosen chart (ϕ, O).*

Proof. The first part is easy.
If $\tilde{\phi}$ is another chart then the coordinate expressions are related by

$$\tilde{\phi} \circ \gamma = \tilde{\phi} \circ \phi^{-1} \circ (\phi \circ \gamma)$$

on the overlap $O \cap \tilde{O}$. The chain rule implies

$$(\tilde{\phi} \circ \gamma_i)'(t_i) = D(\tilde{\phi} \circ \phi^{-1})(p)(\phi \circ \gamma_i)'(t_i),$$

for each of the curves. This implies the result. □

Denote the tangential equivalence relation at p by $\sim_p$.

Definition 2.2. The tangent space T_pM is the set of $\sim_p$ −classes of parametrized curves on M through p.

THEOREM 2.3. *Let (ϕ, U) be a chart on M with $\phi(p) = q$. For each element $v \in \mathbb{R}^m$ let $\gamma_v(t) = \phi^{-1}(q + tv)$ for t close to 0. The map*

$$\Gamma : v \mapsto [\gamma_v]$$

is a bijection of $\mathbb{R}^m$ onto T_pM. The inverse map is given by

$$[\gamma] \mapsto (\phi \circ \gamma)'(0),$$

for each curve γ on M with $\gamma(0) = p$.

The main difficulty of defining tangent vectors to a manifold is due to the fact that an abstract manifold might not be naturally embedded into a fixed finite dimensional linear space. Nevertheless, there is a universal embedding of each differentiable manifold into an infinite dimensional linear space.

Let us denote by $C^\infty(M)$ the linear vector space of smooth functions on M, and by $D(M)$ the dual space of $C^\infty(M)$, that is the space of linear functions on $C^\infty(M)$, and consider the embedding ι of M into $D(M)$ defined by the formula

$$[\iota(p)](f) = f(p); \text{ where } p \in M; f \in C^\infty(M).$$

Having embedded the manifold M into $D(M)$, we can define tangent vectors to M as elements of the linear space $D(M)$.

DEFINITION 2.4. Let M be a differentiable manifold, $p \in M$. We say that a linear function $D \in D(M)$ is a derivation of $C^\infty(M)$ at p if the equality

$$D(fg) = D(f)g(p) + f(p)D(g)$$

holds for every $f, g \in C^\infty(M)$.

Derivations at a point $p \in M$ form a linear subspace $Der_p(M)$ of $D(M)$. Each curve γ passes through p at 0 defines a derivation at the point p by the formula

$$D_\gamma(f) = (f \circ \gamma)'(0).$$

Since two curves define the same derivation if and only if they are equivalent, there is a one-to-one correspondence between the equivalence classes of curves and the derivations obtained as D_γ for some γ. Hence the definition

DEFINITION 2.5. A tangent vector to a manifold M at the point $p \in M$ is a derivation of the form D_γ, where γ is a smooth curve passes through p at 0.
The tangent space T_pM of M at the point p is the set of derivations D_γ along curves in M passing through p at 0.

Definition 2.6. Let $f : M \to N$ be a smooth mapping between the differentiable manifolds M and N, and let $p \in M$. The *derivative* of f at the point p is the linear map $D_p f : T_pM \to T_{f(p)}N$ given in the following way: Let γ be a smooth curve and let $D_\gamma \in T_pM$. Then $T_pf(D)$ is the derivation $D_{(f\circ\gamma)} \in T_{f(p)}N$.
In the first definition of the tangent space we have

$$D_pf([\gamma]) = [f \circ \gamma].$$

Note that D_pf is well defined, that's it doesn't depend on the choice of γ. Indeed, one sees that whenever γ_1 and γ_2 are tangential then so is $f \circ \gamma_1$ and $f \circ \gamma_2$.

3. Vector fields and flows

A vector field on a smooth manifold M is simply a smooth map

$$X : M \to TM,$$

such that for any $p \in M$, $X(p) \in T_pM$. In other words, X is a section for the canonical projection $TM \to M$. We denote by $\Gamma(M)$ the space of vector fields.

Definition 3.1. Let X be a vector field on M and $I \subset \mathbb{R}$ an interval. A smooth map $\gamma : I \to M$ is called an *integral curve* for X if for any $t \in I$ we have

$$D_{\gamma'(t)} = X(\gamma(t)). \tag{1}$$

Associated to a chart $\phi = (x_1, \cdots, x_n)$ on M near p, there is a basis of T_pM, formed by the tangent vectors $(\partial_1(p), \cdots, \partial_n(p))$. The mapping

$$p \mapsto \partial_i(p)$$

gives a local smooth vector field in the domain of the chart for each i. Thus, every smooth vector field X can be written in the form

$$X = \sum_i^n a_i \partial_i,$$

where a_i are smooth functions near p.

Theorem 3.2. *Let X be a smooth vector field on a differentiable manifold M. Then for each point $p \in M$, there exists a unique maximal integral curve $\gamma_p : I \to M$ of the vector field X such that $0 \in I$.*

The real vector space structure of the tangent space at any point makes it possible to give $\Gamma(M)$ the structure of a real vector space where the addition is defined point by point $(X+Y)(p) = X(p)+Y(p)$ and the scaler multiplication $(\lambda \cdot X)(p) = \lambda X(p)$. In particular, we denote by 0 the zero section. Moreover, the space $\Gamma(M)$ has also the structure of $C^\infty(M)-$module. The multiplication with a function is also defined point by point $(f \cdot X)(p) = f(p)X(p)$.

DEFINITION 3.3. A vector field X on M is called complete if, for each $p \in M$, there is an integral curve $\gamma : \mathbb{R} \to M$ of X with $\gamma(0) = p$.

LEMMA 3.4. *Let $M \subset \mathbb{R}^n$ is a compact manifold. Then every vector field on M is complete.*

PROOF. See Exercise **??**. □

Since tangent vectors to a manifold at a point are identified with derivations at this point, vector fields can be regarded as differential operators called *derivations of smooth functions* assigning to a smooth function another smooth function by the formula

$$[X(f)](p) = [X(p)](f); \text{ where } X \in \Gamma(M); \; f \in C^\infty(M); \; p \in M,$$

In this sense, a vector field X is seen as a linear mapping $D_X : C^\infty(M) \to C^\infty(M)$, satisfying the Leibniz rule

$$X(fg) = X(f)g + fX(g).$$

Lie structure. Using this second point of view, we define a *Lie brackets* on $\Gamma(M)$ by

$$[D_X, D_Y] := D_X \circ D_Y - D_Y \circ D_X.$$

None of the operators $D_X \circ D_Y$ and $D_Y \circ D_X$ are vector fields, since they are second order operators, whereas vector fields are first order operators. However, it turns out that their difference $[D_X, D_Y]$ is again a vector field, this is given in the following theorem

THEOREM 3.5. *The Lie bracket $[D_X, D_Y]$ is a smooth vector field on M. That is, there exists a vector field $Z \in \Gamma(M)$ such that $[D_X, D_Y] = D_Z$.*

PROOF. It is sufficient to show that $[D_X, D_Y]$ verifies the Leibniz rule. Let $f, g \in C^\infty(M)$, then we have

$$\begin{aligned}
[D_X, D_Y](fg) &= (D_X \circ D_Y - D_Y \circ D_X)(fg) = D_X(D_Y(fg)) - D_Y(D_X(fg)) \\
&= D_X(D_Y(f)g + fD_Y(g)) - D_Y(D_X(f)g + fD_X(g)) \\
&= D_X(D_Y(f))g + D_Y(f)D_X(g) + D_X(f)D_Y(g) + fD_X(D_Y(g)) \\
&- D_Y(D_X(f))g - D_X(f)D_Y(g) - D_Y(f)D_X(g) - fD_Y(D_X(g)) \\
&= [D_X, D_Y](f)g + f[D_X, D_Y](g).
\end{aligned}$$

□

Flows of a vector field. Let M as above and $X \in \Gamma(M)$. Let $p \in M$, the *maximal existence interval* of p is the open interval $I(p) := \bigcup I$ where the union runs over all $I \subset \mathbb{R}$ open intervals containing 0 such that there exists an integral curve $\gamma : I \to M$ for X with $\gamma(0) = p$.
By Theorem 3.2, there exists an integral curve $\gamma : I(p) \to M$. The *flow* of X is the map $\phi : \mathcal{D} \to M$ defined by

$$\mathcal{D} := \{(t, p) | p \in M, t \in I(p)\},$$

and $\phi(t, p) := \gamma(t)$, where $\gamma : I(p) \to M$ is the unique integral curve.

THEOREM 3.6. *Let $M \subset \mathbb{R}^n$ be a smooth $r-$manifold and $X \in \Gamma(M)$ be a smooth vector field on M. Let $\phi : \mathcal{D} \to M$ be the flow of X. Then the following holds.*

(*i*) *Let $p \in M$ and $s \in I(p)$. Then*

$$I(\phi(s, p)) = I(p) - s, \tag{2}$$

and, for every $t \in \mathbb{R}$ with $s + t \in I(p)$, we have

$$\phi(s + t, p) = \phi(t, \phi(s, p)). \tag{3}$$

(*ii*) *$\mathcal{D}$ is an open subset of $\mathbb{R} \times M$.*
(*iii*) *The map $\phi : \mathcal{D} \to M$ is smooth.*

PROOF. (*i*) The map $\gamma : I(p) - s \to M$ defined by $\gamma(t) := \phi(s + t, p)$ is a solution of the initial value problem $\gamma'(t) = X(\gamma(t))$ with $\gamma(0) = \phi(s, p)$. Hence $I(p) - s \subset I(\phi(s, p))$ and equation (3) holds for every $t \in \mathbb{R}$ with $s + t \in I(p)$. In particular, with $t = -s$, we have $p = \phi(-s, \phi(s, p))$. Thus we obtain equality in equation (2) by the same argument with the pair (s, p) replaced by $(-s, \phi(s, p))$.

(ii) & *(iii)* Let $(t_0, p_0) \in \mathcal{D}$ so that $p_0 \in M$ and $t_0 \in I(p_0)$. Suppose $t_0 \geqslant 0$. Then $K := \{\phi(t, p_0) | 0 \leqslant t \leqslant t_0\}$ is a compact subset of M. (It is the image of the compact interval $[0, t_0]$ under the unique solution $\gamma : I(p_0) \to M$ of equation (1).) Hence, by Lemma 3.7 bellow, there is an $M-$open set $U \subset M$ and an $\varepsilon > 0$ such that

$$K \subset U, \quad (-\varepsilon, \varepsilon) \times U \subset \mathcal{D},$$

and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$. Choose N so large that $t_0/N < \varepsilon$. Define $U_0 := U$ and, for $k = 1, \cdots, N$, define the sets $U_k \subset M$ inductively by

$$U_k := \{p \in U | \phi(t_0/N, p) \in U_{k-1}\}.$$

These sets are open in the relative topology of M. We prove by induction on k that $(-\varepsilon, kt_0/N + \varepsilon) \times U_k \subset D$ and ϕ is smooth on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. For $k = 0$ this holds by definition of ε and U. If $k \in \{1, \cdots, N\}$ and the assertion holds for $k - 1$ then we have

$$\begin{aligned} p \in U_k &\Rightarrow p \in U, \quad \phi(t_0/N, p) \in U_{k-1} \\ &\Rightarrow (-\varepsilon, \varepsilon) \subset I(p), \quad (-\varepsilon, (k-1)t_0/N + \varepsilon) \subset I(\phi(t_0/N, p)) \\ &\Rightarrow (-\varepsilon, kt_0/N + \varepsilon) \subset I(p). \end{aligned}$$

Here the last implication follows from Equation (2). Moreover, for $p \in U_k$ and $t_0/N - \varepsilon < t < kt_0/N + \varepsilon$, we have, by Equation (3), that

$$\phi(t, p) = \phi(t - t_0/N, \phi(t_0/N, p)).$$

Since $\phi(t_0/N, p) \in U_{k-1}$ for $p \in U_k$ the right hand side is a smooth map on the open set $(t_0/N - \varepsilon, kt_0/N + \varepsilon) \times U_k$. Since $U_k \subset U$, ϕ is also a smooth map on $(-\varepsilon, \varepsilon) \times U_k$ and hence on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. This completes the induction. With $k = N$ we have found an open neighborhood of (t_0, p_0) contained in $\mathcal{D}$, namely the set $(-\varepsilon, t_0 + \varepsilon) \times U_N$, on which ϕ is smooth. The case $t_0 \leqslant 0$ is treated similarly. This proves *(ii)* and *(iii)*. □

LEMMA 3.7. *Let M, X, $\mathcal{D}$ be as in Theorem 3.6 and let $K \subset M$ be a compact set. Then there exists an $M-$open set $U \subset M$ and an $\varepsilon > 0$ such that $K \subset U$, $(-\varepsilon, \varepsilon) \times U \subset D$, and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$.*

PROOF. see [**RS18**, Lemma 2.4.10] □

PROPOSITION 3.8. *Let $M \subset \mathbb{R}^n$ be a smooth submanifold and $X \in \Gamma(M)$. Then the following are equivalent.*

(*i*) X *is complete.*
(*ii*) $I(p) = \mathbb{R}$ *for all* $p \in M$.
(*iii*) $\mathcal{D} = \mathbb{R} \times M$.

Proof. Clear. □

Let us denote the space of diffeomorphisms of M by

$$\mathrm{Diff}(M) := \{f : M \to M \,|\, f \text{ is a diffeomorphism}\}.$$

This is a group. The group operation is composition and the neutral element is the identity. Now equation (3) asserts that the flow of a complete vector field $X \in \Gamma(M)$ is a group homomorphism

$$\mathbb{R} \to \mathrm{Diff}(M) : t \to \phi^t.$$

This homomorphism is smooth and is characterized by the equation

$$\frac{d}{dt}\phi^t(p) = X(\phi^t(p)), \;\; \phi^0(p) = p,$$

for all $p \in M$.

Theorem 3.9 (Normal form around a point). *Let* $X \in \Gamma(M)$ *and* $p \in M$ *such that* $X(p) \neq 0$. *Then there exists a local chart* (U, ϕ) *around* p *such that if* $\phi = (x_1, \cdots, x_d)$ *then*

$$X|_U = \frac{\partial}{\partial x_1}.$$

In particular, $\phi^t(x_1, \cdots, x_d) = (t + x_1, \cdots, x_d)$.

Proof. Let (U, ψ) be any local coordinate chart around p with $\psi(p) = 0$. Since we can compose this with a diffeomorphism of the target, we can assume that

$$X(p) = \frac{\partial}{\partial x_1}(p).$$

Define

$$\chi(y_1, \cdots, y_d) = \phi^{y_1}(\psi^{-1}(0, y_2, \cdots, y_d)).$$

The differential $d_0\chi$ of this map at 0 is bijective. Indeed, one can compute

$$d_0\chi = d_p\left((\phi^t)'|_0\right) \circ d_0\psi^{-1}.$$

Using the equality

$$d_p\left((\phi^t)'|_0\right) = d_{\phi^0(p)=p}X \circ d_p\phi^0$$

and the fact that $\phi^0 = id$, we can see that $d_0\chi$ sends the coordinate e_i to the basis $\dfrac{\partial}{\partial x_i}$. □

4. Parallel transport

Let $M \subset \mathbb{R}^n$ be a submanifold. The Euclidean inner product $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ induces an inner product

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}.$$

The set $\{g_p\}_p$ is called the *first fundamental form*.

Note that this inner product induces an orthogonal projection $\Pi_p : \mathbb{R}^n \to T_pM$ for each $p \in M$. The linear map Π_p can be represented by $n \times n$ matrix over $\mathbb{R}$, and it is uniquely determined by the two conditions

$$\Pi_p = \Pi_p^2,$$

and for $v \in \mathbb{R}^n$

$$\Pi_p(v) = v \ \Leftrightarrow\ v \in T_pM.$$

REMARK 4.1. The map $\Pi : M \to \mathbb{R}^{n\times n}$ is smooth.

The differential of Π is the linear map $d_p\Pi : T_pM \to \mathbb{R}^{n\times n}$ which associate to a vector $v = \gamma'(0) \in T_pM$ the matrix

$$d_p\Pi(v) := \frac{d}{dt}|_{t=0}\Pi(\gamma(t)) \in \mathbb{R}^{n\times n}.$$

LEMMA 4.2. *For any $v, w \in T_pM$ we have*

$$d_p\Pi(v) \cdot w = d_p\Pi(w) \cdot v \in T_pM^{\perp}.$$

The collection of symmetric bilinear maps $h_p : (v, w) \mapsto h_p(v, w) := d_p\Pi(v) \cdot w$ is called the *second fundamental form*.

Let $M \subset \mathbb{R}^n$ be a submanifold and $\gamma : I \to M$ a smooth curve. A *vector field along* γ is a section of TM over I, that's a smooth map $X_\gamma : I \to \mathbb{R}^n$ such that $X_\gamma(t) \in T_{\gamma(t)}M$. The derivative $X_\gamma'(t)$ is not in general in the tangent space $T_{\gamma(t)}M$.

DEFINITION 4.3. The covariant derivative of X_γ is the vector field ∇X_γ over γ defined by

$$\nabla X_\gamma := \Pi_p(X_\gamma'(t)) \in T_{\gamma(t)}M.$$

LEMMA 4.4. *Let X_γ as above and $\lambda : I \to \mathbb{R}$ be a smooth function. Then we have*

$$\nabla(\lambda X_\gamma) = \lambda' X_\gamma + \lambda\nabla(X_\gamma).$$

DEFINITION 4.5. Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to M$ be a smooth curve. A vector field X_γ along γ is called *parallel* if

$$\nabla X_\gamma(t) = 0.$$

for all $t \in I$.

In other words, X_γ is parallel if and only if $X'_\gamma(t) \perp T_{\gamma(t)}M$.

EXAMPLE 4.6. In particular, γ' is a vector field along γ and $\nabla\gamma'(t) = \Pi_{\gamma'(t)}(\gamma''(t))$. Hence γ' is a parallel vector field along γ if and only if $\gamma''(t) \perp T_{\gamma(t)}M$ for all $t \in I$.

THEOREM 4.7. *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M$ be a smooth curve. Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$ be given. Then there is a unique parallel vector field X_γ along γ such that $X_\gamma(t_0) = v_0$.*

DEFINITION 4.8 (Parallel transport). Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to M$ be a smooth curve. For $t_0, t \in I$ we define the map

$$\Phi(t, t_0) : T_{\gamma(t_0)}M \longrightarrow T_{\gamma(t)}M$$

by

$$\Phi(t, t_0)(v_0) := X_\gamma(t),$$

where X_γ is the unique parallel vector field along γ satisfying $X_\gamma(t_0) = v_0$. The collection of maps $\Phi(t, t_0)$ for $t, t_0 \in I$ is called parallel transport along γ.

5. Orientability

Let V be a finite dimensional vector space. Two ordered bases $(u_1, ..., u_n)$ and $(v_1, ..., v_n)$ are said to be equally oriented if the transition matrix S, whose columns are the coordinates of the vectors $(u_1, ..., u_n)$ with respect to the basis $(v_1, ..., v_n)$, has positive determinant. Being equally oriented is an equivalence relation among bases, for which there are precisely two equivalence classes. The space V is said to be oriented if a specific class has been chosen, this class is then called the orientation of V, and its member bases are called positive. The Euclidean spaces $\mathbb{R}^n$ are usually oriented by the class containing the standard basis $(e_1, ..., e_n)$. For the null space $V = \{0\}$ we introduce the convention that an orientation is a choice between the signs $+$ and $-$.

Let now a (ϕ, U) be a chart on M with $\phi(p) = q$, we obtain a basis for T_pM by taking the pre-images (e_i) of each of the standard basis vectors $e_1, \ldots, e_m$ for $\mathbb{R}^m$ (in that order), that is, the basis vectors will be the equivalence classes of the curves $t \to \sigma(x_0 + te_i)$. This basis for T_pM is called the standard basis with respect to σ. The compatibility condition between charts (O_i, ϕ_i) and (O_j, ϕ_j) on a set M is that the map $\phi_j \circ \phi_j^{-1} : \phi_j(O_i \cap O_j) \to \phi_i(O_i \cap O_j)$ is a diffeomorphism. In particular, the Jacobian matrix $J(\phi_j \circ \phi_j^{-1})$ of the transition map is invertible, and hence has nonzero determinant. If the determinant is

> 0 everywhere, then we say (O_i, ϕ_i), (O_j, ϕ_j) are oriented-compatible. An oriented atlas on M is an atlas such that any two of its charts are oriented-compatible; a maximal oriented atlas is one that contains every chart that is oriented-compatible with all charts in this atlas.

Definition 5.1. A manifold is called orientable if it admits an oriented atlas.

If an orientation has been chosen we say that M is an oriented manifold. The notion of an orientation on a manifold is crucial, since integration of differential forms over manifolds is only defined if the manifold is oriented.

Let (ϕ, O_i) be a chart on an abstract manifold M, then the tangent space is equipped with the standard basis with respect to ϕ. For each $p \in O_i$ we say that the orientation of T_pM, for which the standard basis is positive, is the orientation induced by ϕ.

Proposition 5.2. *Let M be a connected manifold. Then the following are equivalent*

(i) *M is orientable.*
(ii) *Orientation is preserved moving along loops.*
(iii) *M admits a nowhere vanishing $n-$form.*

Proof. □

Example 5.3. The spheres S^n are orientable. To see this, consider the atlas with the two charts (U_+, ϕ_+) and (U, ϕ_-), given by stereographic projections. (see Example 1.7). Here

$$\phi_- : (U_+ \cap U) = \phi_+(U_+ \cap U) = \mathbb{R}^n \setminus \{0\}.$$

The entries of the Jacobian matrix of $\phi_- \circ \phi_+^{-1}$ at x are given by

$$\frac{\partial}{\partial x_j} \frac{x_i}{\|x\|^2} = \frac{1}{\|x\|^2}\delta_{ij} - \frac{2x_i x_j}{\|x\|^4}. \tag{4}$$

Its determinant is $-\|x\|^{-2n}$. (exercise).

If M is a smooth manifold of dimension n then for each $p \in M$ the tangent space T_pM is a vector space of dimension n, and hence has two choices of orientation. This gives a two-sheeted covering space O_M called the *orientation covering* of M. So one sees that M is orientable if and only if its covering space is disconnected.

An advantage of this covering space point of view is that we immediately have the following result.

PROPOSITION 5.4. *If M is a smooth connected manifold with $\pi_1(M) = 0$ then M is orientable.*

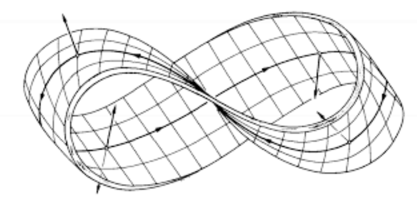
PROOF. Each covering space $\tilde{M} \to M$ is trivial since if $p \in M$ then $\pi_1(\tilde{M}, \tilde{p}) \subset \pi_1(M, p) = 0$. In particular the orientation covering must then consist of two simply-connected components, each diffeomorphic to M. □

Recall That the fundamental group $\pi_1(X; x_0)$ of a pointed topological space $(X; x_0)$ is the set of all homotopy classes of loops in X with base point x_0 together with the multiplication induced by concatenation as follows

$$[\lambda] \cdot [\eta] = [\lambda * \eta],$$

where $\lambda * \eta[t] = \begin{cases} \lambda(2t) & \text{if } t \in [0, 1/2] \\ \eta(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$.

EXAMPLE 5.5. The Möbius strip is a surface with only one side.



It is maybe the simplest example of a non-orientable manifold. To see this, one can remark that the orientation is not preserved along loops, thus it is impossible to make a consistent choice of orientations, because the band is one-sided. Choosing an orientation in one point forces it by continuity to be given in neighboring points, and eventually we are forced to the opposite choice in the initial point.

EXAMPLE 5.6. The real projective space $\mathbb{P}^2$ is another example of a non orientable surface. One way to visualise $\mathbb{P}^2$ is as the semisphere with points on the equator identified with their antipodal points.

EXAMPLE 5.7. In general, the real projective space $\mathbb{P}^n$ is orientable if and only if n is odd or $n = 0$.

6. Tensors

6.1. Exterior product. Let $k \geqslant 0$, and let V be a finite dimensional vector space over $\mathbb{R}$. The exterior product $\Lambda^k V$ of V is defined by

$$\Lambda^k V = \bigotimes^k V / \sim,$$

where the equivalence relation is given by

$\otimes_{i=1}^k x_i \sim \otimes_{i=1}^k y_i \Leftrightarrow \otimes_{i=1}^k x_i = sng(\sigma)\otimes_{i=1}^k y_{\sigma(i)}$, for some permutation σ.

The class of a tensor $x_1 \otimes \cdots \otimes x_k$ is denoted $x_1 \wedge \cdots \wedge x_k$.
Note that the dimension of $\Lambda^k V$ is given by $\binom{n}{k}$, where $n = \dim(V)$.
There is a canonical alternating $k-$linear map $\wedge_k : V^k \to \Lambda^k V$ given by

$$\wedge_k(x_1, \dots, x_n) = x_1 \wedge \cdots \wedge x_n,$$

such that for any alternating $k-$linear map $f : V^k \to W$, there is a unique linear map $g : \Lambda^n V \to W$ such that $f = g \circ \wedge_k$.

LEMMA 6.1. *Assume that $e_1, \dots, e_n$ is a basis of V. Then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | i_1 < \cdots < i_k\}$ is a basis of $\Lambda^k V$. In particular,*

$$\dim \Lambda^k V = \binom{n}{k},$$

where $n = dim(V)$.

PROPOSITION 6.2. *The vectors $v_1, \dots, v_k$ are linearly independent if and only if $v_1 \wedge \cdots \wedge v_k \neq 0$. Two linearly independent $k-$tuples of vectors $v_1, \dots, v_k$ and $w_1, \dots, w_k$ span the same $k-$dimensional linear subspace if and only if $v_1 \wedge \cdots \wedge v_k = c w_1 \wedge \cdots \wedge w_k$ for some $c \in \mathbb{R}^n \setminus \{0\}$.*

COROLLARY 6.3. *The Grassmannian manifold $Gr^k(V)$ can be embedded into the projective space $P(\Lambda^k V)$ by assigning to the $k-$dimensional subspace spanned by the linearly independent vectors $v_1, \cdots, v_k$ the $1-$dimensional linear space spanned by $v_1 \wedge \cdots \wedge v_k$. This embedding is called the* Plücker embedding.

The direct sum $\Lambda^*(V) = \bigoplus_{k=1}^n \Lambda^k V$ is an associative algebra where the product $\wedge$ is defined

$$(v_1 \wedge \cdots \wedge v_k) \wedge (w_1 \wedge \cdots \wedge w_l) = v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l.$$

$\Lambda^* V$ is called the exterior algebra or Grassmannian algebra of V.

DEFINITION 6.4. A (covariant) $k-$tensor on V is a multilinear map $T : V^k \to \mathbb{R}$. The set of $k-$tensors on V is denoted $T^k(V)$.

Let $\phi_1, \dots, \phi_k \in V^*$. The map

$$\phi_1 \otimes \cdots \otimes \phi_k : (v_1, \dots, v_k) \mapsto \phi_1(v_1) \dots \phi_k(v_k)$$

is a $k-$tensor on V. More generally, if $S \in T^k(V)$ and $T \in T^l(V)$ are tensors, we define the tensor product $S \otimes T \in T^{k+l}(V)$ by

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = S(v_1, \dots, v_k) T(v_{k+1}, \dots, vk + l).$$

LEMMA 6.5. *Let $\xi_1, \ldots, \xi_n$ be a basis of V^*. Then the family $\{\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}\}$, where $i_1, \ldots, i_k$ are arbitrary number in $\{1, \ldots, n\}$, form a basis of $T^k(V)$. In particular $\dim T^k(V) = n^k$.*

So one sees that $T^k(V) \cong V^* \otimes \cdots \otimes V^*$.

DEFINITION 6.6. A contravariant $k-$tensor on V is a covariant $k-$tensor on V^*. More generally a tensor of type $\binom{k}{l}$ is a $k-$covariant, $l-$contravariant multilinear map

$$F : \underbrace{V^* \times \cdots \times V^*}_{l \text{ times}} \times \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}.$$

The space of these tensors is denoted $T_l^k(V)$.

LEMMA 6.7.

(1) There is a natural identification $T_1^1(V) \cong$ End V.
(2) There is a natural identification between T_{l+1}^k and the space of multilinear maps

$$\underbrace{V^* \times \cdots \times V^*}_{l \text{ times}} \times \underbrace{V \times \cdots \times V}_{k \text{ times}} \to V$$

PROOF. Exercise. □

DEFINITION 6.8. A $k-$tensor φ is called alternating if for every $v_1, \ldots, v_k \in V$ and any $\sigma \in \Sigma_k$, one have

$$\varphi(v_1, \ldots, v_k) = \operatorname{sign}(\sigma)\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

The space of these is denoted $A^k(V)$. For $\phi_1, \ldots, \phi_k \in V^*$, the map

$$\phi_1 \wedge \cdots \wedge \phi_k : (v_1, \ldots, v_k) \mapsto \sum_{\sigma \in \Sigma} \phi_1(v_{\sigma(1)}) \cdots \phi_k(v_{\sigma(k)})$$

is alternating $k-$tensor.
So we get as before

$$A^k(V) \cong \Lambda^k V^*.$$

REMARK 6.9. There is a canonical map $\Lambda^k V^* \to (\Lambda^k V)^*$ defined by sending an alternating tensor $\varphi_1 \wedge \cdots \wedge \varphi_k$ to the linear form

$$Alt(\varphi_1 \wedge \cdots \wedge \varphi_k)(x_1 \wedge \cdots \wedge x_k) = \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma)\varphi_1(x_{\sigma(1)}) \cdots \varphi_k(x_{\sigma(k)}),$$

which is an isomorphism.

7. Differential forms

Let $M \subset \mathbb{R}^n$ be a smooth submanifold. Let $f \in C^\infty(M)$. The differential of f is a map $df : TM \to \mathbb{R}$, at each fiber, it gives a linear form d_pf of the vector space T_pM. Hence an element of the dual space.

Denote by T^*M the cotangent bundle over M; it is the dual of the tangent bundle. Each fiber of T^*M is isomorphic to $(T_pM)^*$.

DEFINITION 7.1. A differentiable $1-$form on M (also called *covector field*) is a section of the cotangent bundle, that's a smooth map $\omega : M \to T^*M$ such that $\omega(p) \in T_p^*M$ for each $p \in M$. The space of $1-$forms is denoted $\Omega^1(M)$.

As we have seen before, each smooth function f gives rise to a $1-$form df. A differentiable $1-$form is called *exact* if it is equal to df for some $f \in C^\infty(M)$. In particular, if f equals one of the coordinates x_i around a point $p \in M$, we get the $1-$form dx_i. Moreover, any $1-$form ω can be written, locally around p in the form

$$\omega = \sum_{i=1}^{n} a_i dx_i.$$

EXAMPLE 7.2. Let $M = \mathbb{R}^2$ with coordinates x, y, and let $f \in C^\infty(\mathbb{R}^2)$. The differential of x and y equal to dx and dy whose Jacobian matrices at any point are equals to $(1, 0)$ and $(0, 1)$ respectively. Hence the differential of f is given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

And any other covector field ω is of the form $adx + bdy$, with $a, b \in C^\infty(\mathbb{R}^2)$, because the cotangent bundle is trivial.

More generally, we have

DEFINITION 7.3. A differentiable $k-$form on M is a section of the exterior product $\Lambda^k T^*M$ of the cotangent bundle, that's a smooth map $\omega : M \to \Lambda^k T^*M$ such that $\omega(p) \in \Lambda^k T_p^*M$ for each $p \in M$. The space of $k-$forms is denoted $\Omega^k(M)$.

On a local coordinate chart $U \subset M$, let $x_1, \dots, x_d$, then $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is $k-$form, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Moreover, any $k-$form on U can be written uniquely in the form

$$w = \sum_I a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where a_I are smooth functions on U.

EXAMPLE 7.4. Let $M = \mathbb{R}^2$ with coordinates x, y. A differential $2-$form ω on M is of the form $adx \wedge dy$, with $a \in C^\infty(\mathbb{R}^2)$, because the cotangent bundle is trivial and its second exterior power is a line bundle. If $X, Y \in T_pM$, then

$$\omega_p(X, Y) = a(p)(x_1y_2 - x_2y_1).$$

We denote by $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$. The wedge product $\wedge$ gives $\Omega^*(M)$ the structure of an algebra over $C^\infty(M)$.

7.1. Interior product. Let ω a differential $k-$form on a manifold M, and X a smooth vector field. We have

DEFINITION 7.5. The interior product of X and ω is the differential $(k-1)-$form $\iota_X\omega$ defined by

$$\iota_X\omega(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}).$$

If $k = 0$, we set $\iota_X\omega = 0$.

Interior product ι_X with the vector field X can be thought of as a linear map from $\Omega^*(M)$ into itself. This is a degree -1 map, as it decreases the degree of any form by 1. In general, a linear map $L : \Omega^*(M) \to \Omega^*(M)$ is a degree d map, if $L(\Omega^k(M)) \subset \Omega^{k+d}(M)$ for all k.

7.2. Exterior derivative. Let $M \subset \mathbb{R}^n$ be a submanifold. In a local chart around a point $p \in M$, we can write a $k-$form $\omega \in \Omega^k$ in the form

$$\omega = \sum_{i_1 < \cdots < i_d} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_d},$$

We get then

DEFINITION 7.6. The exterior derivative of the $k-$form ω is the $(k+1)-$form $d\omega$ giving locally by

$$d\omega = \sum_{i_1 < \cdots < i_d} da_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_d}.$$

It is a non trivial result that this definition can be globalized to get a morphism

$$d : \Omega^k(M) \to \Omega^{k+1}(M).$$

DEFINITION 7.7. A $k-$form is called *exact* if it equals $d\alpha$ form some $(k-1)-$form α. It is called *closed* if $d\omega = 0$.

Since $d^2 = 0$ by definition, we deduce

LEMMA 7.8. *An exact form is closed. That's the exterior derivative defines a complex*

$$0 \longrightarrow \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \to 0,$$

meaning $Im(d) \subset Ker(d)$. Here $m = dim(M)$.

EXAMPLE 7.9. Reconsider the example 7.2. The $1-$form $\omega = adx + bdy$ is exact if and only if

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}.$$

Indeed, if ω is exact then there is an $f \in C^\infty(\mathbb{R}^2)$ such that $a = \frac{\partial f}{\partial x}, b = \frac{\partial f}{\partial y}$. Hence

$$\frac{\partial a}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial b}{\partial x}.$$

Conversely, assume that $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$. Then choose a smooth two variable function f such that $\frac{\partial f}{\partial x} = a$. Then $\frac{\partial a}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial b}{\partial x}$. So $b - \frac{\partial f}{\partial y}$ doesn't depend on x. Choose a function g on y such that $g'(y) = b - \frac{\partial f}{\partial y}$. Then one sees that $\omega = \mathrm{d}(f+g)$.
Note that this in not true in arbitrary manifold.

Let $f : M \to N$ a smooth map and let $\omega \in \Omega^k(N)$. We define the *pull back* of ω by f to be the $k-$form denoted $f^*\omega$ on M given by

$$f^*\omega(p)(v_1, \cdots, v_k) = \omega(f(p))(d_pf(v_1), \cdots, d_pf(v_k)),$$

for all $v_1, \cdots, v_k \in T_pM$ and all $p \in M$.

8. Integration of differential forms

We will neglect orientation issues, assuming that all domains are positively oriented according to usual conventions and that parameterizations respect these orientations.

We begin by defining the integral of differential forms over boxes (products of intervals). To simplify, we assume that the functions considered are continuously differentiable.

DEFINITION 8.1. The integral of a 1-form $f\,dt$ over an interval $P_1 = [a, b]$ of $\mathbb{R}$ is defined by:

$$\int_{P_1} f\,dt = \int_a^b f(t)\,dt.$$

The integral of a 2-form $f\,ds \wedge dt$ over a box $P_2 = [a,b] \times [c,d]$ in $\mathbb{R}^2$ is defined by the double integral:

$$\int_{P_2} f\,ds \wedge dt := \int_c^d \left(\int_a^b f(s,t)\,ds \right) dt.$$

The integral of a 3-form $f\,ds \wedge dt \wedge du$ over a box $P_3 = [a,b] \times [c,d] \times [e,f]$ in $\mathbb{R}^3$ is defined by the triple integral:

$$\int_{P_3} f\,ds \wedge dt \wedge du = \int_e^f \left(\int_c^d \left(\int_a^b f(s,t,u)\,ds \right) dt \right) du.$$

Remark 8.2. According to the calculation rules we have established for differential forms, which give $dy \wedge dx = -dx \wedge dy$, we have by definition, in case of a swap of the integration variables, for example:

$$\int_{P_2} f\,dt \wedge ds = -\int_{P_2} f\,ds \wedge dt.$$

Definition 8.3. A parameterized curve C (dimension 1) in $\mathbb{R}^2$ is a map

$$\sigma = (x(t), y(t)) : P_1 = [a,b] \to \mathbb{R}^2.$$

A parameterized surface S (dimension 2) in $\mathbb{R}^3$ is a map

$$\sigma = (x(s,t), y(s,t), z(s,t)) : P_2 = [a,b] \times [c,d] \to \mathbb{R}^3.$$

A parameterized volume V (dimension 3) in $\mathbb{R}^3$ is a map

$$\sigma = (x(s,t,u), y(s,t,u), z(s,t,u)) : P_3 = [a,b] \times [c,d] \times [e,f] \to \mathbb{R}^3.$$

We now define the notion of an oriented parameterized domain.

Definition 8.4. Let $\sigma : P_k \to D \subset \mathbb{R}^n$ be a parameterized domain of dimension $1 \leq k \leq n \leq 3$. We say that σ is oriented if the determinant of the matrix of its partial derivatives is positive.

Example 8.5. Here are several examples of parameterized domains.

- The circle with the equation $x^2 + y^2 = 1$ is the oriented parameterized curve in $\mathbb{R}^2$ given by:

  $$\sigma = (\cos(t), \sin(t)) : [0, 2\pi] \to \mathbb{R}^2.$$

- The circle can also be described as the union of two semicircles. The upper semicircle has the oriented parameterization:

  $$\sigma = (-x, \sqrt{1-x^2}) : [-1,1] \to \mathbb{R}^2,$$

  and the lower semicircle has the oriented parameterization:

  $$\sigma = (x, -\sqrt{1-x^2}) : [-1,1] \to \mathbb{R}^2.$$

- The parameterizations of the upper semicircle given by:
$$\sigma = (\sin(t), \cos(t)) : [-\pi, \pi] \to \mathbb{R}^2$$
and
$$\sigma = (x, \sqrt{1 - x^2}) : [-1, 1] \to \mathbb{R}^2$$
are not oriented because they do not rotate in the direct (counterclockwise) sense in the plane.
- The sphere with the equation $x^2 + y^2 + z^2 = 1$ is the oriented parameterized surface (in spherical coordinates) in $\mathbb{R}^3$ given by:
$$\sigma = (\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)) : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3.$$
- The solid cylinder with the equation $x^2 + y^2 \leq R, z \in [0, 1]$ is the oriented parameterized volume (in cylindrical coordinates) in $\mathbb{R}^3$ given by:
$$\sigma = (r\cos(\varphi), r\sin(\varphi), z) : [0, R] \times [0, 2\pi] \times [0, 1] \to \mathbb{R}^3.$$
- The solid ball with the equation $x^2+y^2+z^2 \leq R$ is the oriented parameterized volume (in spherical coordinates) in $\mathbb{R}^3$ given by:
$$\sigma = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)) : [0, \pi]\times[0, 2\pi]\times[0, R] \to \mathbb{R}^3.$$

The integral of a differential form over a general parameterized domain reduces to that of its pullback on the parameter box.

Definition 8.6. Let $\sigma : P_k \to D$ be an oriented parameterized domain of dimension k, chosen from one of the examples above. The integral of a differential k-form $\omega \in \Omega^k$ over D is defined by:
$$\int_\Omega \omega := \int_{P_k} \sigma^*\omega.$$

For the above definition to make sense, we need to define the pullback $\sigma^*\omega$ of a differential form along a parameterization $\sigma : P_k \to D$. This can be done in the following implicit way, which will be made explicit through examples.

Definition 8.7. If $\sigma : P_k \to D$ is a parameterized domain of dimension k, and $\omega \wedge \eta$ is a wedge product of differential forms, we define:
$$\sigma^*(\omega \wedge \eta) = \sigma^*(\omega) \wedge \sigma^*(\eta).$$
If $f \in \Omega^0$ is a function, we define:
$$\sigma^* f = f \circ \sigma.$$
We also have the rule:
$$\sigma^* d\omega = d\sigma^*\omega.$$

EXAMPLE 8.8. If C is a curve parameterized by $\sigma = (x(t), y(t))$ in $\mathbb{R}^2$, we have:

$$\sigma^* dx = d\sigma^* x = d(x(t)) = x'(t)dt.$$

Similarly, we obtain:

$$\sigma^* dy = y'(t)dt.$$

If f is a function, we get:

$$\sigma^* f(t) := f(x(t), y(t)).$$

For example, this gives:

$$\sigma^*(f dx) = \sigma^*(f) \cdot \sigma^*(dx) = f(x(t), y(t))x'(t)dt.$$

PROPOSITION 8.9. *The line integral of a differential 1-form $\omega = f dx + g dy$ along the oriented curve C is given by the formula:*

$$\int_C \omega := \int_{[a,b]} \sigma^* \omega = \int_a^b f(x(t), y(t))x'(t)dt + g(x(t), y(t))y'(t)dt.$$

Morally, during a change of variables $(x(s,t), y(s,t))$, the expression $dx \wedge dy$ is multiplied by the determinant of the variable change in the new coordinates $ds \wedge dt$. The theory of differential forms is entirely developed around this crucial point, which makes it invariant under a change of coordinates. We will quickly reprove this formula with a short calculation.

EXAMPLE 8.10. If S is a surface parameterized by $\sigma = (x(s,t), y(s,t), z(s,t))$, we can compute the pullback of the 2-form $dx \wedge dy$ on the parameter rectangle (s,t) as follows:

$$\begin{aligned}
\sigma^*(dx \wedge dy) &= \sigma^*(dx) \wedge \sigma^*(dy) \\
&= d\sigma^*(x) \wedge d\sigma^*(y) \\
&= dx(s,t) \wedge dy(s,t) \\
&= \left(\frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt\right) \wedge \left(\frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt\right) \\
&= \frac{D(x,y)}{D(s,t)} ds \wedge dt,
\end{aligned}$$

with

$$\frac{D(x,y)}{D(s,t)} = \det \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

being the Jacobian determinant of the parameterization.

More generally, if $\omega = f dx \wedge dy + g dy \wedge dz + h dz \wedge dx$ is a 2-form on $\mathbb{R}^3$, and $\sigma = (x(s,t), y(s,t), z(s,t))$ is a parameterized surface, we obtain for $\sigma^*\omega$ the differential form on the rectangle $[a,b]\times[c,d]$ given by:

$$\begin{aligned}\sigma^*\omega &= \sigma^* f\cdot\sigma^*(dx\wedge dy)+\sigma^* g\cdot\sigma^*(dy\wedge dz)+\sigma^* h\cdot\sigma^*(dz\wedge dx)\\ &= \left[f\circ\sigma\cdot\frac{D(x,y)}{D(s,t)}+g\circ\sigma\cdot\frac{D(y,z)}{D(s,t)}+h\circ\sigma\cdot\frac{D(z,x)}{D(s,t)}\right]ds\wedge dt.\end{aligned}$$

PROPOSITION 8.11. *The integral of a differential 2-form*

$$\omega = f dx\wedge dy + g dy\wedge dz + h dz\wedge dx$$

over an oriented surface S *parameterized by* $\sigma : P_2 : [a,b]\times[c,d]\to\mathbb{R}^3$ *is given by the formula:*

$$\int_S \omega = \int_{P_2}\left[f\circ\sigma\frac{D(x,y)}{D(s,t)}+g\circ\sigma\frac{D(y,z)}{D(s,t)}+h\circ\sigma\frac{D(z,x)}{D(s,t)}\right]ds\wedge dt.$$

It can be shown by a calculation similar to the previous ones that if V is an oriented volume parameterized by $\sigma : [a,b]\times[c,d]\times[e,f]\to\mathbb{R}^3$, we have:

$$\sigma^*(dx\wedge dy\wedge dz) = \frac{D(x,y,z)}{D(s,t,u)}ds\wedge dt\wedge du,$$

with

$$\frac{D(x,y,z)}{D(s,t,u)} = \det\begin{pmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u}\end{pmatrix}$$

being the Jacobian determinant of the parameterization.

PROPOSITION 8.12. *The integral of a differential 3-form* $\omega = f dx\wedge dy\wedge dz$ *over a volume* V *parameterized by* $\sigma : P_3 = [a,b]\times[c,d]\times[e,f]\to\mathbb{R}^3$ *is given by the formula:*

$$\int_V \omega = \int_{P_3} f(s,t,u)\frac{D(x,y,z)}{D(s,t,u)}ds\wedge dt\wedge du.$$

CHAPTER 3

Vector bundles

1. Basic definitions and properties

Let E_p be a vector space of dimension r for any $p \in M$. Let $E = \bigcup_{p \in M} E_p$. Denote by $\pi : E \to M$ the canonical projection which sends $v \in E_p$ to p, and $\iota : M \to E$ the inclusion $p \mapsto 0 \in E_p$.

DEFINITION 1.1. We say that $E \to M$ is a vector bundle of rank r if for each $p \in M$ there are charts (U, ϕ) around p and $(\tilde{U}, \tilde{\phi})$ around $\iota(p)$, with $\tilde{U} = \pi^{-1}(U)$, such that $\tilde{\phi}$ restricts to vector space isomorphisms

$$E_p = \pi^{-1}(p) \to \phi(p) \times \mathbb{R}^r \cong \mathbb{R}^r$$

for all $p \in M$. One calls E the total space and M the base of the vector bundle.

The vector bundle charts may be pictured in terms of a diagram,

$$\begin{array}{ccc} E \supset \tilde{U} & \xrightarrow{\ \tilde{\phi}\ } & \phi(U) \times \mathbb{R}^r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ M \supset U & \xrightarrow{\ \phi\ } & \phi(U) \end{array}$$

EXAMPLE 1.2. Let $M = \mathbb{P}^n$, and consider the line bundle T on $\mathbb{P}^n$ such that its fiber T_p over $p \in \mathbb{P}^n$ is the line l_p in $\mathbb{R}^{n+1}$ defined p, i.e.

$$T = \bigcup_{[l] \in \mathbb{P}^n} l_p.$$

It is a line bundle (a vector bundle of rank 1) over $\mathbb{P}^n$. Indeed, let (O_i, ϕ_i) be one of the charts given in Example 1.11, and let $p_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projection onto the i^{th} coordinate. Then for any $[l] \in O_i$, the restriction of p_i to l gives an isomorphism $l \to \mathbb{R}$. Let $\tilde{\phi} : \tilde{O}_i := \pi^{-1}(O_i) \to \mathbb{R}^n \times \mathbb{R}$ given by $\tilde{\phi}(p, t) = (\phi(p), p_i(t))$ where $t \in l_p$.

THEOREM 1.3. *The union* $TM = \bigcup_{p \in M} T_p M \to M$ *is a vector bundle over* M *of rank* $d = \dim M$.

PROOF. Let (U, ϕ) be a chart on M. Recall that

$$T_p \phi : T_p U = T_p M \to T_{\phi(p)} \phi(U) \cong \mathbb{R}^d$$

is an isomorphism. Now, the pair $(TU, T\phi)$ is a chart on TM. Moreover, the composition

$$T_p\psi \circ (T_p\phi)^{-1} = T_{\phi(p)}(\psi \circ \phi^{-1}),$$

which shows that this composition is smooth. □

EXAMPLE 1.4. Let M be the circle S^1; it can be written as the quotient

$$S^1 = I/\sim, \quad \text{where } I = [0,1],\ 0 \sim 1.$$

Let V be the infinite Mbius band:

$$V = (I \times \mathbb{R})/\sim, \quad \text{where } (0, v) \sim (1, -v)\ \forall\ v \in \mathbb{R}.$$

Then, the projection $\pi : V \to S^1$ onto the first coordinate is well-defined and is a real line bundle (i.e., rank-one bundle) over S^1.

EXAMPLE 1.5. The real projective space of dimension n, denoted $\mathbb{RP}^n$, is the space of real one-dimensional subspaces of $\mathbb{R}^{n+1}$ (or lines through the origin in $\mathbb{R}^{n+1}$) in the natural quotient topology. In other words, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a nonzero vector in $\mathbb{R}^{n+1}$, i.e., an element of $\mathbb{R}^{n+1} - \{0\}$. Two such vectors determine the same one-dimensional subspace in $\mathbb{R}^{n+1}$ and the same element of $\mathbb{RP}^n$ if and only if they differ by a non-zero scalar.

Thus, as sets

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* \equiv (\mathbb{R}^{n+1} - \{0\})/\sim,$$

where

$$\mathbb{R}^* = \mathbb{R} - \{0\}, \quad c \cdot v = cv \in \mathbb{R}^{n+1} - \{0\}, \quad \forall c \in \mathbb{R}^*,$$

$$v \in \mathbb{R}^{n+1} - \{0\},\ v \sim cv\ \forall c \in \mathbb{R}^*.$$

Alternatively, a one-dimensional subspace of $\mathbb{R}^{n+1}$ is determined by a unit vector in $\mathbb{R}^{n+1}$, i.e., an element of S^n. Two such vectors determine the same element of $\mathbb{RP}^n$ if and only if they differ by a non-zero scalar, which in this case must necessarily be ± 1.

Thus, as sets

$$\mathbb{RP}^n = S^n/\mathbb{Z}_2 \equiv S^n/\sim,$$

where

$$\mathbb{Z}_2 = \{\pm 1\}, \quad c \cdot v = cv \in S^n\ \forall c \in \mathbb{Z}_2,\ v \in S^n,\ v \sim cv\ \forall c \in \mathbb{Z}_2,\ v \in S^n.$$

Thus, as sets,

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^* = S^n/\mathbb{Z}_2.$$

It follows that $\mathbb{RP}^n$ has two natural quotient topologies; these two topologies are the same, however. The space $\mathbb{RP}^n$ has a natural smooth structure, induced from that of $\mathbb{R}^{n+1} - \{0\}$ and S^n. Let

$$\gamma_n = \{(\ell, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in \ell\}.$$

The projection

$$\pi : \gamma_n \to \mathbb{RP}^n$$

defines a smooth real line bundle. The fiber over a point $\ell \in \mathbb{RP}^n$ is the one-dimensional subspace ℓ of $\mathbb{R}^{n+1}$! For this reason, γ_n is called the tautological line bundle over $\mathbb{RP}^n$.

Note that $\mathbb{RP}^1 = S^1$ and $\gamma_1 \to S^1$ is the infinite Mbius strip.

2. Sections and homomorphisms

If $\pi : V \to M$ is a vector bundle (real or complex), a section of π or V is a smooth map $s : M \to V$ such that $\pi \circ s = \mathrm{id}_M$, i.e., $s(x) \in V_x$ for all $x \in M$. If s is a section, then $s(M)$ is an embedded submanifold of V: the injectivity of s and ds is immediate from $\pi \circ s = \mathrm{id}_M$, while the embedding property follows from the continuity of π.

Every fiber V_x of V is a vector space and thus has a distinguished element, the zero-vector in V_x, which we denote by 0_x. It follows that every vector bundle admits a section:

$$s_0(x) = (x, 0_x) \in V_x.$$

This map is smooth, since on a trivialization of V over an open subset U of M it is given by the inclusion of U as $U \times \{0\}$ into $U \times \mathbb{R}^k$ or $U \times \mathbb{C}^k$. Thus, M can be thought of as sitting inside of V as the zero section; it is a deformation retract of V.

Definition 2.1. (1) Suppose $\pi : V \to M$ and $\pi' : V' \to N$ are real (or complex) vector bundles. A smooth map $\tilde{f} : V \to V'$ is a *vector bundle homomorphism* if $\tilde{f}$ descends to a map $f : M \to N$, i.e., the diagram

$$\begin{array}{ccc} V & \xrightarrow{\tilde{f}} & V' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & N \end{array}$$

commutes, and the restriction $\tilde{f} : V_x \to V'_{f(x)}$ is linear (or $\mathbb{C}$-linear, respectively) for all $x \in M$.

(2) If $\pi : V \to M$ and $\pi' : V' \to M$ are vector bundles, a vector bundle homomorphism $\tilde{f} : V \to V'$ is an *isomorphism of vector*

bundles if $\pi' \circ \tilde{f} = \pi$, i.e., the diagram

$$\begin{array}{ccc} V & \xrightarrow{\tilde{f}} & V' \\ \pi \downarrow & & \downarrow \pi' \\ M & = & M \end{array}$$

commutes, and $\tilde{f}$ is a diffeomorphism (or equivalently, its restriction to each fiber is an isomorphism of vector spaces). If such an isomorphism exists, then V and V' are said to be isomorphic vector bundles.

Note that the two conditions above on $\tilde{f}$ are equivalent due to the local structures of V and V'.

PROPOSITION 2.2. *If $\pi : V \to M$ is a real (or complex) vector bundle of rank k, V is isomorphic to the trivial real (or complex) vector bundle of rank k over M if and only if V admits k sections $s_1, \ldots, s_k$ such that the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent (over $\mathbb{C}$, respectively) in V_x for all $x \in M$.*

PROOF. We consider the real case; the proof in the complex case is nearly identical.

(1) Suppose $h : M \times \mathbb{R}^k \to V$ is an isomorphism of vector bundles over M. Let $e_1, \ldots, e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. Define sections $s_1, \ldots, s_k$ of V over M by

$$s_l(x) = h(x, e_l) \quad \forall l = 1, \ldots, k,\ x \in M.$$

Since the maps $x \mapsto (x, e_l)$ are sections of $M \times \mathbb{R}^k$ over M and h is a bundle homomorphism, the maps s_l are sections of V. Since the vectors (x, e_l) are linearly independent in $x \times \mathbb{R}^k$ and h is an isomorphism on every fiber, the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent in V_x for all $x \in M$, as needed.

(2) Suppose $s_1, \ldots, s_k$ are sections of V such that the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent in V_x for all $x \in M$. Define the map

$$h : M \times \mathbb{R}^k \to V \quad \text{by} \quad h(x, c_1, \ldots, c_k) = c_1 s_1(x) + \ldots + c_k s_k(x) \in V_x.$$

Since the sections $s_1, \ldots, s_k$ and the vector space operations on V are smooth, the map h is smooth. It is immediate that $\pi(h(x, c)) = x$ and the restriction of h to $x \times \mathbb{R}^k$ is linear; thus, h is a vector bundle homomorphism. Since the vectors $s_1(x), \ldots, s_k(x)$ are linearly independent in V_x, the homomorphism h is injective and thus an isomorphism on every fiber. We conclude that h is an isomorphism between vector bundles over M.

□

CHAPTER 4

Riemannian manifolds

The Riemannian manifold is the essential object studied in Riemannian geometry. It is a variety on which it is possible to perform length calculations. Technically, a Riemannian manifold is a differential manifold endowed with an additional structure called a Riemannian metric allowing to calculate the scalar product of two vectors tangent to the manifold at the same point.

Let M be a smooth manifold. A $\binom{k}{l}-$*tensor field* on M is section of the tensor bundle $\mathcal{T}_l^k M = \amalg_{p\in M} T_l^k(T_pM)$. A $\binom{0}{1}-$*tensor field* is a vector field and a $\binom{k}{0}-$*tensor field* which is alternating is $k-$form.

A Riemannian metric on an abstract manifold M is a $2-$tensor field $g \in \mathcal{T}^2(M)$ that is symmetric (i.e., $g(X,Y) = g(Y,X)$) and positive definite (i.e., $g(X,X) > 0$ if $X \neq 0$). A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $\langle X, Y\rangle = g(X,Y)$ for $X, Y \in T_pM$. A manifold together with a given Riemannian metric is called a Riemannian manifold.

LEMMA 0.1. *Every smooth manifold can be given a Riemannian metric.*

PROOF. Consider a partition of unity $\{f_\alpha\}_{\alpha\in I}$. Let $\{(U_i, \phi_i)\}$ be a atlas on M. Then define

$$g = \sum_\alpha f_\alpha \phi^* g_i,$$

where g_i is the canonical metric on $\phi_i(U_i)$. □

Given a Riemannian metric g on M, we define the length or norm of a tangent vector at p, as in the Euclidean case, by $|X|_p = \sqrt{g_p(X,X)}$. We define the angle between two nonzero vectors $X, Y \in T_pM$ to be the unique $\theta \in [0,\pi]$ satisfying $\cos\theta = g_p(X,Y)/(|X|_p|Y|_p)$.

Given a smooth curve $\gamma : [a,b] \to M$. For any $t \in [a,b]$, we have a tangent vector $\gamma'(t) \in T_{\gamma(t)}M$. So get a nonnegative continuous function on $[a,b]$ given by $t \mapsto |\gamma'(t)|_{\gamma(t)}$ (one should assume that g is continuous). So we define the length of the curve γ to be the integral of this function

$$\mathscr{L}(\gamma) = \int_a^b |\gamma'(t)|_{\gamma(t)} dt.$$

Assume now that M is connected. Note that since M is locally homeomorphic to an open of $\mathbb{R}^d$, then it is locally path connected, and so it is path connected since it is connected.
Define

$$d(p,q) = \inf \left\{ \mathscr{L}(\gamma) | \gamma \text{ is a pieacewise } C^1 \text{ curve connecting } p \text{ and } q \right\}.$$

PROPOSITION 0.2. *d is a metric on M.*

Let (M, g) be a Riemannian manifold. If $(U, \varphi = (x^1, \ldots, x^n))$ is a chart of M, a local expression for g can be given as follows. Let $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$ be the coordinate vector fields, and let $\{dx^1, \ldots, dx^n\}$ be the dual 1-forms. For $p \in U$ and $u, v \in T_pM$, we write

$$u = \sum_{i=1}^n u^i \frac{\partial}{\partial x^i}\bigg|_p$$

and

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\bigg|_p.$$

Then, by bilinearity,

$$gp(u,v) = \sum_{i,j} u^i v^j gp\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_{i,j} g_{ij}(p) u^i v^j,$$

where we have set

$$g_{ij}(p) = gp\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Note that $g_{ij} = g_{ji}$. Hence we can write

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j = \sum_{i<j} \tilde{g}_{ij}\, dx^i\, dx^j,$$

where $\tilde{g}_{ii} = g_{ii}$, and $\tilde{g}_{ij} = 2g_{ij}$ if $i < j$.

Next, let $(U_0, \varphi_0 = (x_0^1, \ldots, x_0^n))$ be another chart of M such that $U \cap U_0 \neq \emptyset$. Then

$$\frac{\partial}{\partial x_0^i} = \sum_k \frac{\partial x^k}{\partial x_0^i} \frac{\partial}{\partial x^k},$$

so the relation between the local expressions of g with respect to (U, φ) and (U_0, φ_0) is given by

$$g_{i_0 j_0} = g\left(\frac{\partial}{\partial x_0^i}, \frac{\partial}{\partial x_0^j}\right) = \sum_{k,l} \frac{\partial x^k}{\partial x_0^i}\frac{\partial x^l}{\partial x_0^j} g_{kl}.$$

EXAMPLE 0.3. The canonical Euclidean metric is expressed in Cartesian coordinates by

$$g = dx^2 + dy^2.$$

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ yields that

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$

and

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

Thus, we have

$$g = dx^2 + dy^2$$

$$= \left(\cos^2\theta\, dr^2 + r^2\sin^2\theta\, d\theta^2 - 2r\sin\theta\cos\theta\, dr\, d\theta\right)$$
$$+ \left(\sin^2\theta\, dr^2 + r^2\cos^2\theta\, d\theta^2 + 2r\sin\theta\cos\theta\, dr\, d\theta\right)$$

$$= dr^2 + r^2 d\theta^2.$$

Let (M, g) and (M', g') be Riemannian manifolds. An isometry between (M, g) and (M', g') is a diffeomorphism $f : M \to M'$ whose differential is a linear isometry between the corresponding tangent spaces, namely,

$$g_p(u, v) = g'_{f(p)}(df_p(u), df_p(v)),$$

for every $p \in M$ and $u, v \in T_pM$. We say that (M, g) and (M', g') are isometric Riemannian manifolds if there exists an isometry between them. This completes the definition of the category of Riemannian manifolds and isometric maps.

Note that the set of all isometries of a Riemannian manifold (M, g) forms a group, called the isometry group of (M, g), with respect to the operation of composition of mappings, which we will denote by $\mathrm{Isom}(M, g)$.

Here we quote without proof the following important theorem:

THEOREM 0.4. *The isometry group of a Riemannian manifold has the structure of a Lie group with respect to the compact-open topology. Its isotropy subgroup at an arbitrary fixed point is compact.*

A local isometry from (M, g) into (M', g') is a smooth map $f : M \to M'$ satisfying the condition that every point $p \in M$ admits a neighborhood U such that the restriction of f to U is an isometry onto its image. In particular, f is a local diffeomorphism.

EXAMPLE 0.5 (The Euclidean Space). The Euclidean space is $\mathbb{R}^n$ equipped with its standard scalar product. The essential feature of $\mathbb{R}^n$ as a smooth manifold is that, since it is the model space for finite-dimensional smooth manifolds, it admits a global chart given by the identity map. Of course, the identity map establishes canonical isomorphisms of the tangent spaces of $\mathbb{R}^n$ at each of its points with $\mathbb{R}^n$ itself.

Therefore, an arbitrary Riemannian metric in $\mathbb{R}^n$ can be viewed as a smooth family of inner products in $\mathbb{R}^n$. In particular, by taking the constant family given by the standard scalar product, we get the canonical Riemannian structure in $\mathbb{R}^n$. In this book, unless explicitly stated, we will always use its canonical metric when referring to $\mathbb{R}^n$ as a Riemannian manifold.

If $(x^1, \dots, x^n)$ denote the standard coordinates on $\mathbb{R}^n$, then it is readily seen that the local expression of the canonical metric is

$$dx_1^2 + \cdots + dx_n^2. \tag{5}$$

More generally, if a Riemannian manifold (M, g) admits local coordinates such that the local expression of g is as in (1.3.1), then (M, g) is called flat and g is called a flat metric on M. Note that, if g is a flat metric on M, then the coordinates used to express g as in (5) immediately define a local isometry between (M, g) and Euclidean space $\mathbb{R}^n$.

CHAPTER 5

Exercises

EXERCISE 0.1.
Prove that the collection $\mathcal{A} = \{(\phi_\pm, U_\pm)\}$ given in Example (1.7) is an atlas on S^n.

EXERCISE 0.2.
Assume that $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open non-empty sets, and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous map. Denote by M the graph of f, i.e.

$$M = \{(x, y) |\ y = f(x)\}.$$

Let $U = (W \times V) \cap M$ and let $\phi(x, y) = x$ be the projection of U onto W. Show that the pair (ϕ, U) is a chart on M.

EXERCISE 0.3. Consider the cylinder $\mathcal{C} = \{(x, y, z) \in \mathbb{R}\ |x^2 + y^2 = 1\} \subset \mathbb{R}^3$ with the induced topology, and let $\varphi_1 : \mathbb{R}^2 \to \mathcal{C}$ and $\varphi_2 : \mathbb{R}^2 \to \mathcal{C}$ given by

$$\varphi_1(u, v) = (\frac{1 - u^2}{1 + u^2}, \frac{2u}{1 + u^2}, v).$$

$$\varphi_2(u, v) = (\frac{u^2 - 1}{1 + u^2}, \frac{-2u}{1 + u^2}, v).$$

Denote by V_i the image of φ_i (for $i = 1, 2$).

(1) Show that φ_1 is injective.
(2) Is φ_1 surjective? Justify your answer.
(3) Describe precisely V_1 the image of φ_1, and deduce that it is an open in $\mathcal{C}$.
(4) Deduce that $\varphi_1 : \mathbb{R}^2 \to V_1$ is bijective.
(5) Show that the map $\varphi_1^{-1} : V_1 \to \mathbb{R}^2$ given by

$$(x, y, z) \mapsto (\frac{y}{1 + x}, z)$$

is the inverse map of φ_1.

$$\left(\begin{array}{c}\text{Use the fact that } x^2 + y^2 = 1 \\ \text{which induces } \frac{y^2}{(1+x)^2} = \frac{1-x}{1+x}\end{array}\right)$$

(6) Show that $\varphi_1 : \mathbb{R}^2 \to V_1$ is a homeomorphism and deduce that (φ_1^{-1}, V_1) is a chart on $\mathcal{C}$.

(7) Prove that the collection $\mathcal{A} = \{(\varphi_1^{-1}, V_1), (\varphi_2^{-1}, V_2)\}$ is an atlas on $\mathcal{C}$ (we admit that (φ_2^{-1}, V_2) is a chart).
(8) Deduce that $\mathcal{C}$ is an abstract manifold. What is its dimension?

EXERCISE 0.4. Let $T^n = S^1 \times \cdots \times S^1$ be the real n-torus. Let $O = \{(x_1, \cdots, x_n) \in T^n | \forall i : x_i \neq (1, 0)\}$.
(1) Show that O is open in T^n with respect to the induced topology of $\mathbb{R}^{2n}$.
(2) Show that the map $(x_1, \cdots, x_n) \mapsto (\phi(x_1), \cdots, \phi(x_n))$ is a bijection : $O \to \mathbb{R}^n$, where $\phi(a, b) = \frac{b}{1-a}$
(3) Show that (ϕ, O) is a chart on T^n.

EXERCISE 0.5. Consider the projective space $\mathbb{P}^n$ given in Example (1.11). Show that the collection of pairs (ϕ_i, O_i), for $i = 1, \ldots, n+1$, is an atlas on $\mathbb{P}^n$.

EXERCISE 0.6. Let $\mathbb{P}^1$ be the real projective line, and S^1 the unit circle in $\mathbb{R}^2$. Consider the relation $\star$ on S^1 given by

$$v \star w \ \Leftrightarrow \ v = w \text{ or } v = -w.$$

(1) Determine the equivalence class of $v \in S^1$.
(2) For any $v = (x, y) \in \mathbb{R}^2 \setminus 0$, show that there are exactly two real numbers λ_1 and λ_2 such that $\lambda_1 v$ and $\lambda_2 v$ are in S^1 (λ_1 and λ_2 must be determined).
(3) Deduce a map $\varphi : \mathbb{P}^1 \to S^1/\star$ and show that it is bijective.

Consider now the circle S^1 as a subset of $\mathbb{C}$ via the natural identification $\mathbb{R}^2 \cong \mathbb{C}$. Let $\phi : S^1/\star \to S^1$ be the map given by $\phi([z]) = z^2$, where $[z]$ is the equivalence class of z modulo $\star$.
(4) Show that ϕ is well defined and bijective.
(5) Deduce that $\mathbb{P}^1$ is bijective to S^1.

EXERCISE 0.7. Consider the subset $\mathcal{T} = \{(x, y, z) \in \mathbb{R}^3 \,|(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\} \subset \mathbb{R}^3$ with the induced topology, and let $f : S^1 \times S^1 \to \mathbb{R}^3$ the map given by given by

$$f((a, b), (c, d)) = ((c + 2)a, (c + 2)b, d)$$

(1) Show that f is injective.
(2) Show that $\text{Im}(f) = \mathcal{T}$ and deduce that $f : S^1 \times S^1 \to \mathcal{T}$ is smooth diffeomorphism.
(3) Define an atlas on S^1.
(4) Deduce an atlas on $\mathcal{T}$.
(5) What is the dimension of $\mathcal{T}$.

EXERCISE 0.8. If $f \in C^\infty(M)$ is a constant function and D is a derivation at a point $p \in M$, then $D(f) = 0$.

EXERCISE 0.9. Let $M = S^2$ be the unit sphere. Determine the tangent space. Give an example of a vector field X and an integral curve for this X.

EXERCISE 0.10. Let $M = S^2$ be the unit sphere. Find the directional derivative at $p = (0, 1, 0)$ in the following cases : $X = (2, 1, 0)$, $Y = (1, -1, 2)$ and $Z = (0, 1, 1)$.

EXERCISE 0.11 (Bump function). Prove that there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with value in $[0, 1]$ such that $\varphi(x) = 1$ if $\|x\| \leqslant r$ and $\varphi(x) = 0$ if $\|x\| \geqslant s$, where $0 < r < s$.

EXERCISE 0.12. If two functions $f, g \in C^\infty(M)$ coincide on a neighborhood U of $p \in M$ and D is a derivation at p then $D(f) = D(g)$.

EXERCISE 0.13. Show that the universal embedding of an abstract manifold M in $D(M)$ is injective.

EXERCISE 0.14. Show that for any two vector fields X an Y , we have

$$\iota_X \circ \iota_Y = \iota_Y \circ \iota_X.$$

EXERCISE 0.15. Let $U \subset \mathbb{R}^3$ be a non empty open subset, and consider the vector field $X = x^2 y \frac{\partial}{\partial x}$ and the $2-$form $\omega = z\mathrm{d}x \wedge \mathrm{d}y + x\mathrm{d}y \wedge \mathrm{d}z$.

(1) Compute $\tilde{\omega} := \mathrm{d}\omega$.
(2) Compute $\iota_X \tilde{\omega}$.
(3) Is $\tilde{\omega}$ exact ? Is it closed? Justify your answer.

EXERCISE 0.16. Show that S^n is orientable.

EXERCISE 0.17. Let $M = \mathbb{P}^n$, and consider the line bundle T on $\mathbb{P}^n$ such that its fiber T_p over $p \in \mathbb{P}^n$ is the line l_p in $\mathbb{R}^{n+1}$ defined p, i.e.

$$T = \bigcup_{[l] \in \mathbb{P}^n} l_p.$$

Show that T is a line bundle (a vector bundle of rank 1) over $\mathbb{P}^n$.

1. Solutions

SOLUTION 1.1. Application exercise.

SOLUTION 1.2. It is enough to show that ϕ is a homeomorphism (its inverse is given by $\phi^{-1}(x) = (x, f(x))$).

SOLUTION 1.3.

(1) Let $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ s.t. $\varphi_1(u_1, v_1) = \varphi_1(u_2, v_2)$, then $\frac{1-u_1^2}{1+u_1^2} = \frac{1-u_2^2}{1+u_2^2}$, $\frac{2u_1}{1+u_1^2} = \frac{2u_2}{1+u_2^2}$ and $v_1 = v_2$. The first one implies $u_1^2 = u_2^2$, replacing this in the second one we get $u_1 = u_2$.

(2) No, because $(-1, 0, v)$ doesn't belong to its image, for any v.

(3) $V_1 = \{(x, y, z) | x \neq -1, x^2 + y^2 = 1\}$. So geometrically, it is the cylinder deprived of the line with equations $x = -1,\ y = 0$. Since this line is closed, V_1 is open.

(4) By equation (1), φ_1 is injective, and by definition of V_1, φ_1 is surjective on V_1. So it is bijective.

(5) We have for all $(u, v) \in \mathbb{R}^2$ and all $(x, y, z) \in V_1$:

$$\begin{aligned}\varphi_1^{-1} \circ \varphi_1(u, v) &= \varphi_1^{-1}(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}, v) \\ &= (\frac{\frac{2u}{1+u^2}}{1 + \frac{1-u^2}{1+u^2}}, v) = (u, v).\end{aligned}$$

and

$$\begin{aligned}\varphi_1 \circ \varphi_1^{-1}(x, y, z) &= \varphi_1(\frac{y}{1+x}, z) \\ &= (\frac{1 - (\frac{y}{1+x})^2}{1 + (\frac{y}{1+x})^2}, \frac{2(\frac{y}{1+x})}{1 + (\frac{y}{1+x})^2}, z) \\ &= (\frac{1 - (\frac{1-}{1+x})}{1 + (\frac{1-x}{1+x})}, \frac{2(\frac{y}{1+x})}{1 + (\frac{1-x}{1+x})}, z) \\ &= (x, y, z).\end{aligned}$$

(6) By the previous questions, φ_1 is bijective. Moreover, from the expressions of φ_1 and φ_1^{-1}, we see that both are continuous, since they are given by well defined rational polynomials. Since $\varphi_1^{-1} : V_1 \to \mathbb{R}^2$ and both V_1 and $\mathbb{R}^2$ are open, we deduce that (φ_1^{-1}, V_1) is a chart on $\mathcal{C}$.

(7) One can see that V_2 is the cylinder $\mathcal{C}$ without the line whose equations are $x = 1$ and $y = 0$. So clearly $V_1 \cap V_2 = \mathcal{C}$. Now the composition $\varphi_1^{-1} \circ \varphi_2$ is given on $\varphi_2^{-1}(V_1 \cap V_2) = \{(u, v) | u \neq 0\} \subset \mathbb{R}^2$ by

$$(u, v) \mapsto (\frac{\frac{-2u}{1+u^2}}{1 + \frac{u^2-1}{1+u^2}}, v) = (\frac{-1}{u}, v),$$

which is clearly smooth.

(8) We have an atlas $\mathcal{A}$ on $\mathcal{C}$, so we have a maximal atlas $\mathcal{A}$, and $\mathcal{C}$ is clearly Hausdorff, so $(\mathcal{C}, \mathcal{A})$ is an abstract manifold. Its is dimension is 2.

SOLUTION 1.4. (1) Let $N = (1, 0)$. Then $O = (\mathbb{R}^2 \setminus N)^n \cap T^n$. So it is open.
(2) It is sufficient to show that $\phi : S^2 \setminus N \to \mathbb{R}$ is bijective.
(3) From the formula of ϕ and ϕ^{-1}, we see that they are continuous.

SOLUTION 1.5. We have

$$\phi_i^{-1} : \mathbb{R}^n \to O_i,\ (x_1, \cdots, x_n) \mapsto [x_1 : \cdots : 1 : \cdots : x_n],$$

where 1 is in i^{th} position. Hence $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \setminus \{x_j \neq 0\} \to \mathbb{R}^n$

$$\phi_j \circ \phi_i^{-1}(x_1, \cdots, x_n) = (x_1/x_j, \ldots, 1/x_j, \cdots, x_n/x_j),$$

which is smooth.

SOLUTION 1.6.
(1) $\bar{v} = \{v, -v\}$
(2) $\lambda_1 = \dfrac{1}{\parallel v \parallel} = -\lambda_2$, clearly $\parallel \lambda_1 v \parallel = \parallel \lambda_2 v \parallel = 1$.
(3) given $[x : y] \in \mathbb{P}^1$, define $\varphi([x : y]) = \overline{\lambda_1 v}$, where $v = (x, y)$. Clearly this is well defined. In fact, geometrically, it sends a line l through the origin in $\mathbb{R}^2$ to the class given by the two points $l \cap S^1$, this is a bijective correspondence.
(4) Since $\bar{z} = \{z, -z\}$, we see that $\phi([z]) = z^2 = (-z)^2 = \phi([-z])$, so ϕ is well defined. Now, since $\mathbb{C}$ is algebraically closed, ϕ is surjective. If $\phi(w) = \phi(z)$, then $w^2 = z^2$, so $w = \pm z$, hence $[w] = [z]$, hence ϕ is injective, so it is bijective.
(5) The map $\phi \circ \varphi : \mathbb{P}^1 \to S^1$ is the required bijection.

SOLUTION 1.7.
(1) Easy computation.
(2) Easy.
(3) $\mathcal{A} = \{(U_+, \phi_+), (U_-, \phi_-)\}$, where $U_\pm = S^1 \setminus \{(\pm 1, 0)\}$, $\phi_\pm(x, y) = \dfrac{y}{1 \pm x}$.
(4) An atlas on $\mathcal{T}$ is given by $\{(U_\pm \times U_\pm, \phi_\pm \times \phi_\pm)\}$ which contains four charts.
(5) For example $\phi_+ \times \phi_+ : U_+ \times U_+ \to \mathbb{R}^2$, so $\dim \mathcal{T} = 2$.

SOLUTION 1.8. Because of linearity, it is enough to show that $D(\underline{1}) = 0$, where $\underline{1}$ is the constant 1 function on M. This comes from

$$D(\underline{1}) = D(\underline{1} \cdot \underline{1}) = D(\underline{1})\underline{1}(p) + \underline{1}(p)D(\underline{1}) = 2D(\underline{1}).$$

SOLUTION 1.9. The tangent space at $p = (a, b, c) \in S^2$ is the plan of $\mathbb{R}^3$ whose normal vector is (a, b, c). Thus T_pS^2 has equation $ax + by + cz = 0$. Let X be the map $S^2 \to \mathbb{R}^3$ given by $X(p) = (-b, a, 0)$. Clearly X is a vector field and an integral curve for X is given by $[0, 2\pi] \ni t \mapsto (\cos t, \sin t, 0) \in S^2$.

SOLUTION 1.10. The directional derivative D_X is defined by $D_X : C^\infty(S^2) \to \mathbb{R}$ as

$$D_X(f) = d_pf(X) = (\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p))X = 2\frac{\partial f}{\partial x}(p) + \frac{\partial f}{\partial y}(p).$$

SOLUTION 1.11. The function $\psi : \mathbb{R} \to \mathbb{R}$ which equals $\psi(x) = e^{-1/x}$ if $x > 0$ and 0 otherwise, is a smooth function. Define

$$\varphi(x) = \frac{\psi(s - \|x\|)}{\psi(s - \|x\|) + \psi(\|x\| - r)}$$

SOLUTION 1.12. We can define a smooth function h on M which is zero outside U and such that $h(p) = 1$. In this case $h(f - g)$ is the constant 0 function on M. Thus we have

$$0 = D(0) = D(h(f-g)) = D(h)(f(p)-g(p))+h(p)D(f-g) = D(f)-D(g).$$

SOLUTION 1.13. Use the bump function to show that the universal embedding of an abstract manifold M in $D(M)$ is injective.

SOLUTION 1.14. We have $\omega(X, Y, \cdots) = -\omega(Y, X, \cdots)$. this implies the result.

SOLUTION 1.15.

(1) $\tilde{\omega} = \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
(2) $\iota_X\tilde{\omega}(Y, Z) = \tilde{\omega}(X, Y, Z) = 2dx(X)dy(Y)dz(Z) - 2dx(X)dy(Z)dz(Y) = 2x^2y(dy(Y)dz(Z) - dy(Z)dz(Y))$, So $\iota_X\tilde{\omega} = 2x^2y\mathrm{d}y \wedge \mathrm{d}z$.
(3) By definition, $\tilde{\omega} = d\omega$, so it is exact. It is closed, since it is three form on three dimensional space.

SOLUTION 1.16. Consider the atlas with the two charts (U_+, ϕ_+) and (U_-, ϕ_-), given by stereographic projections. (see Example 1.7). Here

$$\phi_- : (U_+ \cap U_-) = \phi_+(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}.$$

The entries of the Jacobian matrix of $\phi_- \circ \phi_+^{-1}$ at x are given by

$$\frac{\partial}{\partial x_j}\frac{x_i}{\|x\|^2} = \frac{1}{\|x\|^2}\delta_{ij} - \frac{2x_ix_j}{\|x\|^4}. \tag{6}$$

Its determinant is $-\|x\|^{-2n}$. Indeed, one can see that this Jacobian matrix can be written in the form

$$\frac{1}{\|x\|^2}\left(I_n - \frac{1}{\|x\|^2}x^t\, x\right).$$

Since $xx^t = \|x\|^2$, one sees by multiplying by x from the right that x is a simple eigenvector for this matrix with eigenvalue $-1/\|x\|^2$, and if $y_1, \cdots, y_{n-1}$ are a basis for the orthogonal subspace $\{x\}^\perp$ of x, then by multiplying the Jacobian by y_i, we see that they are eigenvector with eigenvalue $1/\|x\|^2$. Since the determinant is the multiplication of the eigenvalues, the result follows.
Now the atlas $\{(U_+, \phi_+), (U_-, \tilde{\phi}_-)\}$, where

$$\tilde{\phi}_-(x_0, \cdots, x_n) = (\frac{-x_1}{1 \pm x_0}, \frac{x_2}{1 \pm x_0}, \cdots, \frac{x_n}{1 \pm x_0}),$$

has a determinant $\|x\|^{-2n} \geqslant 0$.

Solution 1.17.

Let (O_i, ϕ_i) be one of the charts given in Example 1.11, and let $p_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projection onto the i^{th} coordinate. Then for any $[l] \in O_i$, the restriction of p_i to l gives an isomorphism $l \to \mathbb{R}$. Now take $\tilde{\phi} : \tilde{O}_i := \pi^{-1}(O_i) \to \mathbb{R}^n \times \mathbb{R}$ given by $\tilde{\phi}(p, t) = (\phi(p), p_i(t))$ where $t \in l_p$. This gives a local trivialization.

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