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THÈSE

Pour lóbtention du diplôme de doctorat LMD

Filière : Mathématiques

Spécialité : Equations Différentielles et Applications

Etude de quelques classes d'équations différentielles fractionnaires avec impulsions

Présentée Par : IRGUEDI Aida
Date de discussion : 08/06/2023

Devant le jury composé de:

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Contents

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ملخص

في هذه الأطروحة، ندرس وجود و وحد انيةالحلول لمسائل القيمة الأبتدائية والحدية للمعادلات التفاضلية الكسرية المندفعة التي تتضمن المشتق الكسري لكابوتو- هادمارد .تستند نتائجنا الي مبدا تقلص لباناك، ونظرية النقطة الثابتة لشيفر، و البديل غير الخطي ليراي شودر و نظرية مونش جنبا الي جنب مع تقنية مقاييس عدم التوافق. أخيراً ، قدمنا أمثلة لتوضيح نتائجنا.

الكلمات المفتاحية : المعادلات التفاضلية الكسرية، المشتق الكسري لكابوتو-هادمارد، الشروط النبضية ، شروط متكاملة، نظرية القياس ، ووجود ووحدانية، نظريات النقطة الثابتة ، الفضاء باناخ

Abstract

In this thesis, we study the existence and uniqueness of solutions for initial and boundary value problems of impulsive fractional differential equations involving the Caputo-Hadamard fractional derivative.Our results are based on the Banach contraction principle, Schaefer fixed point theorem, nonlinear alternative of Leray-Schauder and the Mönch's fixed point theorem combined with the technique of measures of noncompactness. Finally, we presented examples to illustrate of our results.

Keywords: Fractional differential equations, Caputo-Hadamard fractional derivative , impulsive conditions, integral conditions, Measure of noncompactness, existence and uniqueness, fixed point theorems, Banach space.

Résumé

Dans cette thèse, on a étudié l'existence et l'unicité des solutions pour les problémes da valeur initiale et aux limites concernant les équations différentielles fractionnaires impulsives avec la dérivée Caputo-Hadamard. Nos révsultats ont basée sur le principe de contraction de Banach, le théoréme de point fixe de Schaefer, l'alternative non linéaire de t Leray-Schauder est le théorème à point fixe du Mönch combiné avec la technique de mesures de non-composidité. Nous avons fourni des exemples illustrative de nos résultats.

Mots clés: Equation différentiales fractionaires , dérivée fractionnaire de Hadamard-Caputo, impulsives, conditions intégrales, meusure de non compacité, existence and unicité, théorémes de point fixe, espace de Banach.

List of Symbols

We use the following notations this thesis.

Acronyms

FC: ***F***ractional **C**alculus.

FDE: ***F***ractional **D**ifferential **E**quation.

FD: ***F***ractional **D**erivative.

FI: ***F***ractional **I**ntegral.

IVP: **I**nitial **V**alue **P**roblem.

BVP: **B**oundary **V**alue **P**roblem.

MNC: **M**easure of **N**on-**C**ompactness.

Notation

N: **S**et of natural numbers.

N^*: **S**et of natural numbers with nonzero.

R: **S**et of real numbers.

R^+: **S**et of positive real numbers.

C: **S**et of complex numbers.

$\Re(\nu)$: **A**real part of a complex number ν.

δ: **D**irac Delta Function.

$\Gamma(\cdot)$: **G**amma Function.

$B(\cdot,\cdot)$: **B**eta Function.

$\in$: **B**elongs .

sup: **S**upremum.

max: **M**aximum.

convA: The closed convex hull of a set A.

χ : **B**anach **S**paces.

$\|,\|$: Norm in the spaces χ.

$C(J,\chi)$: **S**pace of all continuous functions in J.

$C^n(J,\chi)$: **S**pace of n time continuously. differentiable functions in J.

$AC(J,\chi)$: **S**pace of absolutely continuous functions in J.

$L^1(J,R)$: **S**pace of Lebesgue integrable functions in J.

$L^1(J,\chi)$: **S**pace of Bochner -integrable functions in J.

$L^p(J,\chi)$: **S**pace of measurable functions z
pace of measurable functions z, for any $1 < p < \infty$.

$L^\infty(J,R)$: **S**pace of functions z that are essentially bounded in J .

$I^\nu\Psi$: **R**iemann-**L**iouville fractional integral of order $\Re(\nu) > 0$.

${}_HI^\nu\Psi$: **H**adamard fractional integral of order $\Re(\nu) > 0$.

${}^{RL}D^\nu\Psi$: **R**iemann-**L**iouville fractional derivative of orde $\Re(\nu) > 0$.

${}^CD^\nu\Psi$: **C**aputo fractional derivative of orde $\Re(\nu) > 0$.

${}^HD^\nu\Psi$: **H**adamard fractional derivative of orde $\Re(\nu) > 0$.

${}^C_HD^\nu\Psi$: **C**aputo-**H**adamard fractional derivative of orde $\Re(\nu) > 0$.

Introduction

Fractional calculus is a branch of mathematics which examines how use the concepts of ordinary derivation and integration to an arbitrary real or even complex order. Its inception dates back to the end of the 17 th century [38], when Newton and Liebniz created the fundamentals of differential and integral calculus. In 1695 Gottfried Leibniz announced the key question can the meaning of derivatives with integer order be applied to derivatives with non-integer order ? On September 30, 1695, the Hospital questioned Leibniz in a letter to L'hopital what significance $\frac{d^{\nu}\Psi(\xi)}{d\xi^{\nu}}$ may have if n were the fraction $\frac{1}{2}$.

The notion of fractional Calculus provides a useful instrument in applied mathematics for studying a wide range of problems in mathematical modeling of systems, control theory, chemistry, viscoelasticity, rheology, physics, biology, see, for example [52]-[60].There has been significant progress in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives, as shown in ([26, 23, 24]), and [25], where the Caputo derivative is very useful in many problems because it satisfies its initial data. An other fractional derivative that appears beside Riemann Liouville and Caputo derivatives in the literature is the Hadamard fractional derivative, which was introduced in 1892 [31]. This derivative differs from the Riemann-Liouville derivative, but both approaches have drawbacks, one of which is that the derivative of a constant is not equal to zero, and it differs from the Caputo derivative in two ways: first, its kernel contains a logarithmic function of arbitrary exponent, and second, the Hadamard derivative of a constant does not equal zero. The reader can find detailed and properties of Hadamard fractional derivative and integral in [49, 50, 47].

The Caputo-Hadamard approach was proposed by Jarad in 2012 [20], and its formula is obtained by permuting the integration operator and that of the Hadamard derivation.The difference between the Caputo-Hadamard approach and the Hadamard approach is that in the first definition, the derivative of a constant is zero. The most significant advantage of the Caputo-Hadamard derivation is that it can be applied to systems with any initial conditions, as opposed to the Hadamard derivation, which requires the presence of the zero initial condition at point one.

Fractional differential equations appear in a variety of fields, including mathematics, bioengineering, chemical, physical, biological, and engineering, and have thus emerged as an important area of study in recent decades; see the monographs of Atangana [2], Kilbas et al. [3], Podlubny [29], and Zhou [61], as well as the references therein.

The theory of impulsive differential equations, on the other hand, has advanced rapidly over time and has played a significant role in modern applied mathematical models of real processes emerging in phenomena in physics, population dynamics, chemical technology, biotechnology, and economics. Some researchers have concentrated on impulsive differential equations with Hadamard and Caputo-Hadamard derivatives (see [58, 7, 11, 8, 48] and the references therein)

Differential equations with delays are a particular class of functional differential equations that are used in a wide range of biological and physical applications. These equations typically need us to deal with variable or state-dependent delays. Recently, the study of functional differential equations with state dependent delay has received a lot of attention due to its frequent and important applications in the mathematical modeling of a variety of several real of problems, have provided uniqueness and existence of solutions findings for different problems for finite and infinite delay applying some fixed point theorems see for instance the monographs by Baghli and Benchohra (alt [12, 44, 13]).

Fixed point theory gives the tools needed to prove existence theorems for a many range of nonlinear problems.Fixed point theorems are often dependent on certain properties (such as complete continuity, monotonous, contraction, ...) that the application in question must place the theory itself is a lovely blend of analysis, topology, and geometry.The theory of fixed points has been revealed as a highly strong and significant instrument in the study of modern mathematics over the last 80 years or so. Fixed point approaches, in particular, have been used in domains as diverse as economics, engineering, game theory, biology and chemistry etc. Fixed point theory is significant in functional analysis, approximation theory, differential equations, and applications such as boundary value problem. The well-known fixed point theorems are Banach's theorem, the nonlinear alternative of Leary-Schauder and Schaefer's theorem. Fixed point theory like, Monch's fixed point theorem combined with Kuratowski's noncompactness measure. In this approach was primarily introduced in Banas and Goebel's monograph [32], and it has ago been developed and applied in numerous articles. For examples, see Banas and Sadarangani[33],Guo et al. [16], Lakshmikantham and Leela [59], Monch [21], and Szufla [54]. The authors recently extended the measure of noncompactness to various classes of fractional differential equations in Banach spaces in [57, 15, 41, 42, 43].

This thesis, which discusses the existence of solutions for different classes of impulsive differential equations with fractional orders, are divided into an introduction and four chapters:

In Chapter 1, we give some notations and definitions of fractional calculus, measures of noncompactness, some fixed point theorems and the concept of solutions for impulsive differential equations in the fractional case. In the first section of this chapter we give some notations and essential properties of functional analysis. In the second section, main properties of measures of noncompactness. The third section, we recall some fixed point theorems. The fourth part is devoted to the concepts of fractional calculus and some fractional differential properties that are important to this thesis. In the last section we give a brief description of (IDE), then we present the two approaches considered in the literature concerning the concept of solutions for (IDE) in the fractional case.

In Chapter 2, we establish some existence results for two kinds of initial value problems for impulsive fractional differential equations of the Caputo-Hadamard type.
In the first part, we discuss the following problem:

$$\begin{cases} {}^{C}_{H}D^{\nu}z(\xi) = \Psi(\xi, z_{\xi}), \quad \xi \in J = [\mathtt{a}, \mathtt{T}], \xi \neq \xi_{\kappa}, \mathtt{a} > 0, \\ \Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \\ z(\xi) = \Phi(\xi), \quad \xi \in (\mathtt{a} - \tau, \mathtt{a}], \end{cases} \tag{$P1$}$$

where ${}^{C}_{H}D^{\nu}$ is the Caputo-Hadamard fractional derivative of order $0 < \nu \leq 1$, $\mathtt{a} > 0$,. $\kappa = 1, ..., w$.
We start by studying $(P1)$ when Ψ, I_{κ} and Φ are real valued functions, where we prove some existence results using Banach, Schaefer and Leary-Schauder's fixed point theorems. Then we treat $(P1)$ in a general Banach space by means of Monch's fixed point theorem and Kuratowski's measure of non-compactness.

In the second part of this chapter, we deal with the following impulsive fractional neutral problem:

$$\begin{cases} {}^{C}_{H}D^{\nu}\left[z(\xi) - g(\xi, z_{\xi})\right] = \Psi(\xi, z_{\xi}), \quad \xi \in [\mathtt{a}, \mathtt{T}], \xi \neq \xi_{\kappa}, \\ \Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}\left(z\left(\xi_{\kappa}^{-}\right)\right), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \\ z(\xi) = \Phi(\xi), \quad \xi \in (\mathtt{a} - \tau, \mathtt{a}], \end{cases} \tag{$P2$}$$

Our results are based on some classical fixed point theorems. All our theoretical results are illustrated by numerical examples.

In chapter 3, we look at the existence and uniqueness of solutions for a classes of functional impulsive fractional differential equations with Caputo-Hadamard derivative.Here results are discussed, the based on fixed point theorems of the Banach, Schaufer and Leary-Schauder.

In section 3.2, we are interested in the existence and uniqueness of solutions for the following initial value problem :

$$\begin{cases} {}^{C}_{H}D^{\nu}z(\xi) = \Psi(\xi, z_{\xi}), \quad \xi \in J = [\mathtt{a}, \mathtt{T}], \xi \neq \xi_{\kappa}, \\ \Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \\ \Delta z' \mid_{\xi=\xi_{\kappa}} = \overline{I}_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \\ z(\xi) = \Phi(\xi), \quad \xi \in (\mathtt{a} - \tau, \mathtt{a}], \ z'(\mathtt{a}) = \gamma, \end{cases} \tag{P3}$$

where ${}^{C}_{H}D^{\nu}$ is the Caputo-Hadamard fractional derivative of order $1 < \nu \leq 2$, $\mathtt{a} > 0$, $\Psi : J * C_{\tau} \to R$, $\gamma \in R$ are given continuous functions, $\Phi \in C([\mathtt{a} - \tau, \mathtt{a}], R)$ and $I_{\kappa}, \overline{I_{\kappa}} \in C(R, R)$, $\kappa = 1, 2, ..., w$, $\mathtt{a} = \xi_0 < \xi_1 < ... < \xi_w < \xi_{w+1} = \mathtt{T}$. For any continuous functions z defined on $J' = J \setminus \{\xi_1, ..., \xi_w\}$, $\Delta z|_{\xi=\xi_{\kappa}} = z(\xi_{\kappa}^{+}) - z(\xi_{\kappa}^{-})$ and $\Delta z'|_{\xi=\xi_{\kappa}} = z'(\xi_{\kappa}^{+}) - z'(\xi_{\kappa}^{-})$, $z(\xi_{\kappa}^{+})$, $z(\xi_{\kappa}^{-})$ represent the right and the left limits of $z(\xi)$ at $\xi = \xi_{\kappa}$.

in section 3.3, we present the existence and uniqueness of solutions to the boundary boundary value problem for impulsive fractional differential equations with integral conditions of the following form :

$$\begin{cases} {}^{C}_{H}D^{\nu}z(\xi) = \Psi(\xi, z_{\xi}), \quad \xi \in J = [\mathtt{a}, \mathtt{T}], \ \xi \neq \xi_{\kappa}, \\ \Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \\ \Delta z' \mid_{\xi=\xi_{\kappa}} = \overline{I}_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \\ z(\xi) = \Phi(\xi), \qquad \xi \in [\mathtt{a} - \tau, \mathtt{a}], \ \ z'(\mathtt{T}) = \int_{\mathtt{a}}^{\mathtt{T}} \hbar(s, z(s))ds, \end{cases} \tag{P4}$$

where Ψ, Φ and I_{κ}, $\overline{I_{\kappa}}$ are as in problem $(P3)$, and $\hbar : J * R \to R$ are given continuous functions.

Finally, we will conclude our thesis with a conclusion of our findings as well as a bibliography based on this work

Chapter 1

Preliminaries

In this chapter, we give some notations and definitions and som of the basic properties for measures of noncompactness, some fixed point theorems and we recall some properties of fractional calculus and refer the reader to [1, 27, 46, 29].

1.1 Some Functional Analysis Results

In this section, we present certain fundamental suitable function spaces which are very important for the study of impulsive differential equations [3].
Let $J = [\mathtt{a}, \mathtt{T}]$, $(0 < \mathtt{a} < \mathtt{T} < +\infty)$ be a bounded interval in R.
By $(\chi\ , \|.\|)$ the Banach space.

1.1.1 Space of Integral Functions

Denote by $L^1(J, R)$ the Banach space of functions z Lebesgues integrable with the norm

$$\|\Psi\|_{L^1} = \int_{\mathtt{a}}^{\mathtt{T}} |\Psi(\xi)|\, d\xi.$$

while $L^P(J, R)$ denote the space of Lebesgue integrable functions on J where $|z|^p$ belongs to $L^1(J, R)$, endowed with the norm

$$\|\Psi\|_{L^P} = \left(\int_{\mathtt{a}}^{\mathtt{T}} |\Psi(\xi)|^p\, d\xi\right)^{\frac{1}{p}}, \quad 1 < p < \infty$$

* In particular, if $p = 1$, then $L^1(J, R)$ is the space of equivalence classes of measurable integrable functions in J provided with the norm

$$\|\Psi\|_{L^1} = \int_{\mathtt{a}}^{\mathtt{T}} |\Psi(\xi)|\, d\xi.$$

* if $p = \infty$, let $L^{\infty}(J, R)$ be the space of measurable functions Ψ bounded almost every where (p.p) on Ω, with the norm

$$\|\Psi\|_{L^{\infty}} = \inf\{d > 0, |\Psi(\xi)| \leq d,\ \xi \in J\}$$

Remark 1.1. Let $\Psi \in L^{\infty}$, sach that

$$|\Psi(\xi)| \leq \|\Psi\|_{\infty}\ p.p.for\ all\, C$$

Definition 1.2. A function $\Psi : J \to \chi$ is called strongly measurable if there exists a sequence of simple functions (Ψ_n) such that

$$\lim_{n\to\infty} \|\Psi_n(\xi) - \Psi(\xi)\| = 0$$

for any sequence of simple functions (Ψ_n) then

$$\| \int_J \Psi(\xi) d\xi \| \leq \int_J \|\Psi(\xi)\| d\xi$$

1.1.2 Space of Continuous Functions

- Let $C^n(J;\chi)$ a space of functions $z : J \to \chi$ where z is n time continuously differentiable in J,

$$\|z\|_{C^n} := \sum_{\kappa=0}^{n} \|z^{\kappa}\|_C := \sum_{\kappa=0}^{n} \max_{\xi\in J} |z^{\kappa}(\xi)|\ , n \in N.$$

- In particular if $n = 0$, $C^0(J;\chi) = C(J;\chi)$ the space of all continuous functions Ψ with the norm :

$$\|z\|_C := \max_{\xi\in J} |z(\xi)|.$$

- For any continuous function z defined on $[\mathtt{a} - \tau, \mathtt{T}]$, $\xi \in J$, note by z_ξ element of $C_\tau := C([\mathtt{a} - \tau, \mathtt{a}], R)$, is defined as

$$z_\xi(s) = z(\xi + s)\ , s \in [\mathtt{a} - \tau, \mathtt{a}].$$

  Here $z_\xi(.)$ represents the history of the state from time $\xi - \tau$ up to ξ.

1.1.3 Spaces of Absolutely Continuous Functions

An absolutely continuous function is important in differential equation theory because, even if it is not differentiable at all points, it can be recovered by integration its derivative. It does, in fact, have this property, making it the weakest acceptable type of solution in a discontinuous (impulsive) differential equation.

A function $\Psi : J \to \chi$ is said to be absolutely continuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{\kappa=1}^{n}(d_\kappa - c_\kappa) < \delta \Rightarrow \sum_{\kappa=1}^{n}\|\Psi(d_\kappa) - \Psi(c_\kappa)\| < \varepsilon,$$

for every finite number of nonoverlapping intervals $(c_\kappa; d_\kappa)$, $\kappa = 1, ..., n$, with $[c_\kappa; d_\kappa] \subset J$.

A function $\Psi : J \to \chi$.

1- If the function Ψ is absolutely continuous on J then it is continuous.

2- If Ψ is absolutely continuous, then Ψ is almost everywhere differentiable.

3- Every absolutely continuous function is the inde? nite integral of its derivative.

4- If Ψ satisfies the Lipschitz condition, then it is absolutely continuous.

Definition 1.3. The space $AC(J;\chi)$ consisting of functions absolutely continuous. we denote by $AC(J,\chi)$ the space of primitives of Lebesgue summable functions :

$$\Psi(\xi) \in AC(J,\chi) \Leftrightarrow \Psi(\xi) = c + \int_{\mathtt{a}}^{\xi}\Phi(s)ds \ \ , (\Phi(\xi) \in L^1(J,\chi)).$$

Thus, and therefore an absolutely continuous function $\Psi(\xi)$ has a summable derivative $\Psi'(\xi) = \Phi(\xi)$ almost everywhere on J, and therefore $c = \Psi(\mathtt{a})$.

Definition 1.4. The space $AC^n(J,\chi)$ consists of all functions $\Psi(\xi)$ which have continuous derivatives up to order $(n-1)$ such that $\Psi^{(n-1)}(\xi) \in AC(J,\chi)$:

$$AC^n(J,\chi) = \{\Psi : J \to \chi \text{ and } \Psi^{(n-1)} \in AC(J,\chi), n \in N^*\},$$

and

$$AC^n_\delta(J,\chi) = \left\{z : J \to \chi; \delta^{n-1}z(\xi) \in AC(J,\chi)\right\},$$

where $\delta = \xi\frac{d}{d\xi}$ is the Hadamard derivative.

Let $AC^n(J,\chi)$ the space of all those and let only those functions $\Psi(\xi)$ which can be represented in the form

$$\Psi(\xi) = \int_{\mathtt{a}}^{\xi}\frac{(\xi-\mathtt{a})^{n-1}}{(n-1)!}\Phi(s)ds + \sum_{j=1}^{n}c_j(\xi-\mathtt{a})^j, \tag{1.1}$$

where $\Phi(\xi) \in L^1(J,\chi)$, $c_j \in R$, $j = 0,1,...,n-1$, it follows from (1.1) that

$$\Phi(\xi) = \Psi^n(\xi),\ c_j = \frac{\Psi^j(\mathtt{a})}{j!},\ \ j = 1,....,n$$

1.2 Measure of Non-Compactness

We recall here some notations, concepts and properties of measure of noncompactness MNC, for further details on the topic, we refer the reader to [9, 35, 16].

1.2.1 Some properties of measure of non-compactness

Now, let us recollect some basic information on the measure of noncompactness in sense Kuratowski.

Definition 1.5. [9] Let χ be a Banach space and Ω_χ the bounded subsets of χ. The Kuratowski measure of noncompactness is the map $\mu : \Omega_\chi \to R^+$ defined by

$$\mu(A) = \inf\{\varepsilon > 0 : A \subset \cup_{j=1}^n A_j,\ \ diam(A_j) \le \varepsilon\},\ \ here\ A \in \Omega_\chi,$$

where the diameter of A_j is defined by

$$diam(A_j) = \sup\{\|y - z\| : y, z \in A_j\},\ \ j = 1, ..., n.$$

The Kuratowski measure of noncompactness satisfies some properties (see [35]).

- √ $\mu(A) = 0 \Leftrightarrow \bar{A}$ is compact (A is relatively compact).
- √ $\mu(A) = \mu(\bar{A})$.
- √ $A_1 \subset A_2 \Rightarrow \mu(A_1) \le \mu(A_2)$.
- √ $\mu(A_1 + A_2) \le \mu(A_1) + \mu(A_2)$.
- √ $\mu(cA) = |c|\mu(A)$, $c \in R$.
- √ $\mu(convA) = \mu(A)$.
- √ $\mu(A + c) = \mu(A)$, for any $c \in E$.
- √ $\mu(A_1 \cup A_2) = \max\{\mu(A_1), \mu(A_2)\}$.

Theorem 1.6. *[9]*
Let χ be a Banach space and A a subset of χ ,and let μ be a non-compactness measure Kuratowski. Then

$$\mu(A) = \mu(\overline{A}) = \mu(convA).$$

Properties 1.7. *If $V \subset C(J,\chi)$ is bounded, and equicontinuous, then $convV \subset C(J,\chi)$ is also bounded, and equicontinuous.*

Definition 1.8. A function $\Psi : \mathbf{J} * \chi \to \chi$ has the Carathéodory property if :

(i) $\xi \to \Psi(\xi, z)$ is measurable for each $z \in R$.

(ii) $z \longrightarrow \Psi(\xi, z)$ is a continuous for each $\xi \in J$,

For any $V \subset C(J,\chi)$ we define

$$\int_J V(s)ds = \left\{ \int_J z(s)ds : z \in V \right\}, \text{for } \xi \in J,$$

where $V(s) = \{z(\xi) \in \chi \ : \ z \in V\}$.

Lemma 1.9. *[16] Let $V \subset C(J,\chi)$ is a bounded and equicontinuous set, then*

1. *the function $\xi \to \mu(V(\xi))$ is a continuous in J and*

$$\mu(V) = \sup_{a \leq \xi \leq T} \mu(V(\xi))$$

2.

$$\mu\left(\left\{ \int_J z(\xi)d\xi \ : \ z \in V \right\} \right) \leq \int_J \mu(V(\xi))d\xi. \tag{1.2}$$

1.2.2 Arzela-Ascoli theorem

Lemma 1.10. *[16] Let A be a subset $C(J,\chi)$, then A is relatively compact in $C(J,\chi)$ if and only if the following conditions hold:*

1. *The set A is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|t_1 - t_2| \leq \delta \Rightarrow \|\Psi(t_1) - \Psi(t_2)\| \leq \varepsilon, \ \ for\ every\ t_1, t_2 \in J\ and\ \Psi \in A.$$

2. *The set A is uniformly bounded if there exists $M > 0$ such that*

$$\|\Psi(\xi)\| < M \ \ for\ every\ \xi \in J \ \ and \ \ \Psi \in A.$$

1.3 Basic results from nonlinear functional analysis and Fixed points theory

In this section, we give several fixed point theorems that we will use to obtain varied results.

1.3.1 Banach fixed point theorem

Theorem 1.11. *[4] Let D be closed subset of a Banach space χ, and $F : D \to D$ be a strict contraction*

$$\|Fz_1 - Fz_2\| \leq K\|z_1 - z_2\|, \ \textit{for some } K \in (0,1), \ \forall z_1, z_2 \in D$$

.

Then, F has a unique fixed point.

1.3.2 Schaeferś fixed point theorem

Theorem 1.12. *[4] Let χ be a Banach space, and $F : \chi \to \chi$ be a completely continuous operator, if the set*

$$\varepsilon(F) = \{z \in \chi : \ z = \lambda F(z), \ \textit{for some } \lambda \in [0,1]\},$$

is bounded, then F has a fixed point in χ.

1.3.3 Leray-Schauder nonlinear alternative

Theorem 1.13. *[3] Let χ be a Banach space, and U a bounded open subset of χ with $0 \in U$. Then, every completely continuous map $F : \overline{U} \to \chi$ has at least one of the following two properties:*

(i) F has a fixed points in $\overline{U}$.

(ii) There exist $z \in \partial U$ and $\lambda \in [0,1]$ with $z = \lambda F(z)$.

1.3.4 Mönch fixed point theorem

Theorem 1.14. *[21]*

Let D be a bounded closed and convex subset of a Banach space χ such that $0 \in D$, and let F be a continuous mapping of D into itself. If the implication

$$V = \overline{canv} F(V) \ \ \textit{or} \ \ V = F(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every subset V of D, then F has a fixed point.

1.4 Concepts of Fractional Calculus

In this section, we begin by introducing some important concepts of fractional calculus: fractional integral and fractional derivatives in the sens of Riemann-Liouville, Caputo, Hadamard and Caputo- Hadamard, as well as their essential properties. For more details, we refer the reader to [27, 29].

1.4.1 Special Functions

Here, we give some information on two of the most important functions that have been found to be useful in providing solutions to the problems of fractional calculus. The basic function is the Gamma function. This function generalizes the factorial $n!$, and so allows n to take non-integer values, used in multiple differentiation and repeated integrations. The second function is the Beta function, and then we go through some of their properties. More details about these functions can be found in [56, 46, 53].

Definition 1.15. For any complex number ν such as $\Re(\nu) > 0$, the function Gamma $\Gamma(\nu)$ is defined as:

$$\Gamma(\nu) = \int_0^{+\infty} e^{-\xi}\xi^{\nu-1}d\xi, \ \ \nu > 0,$$

where $\xi^{\nu-1} = e^{(\nu-1)\log\xi}$.

Important properties of the Gamma function :

1.
$$\Gamma(\nu) = \frac{\Gamma(\nu+1)}{\nu}. \tag{1.3}$$

Proof: by integrating by parts

$$\begin{aligned}
\Gamma(\nu) &= \int_0^{+\infty} e^{-\xi}\xi^{\nu-1}d\xi \\
&= \left[\frac{e^{-\xi}\xi^{\nu}}{\nu}\right]_0^{+\infty} + \int_0^{+\infty} \frac{e^{-\xi}\xi^{\nu-1}}{\nu}d\xi \\
&= \frac{\Gamma(\nu+1)}{\nu}.
\end{aligned}$$

2. Let n an integer, we get closer and closer as

$$\Gamma(\nu+n) = \nu(\nu+1)\ldots(\nu+n-1)\Gamma(\nu)$$

Proof: we know that

$$\begin{aligned}
\Gamma(\nu) &= \frac{1}{\nu}\Gamma(\nu+1) \\
&= \frac{1}{\nu}\frac{1}{\nu+1}\Gamma(\nu+2) \\
& \quad \frac{1}{\nu}\frac{1}{\nu+1}\frac{1}{\nu+2}\Gamma(\nu+3)
\end{aligned}$$

By repeating the process n times, we get the following relation

$$\Gamma(\nu) = \frac{1}{\nu}\frac{1}{\nu+1}\frac{1}{\nu+2}\frac{1}{\nu+3}...\frac{1}{\nu+n-1}\Gamma(\nu+n).$$

3.

$$\Gamma(n) = (n-1)!, \quad n \geq 1.$$

Proof: Let $\Gamma(1) = 1$, and using the above property, we obtain values for $\nu = 1, 2, 3, ...$

$$\Gamma(2) = 1\Gamma(1) = 1!.$$

$$\Gamma(3) = 2\Gamma(2) = 2!.$$

$$\Gamma(4) = 3\Gamma(3) = 3!.$$

$$........$$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!.$$

4. Using the identity (1.3) we obtain :

$$\begin{aligned}
\Gamma(n+\frac{1}{2}) &= \Gamma\left(1+(n-\frac{1}{2})\right) = (n-\frac{1}{n})\Gamma(n-\frac{1}{2}) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)...\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}),
\end{aligned}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!}{2^n}$$

$$\Gamma(\frac{1}{2}) = \frac{2n!}{n!2^{2n}}\Gamma(\frac{1}{2})$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n-1)!}{2^n}, \; n \in N.$$

Some particular values of $\Gamma(\nu)$:

* $\Gamma(0) = +\infty$.

* $\Gamma(1) = \Gamma(2) = \int_0^{+\infty} e^{-\xi}\xi^{1-1}du = 1$.

* $\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} e^{-\xi^2}d\xi = \sqrt{\pi}$ (Gaussian integral).

* $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$.

Definition 1.16. [10] The Beta function, be defined as

$$\beta(\nu_1, \nu_2) = \int_0^1 (1-\xi)^{\nu_1-1}\xi^{\nu_2-1}d\xi, \quad \Re(\nu_1) > 0, \Re(\nu_2) > 0.$$

Relation between the Gamma and the Beta function.

[10] The basic properties the following

$$\mathrm{B}(\nu_1, \nu_2) = \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_2)} = \mathrm{B}(\nu_1, \nu_2), \quad for \;\; each \; (\nu, \nu_1) \in R_+^*.$$

1.4.2 Fractional Integral and Derivative of Riemann-liouville

In this part, we give defined the Riemann-Liouville. As we know his approach is inspired by the Cauchy formula which gives the successive primitives (n times) the repeated integral of continuous function which is given by the equation :

$$(I_{\mathsf{a}}^1\Psi)(\xi) = \int_{\mathsf{a}}^{\xi} \Psi(s)ds$$

$$(I_{\mathsf{a}}^2\Psi)(\xi) = \int_{\mathsf{a}}^{\xi} \left(\int_{\mathsf{a}}^{s} \Psi(\xi)d\xi\right) ds$$

$$(I_{\mathsf{a}}^2\Psi)(\xi) = \int_{\mathsf{a}}^{\xi} (\xi - s)\Psi(s)ds$$

$$(I_{\mathsf{a}}^n\Psi)(\xi) = \int_{\mathsf{a}}^{\xi} \frac{(\xi-s)^{n-1}}{(n-1)!}\Psi(s)ds\,, \;\; n \in N^*. \tag{1.4}$$

Fractional integral Riemann-liouville

Definition 1.17. The Riemann-Liouville fractional integral of order $\Re(\nu) > 0$ of all function $\Psi : J \to R$, be defined as

$$(I_a^\nu f)(\xi) = \frac{1}{\Gamma(\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu-1} \Psi(s) ds.$$

Example 1.18. The Riemann-Liouville fractional integral of function $\Psi(\xi) = (\xi - \mathtt{a})^{\nu_1}$,$\nu_1 \in R$, then we have

$$(I_a^\nu f)(\xi) = \frac{1}{\Gamma(\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu-1} (s - \mathtt{a})^{\nu_1} ds, \;\; \xi > \mathtt{a}.$$

by setting $s = \mathtt{a} + z(\xi - \mathtt{a})$ we find

$$\begin{array}{rcl} \frac{1}{\Gamma(\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu-1} (s - \mathtt{a})^{\nu_1} ds & = & \frac{1}{\Gamma(\nu)} \int_0^1 (\xi(1-z) - \mathtt{a}(1-\xi))^{\nu-1} z(\xi - \mathtt{a})^{\nu_1} (\xi - \mathtt{a}) dy \\ & = & \frac{1}{\Gamma(\nu)} \int_0^1 ((1-z)(\xi - \mathtt{a}))^{\nu-1} z^{\nu_1} (\xi - \mathtt{a})^{\nu_1} (\xi - \mathtt{a}) dy \\ & = & \frac{(\xi-\mathtt{a})^{\nu+\nu_1}}{\Gamma(\nu)} \int_0^1 (1-z)^{\nu-1} z^{\nu_1} dy \end{array}.$$

Since

$$\mathrm{B}(\nu_1, \nu_2 + 1) = \frac{\Gamma(\nu_1)\Gamma(\nu_2 + 1)}{\Gamma(\nu_1 + \nu_2 + 1)},$$

we find

$$\frac{1}{\Gamma(\nu_1)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu_1 - 1} (s - \mathtt{a})^{\nu_2} ds = \frac{\Gamma(\nu_2 + 1)(\xi - \mathtt{a})^{\nu_1 + \nu_2}}{\Gamma(\nu_1 + \nu_2 + 1)}.$$

Example 1.19. Let $\Psi(\xi) = e^{\lambda\xi}$ where $\lambda > 0$, then we have

$$\begin{array}{rcl} I_{\mathtt{a}}^{\nu} e^{\lambda\xi} & = & \frac{1}{\Gamma(\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{(\nu-1)} \exp(\lambda s) ds \\ & = & \frac{1}{\Gamma(\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{(\nu-1)} \sum_{\kappa \geq 0} \frac{(\lambda s)^\kappa}{\kappa!} ds \\ & = & \sum_{\kappa \geq 0} \frac{\lambda^\kappa}{\kappa! \Gamma(\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu-1} s^\kappa ds \end{array}.$$

We make the change of variable:

$$s = tz, \;\; ds = tdz$$

$$I_{\mathtt{a}}^{\nu} e^{\lambda\xi} = \sum_{\kappa \geq 0} \frac{\lambda^\kappa}{\kappa! \Gamma(\nu)} \int_0^1 (\xi - tz)^{\nu-1} (tz)^\kappa ds$$

$$= \sum_{\kappa \geq 0} \frac{\lambda^\kappa \Gamma(\kappa + 1)}{\Gamma(\nu + \kappa + 1)} \xi^{\nu+\kappa}.$$

Theorem 1.20. *Let $\Psi \in C(J, R)$. For all $\nu_1 > 0$ and $\nu_2 > 0$, we have*

$$(I^{\nu_1} I^{\nu_2} \Psi)(\xi) = (I^{\nu_1+\nu_2}\Psi)(\xi).$$

Proof: By defined integral Riemann-liouville, we have

$$(I^{\nu_1}(I^{\nu_2}\Psi)(\xi) = \frac{1}{\Gamma(\nu_1)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu_1 - 1} (I^{\nu_2})\Psi(s)ds$$

$$= \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{\nu_1-1} ds \int_{\mathtt{a}}^{s} (s - x)^{1-\nu_2}\Psi(x)dx,$$

if we apply the Dirichlet equality, we find

$$I^{\nu_1}(I^{\nu_2}\Psi)(\xi) \quad = \quad \tfrac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\mathtt{a}}^{\xi} \int_{\mathtt{a}}^{s} (\xi - s)^{\nu_1-1}(s - x)^{\nu_2-1}\Psi(x)dsdx$$

$$s = x + z(\xi - x)$$

$$ds = (\xi - x)dx, \quad \xi - s = (1 - z)(\xi - x)$$

Hence

$$(I^{\nu_1} I^{\nu_2}\Psi)(\xi) \quad = \quad \tfrac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{\mathtt{a}}^{\xi} (\xi - x)^{\nu_1+\nu_2-1}\Psi(x) \int_0^1 (1 - z)^{\nu_1-1} z^{\nu_2-1} dxdz$$

$$\int_0^1 (1 - z)^{\nu_1-1} z^{\nu_2-1} dz = \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1 + \nu_2)}$$

$$I^{\nu_1} I^{\nu_2}\Psi(\xi) = \frac{1}{\Gamma(\nu_1 + \nu_2)} \int_{\mathtt{a}}^{\xi} (\xi - x)^{\nu_1+\nu_2-1}\Psi(x)dx = (I^{\nu_1+\nu_2}\Psi)(\xi).$$

Fractional derivative Riemann-liouville

Definition 1.21. The Riemann-Liouville fractional derivative of order $\Re(\nu) > 0$ of function Ψ, be defined as

$$({}^{RL}D^{\nu}\Psi)(\xi) = \left(\frac{d}{d\xi}\right)^n (I^{n-\nu}\Psi)(\xi)$$

$$= \frac{1}{\Gamma(n - \nu)} \left(\frac{d}{d\xi}\right)^n \int_{\mathtt{a}}^{\xi} (\xi - s)^{n-\nu-1}\Psi(s)ds, \;\; \xi > \mathtt{a}, \;\; n = [\Re(\nu)] + 1.$$

In particular, where $n \in N^*$, then

$$({}^{RL}D^{0}\Psi)(\xi) = I^0\Psi(\xi), \;\; (D^n\Psi)(\xi) = I^n\Psi(\xi).$$

If $0 < \Re(\nu) < 1$, then

$$(D^n\Psi)(\xi) = \frac{1}{\Gamma(1 - \nu)} \frac{d}{d\xi} \int_{\mathtt{a}}^{\xi} (\xi - s)\Psi(s)ds, \;\; \xi > \mathtt{a}.$$

Remark 1.22. Fractional Riemann-Liouville derivative of constant is qeneral non zero, we have

$$^{RL}D_{\mathtt{a}}^{\nu}c = \frac{c}{\Gamma(1-\nu)}(\xi - \mathtt{a})^{-\nu}.$$

Theorem 1.23. *If $\Re(\nu) > 0$ and Ψ be a summable function, then the following equalities*

$$^{RL}D_{\mathtt{a}}^{\nu}(I_{\mathtt{a}}^{\nu}\Psi)(\xi) = \Psi(\xi).$$

Proof: Using the Riemann-Liouville fractional inteqral and derivative, we find

$$\begin{aligned} D_{\mathtt{a}}^{\nu}(I_{\mathtt{a}}^{\nu}\Psi)(\xi) &= \tfrac{1}{\Gamma(\nu)\Gamma(n-\nu)}\left(\tfrac{d}{d\xi}\right)^{n}\int_{\mathtt{a}}^{\xi}(\xi-s)^{n-\nu-1}ds\int_{\mathtt{a}}^{s}(s-x)^{\nu-1}\Psi(x)dx \\ &= \tfrac{1}{\Gamma(\nu)\Gamma(n-\nu)}\left(\tfrac{d}{d\xi}\right)^{n}\int_{\mathtt{a}}^{\xi}\int_{\mathtt{a}}^{s}(\xi-s)^{n-\nu-1}\Psi(x)(s-x)^{\nu-1}dsdx \\ &= \tfrac{1}{\Gamma(\nu)\Gamma(n-\nu)}\left(\tfrac{d}{d\xi}\right)^{n}\int_{\mathtt{a}}^{\xi}\left(\Psi(x)\int_{\mathtt{a}}^{s}(\xi-s)^{n-\nu-1}(s-x)^{\nu-1}ds\right)dx \end{aligned}$$

by setting $s = x + z(\xi - x)$, we find

$$\begin{aligned} \int_{\mathtt{a}}^{s}(\xi-s)^{n-\nu-1}(s-x)^{\nu-1}dx &= \int_{0}^{1}(\xi-x-z(\xi-x))^{n-\nu-1}(x+z(\xi-x)-x)^{\nu-1}(\xi-x)dz \\ &= (\xi-x)^{n-1}\int_{0}^{1}(1-z)^{n-\nu-1}z^{\nu-1}dz \\ &= (\xi-x)^{n-1}\mathrm{B}(n-\nu,\nu). \end{aligned}$$

Hence

$$\begin{aligned} D_{\mathtt{a}}^{\nu}(I_{\mathtt{a}}^{\nu}\Psi(\xi)) &= \tfrac{\mathrm{B}(n-\nu,\nu)}{\Gamma(\nu)\Gamma(n-\nu)}\left(\tfrac{d}{d\xi}\right)^{n}\int_{\mathtt{a}}^{\xi}\Psi(x)(\xi-x)^{n-1}dx \\ &= \tfrac{1}{\Gamma(n)}\left(\tfrac{d}{d\xi}\right)^{n}\int_{\mathtt{a}}^{\xi}\Psi(x)(\xi-x)^{n-1}dx. \end{aligned}$$

Now, by using the equation (1.4), we find that

$$D_{\mathtt{a}}^{\nu}(I_{\mathtt{a}}^{\nu}\Psi(\xi)) = \Psi(\xi).$$

Let $\Re(\nu) > 0$, then equation

$$(^{RL}D_{\mathtt{a}_+}^{\nu}z)(\xi) = 0 \Longleftrightarrow z(\xi) = \sum_{j=0}^{n-1}c_j(\xi-\mathtt{a})^{\nu-j},\ n = [\Re(\nu)] + 1,$$

where $c_j \in R, j = 0, ..., n-1$ are arbitrary constants.
In particular,if $0 < \Re(\nu) \leq 1$, we have

$$(^{RL}D_{\mathtt{a}}^{\nu}z)(\xi) = 0 \Longleftrightarrow z(\xi) = c(\xi-\mathtt{a})^{\nu-1},\ \ c \in R.$$

1.4.3 Fractional Derivative of Caputo

The Caputo fractional derivative is one of the most used definitions of a fractional derivative along with the Riemann-Liouville. Whereas the Riemann-Liouville definition of a fractional derivative is usually employed in mathematical texts and not so frequently in applications,the Caputo approach appears often while modeling applied problems by means of fractional derivatives and fractional order differential equations. In this subsection we give the definition of the fractional derivative in the sense of Caputo as well as some essential properties.

Definition 1.24. The fractional derivative of Caputo of order $\Re(\nu) > 0$ of function Ψ defined on J is given by

$$({}^{C}D_{\mathtt{a}}^{\nu}\Psi)(\xi) = \left({}^{C}D_{\mathtt{a}}^{\nu}\left[\Psi(\xi) - \sum_{\kappa=1}^{n}\frac{\Psi^{\kappa}(\mathtt{a})}{\kappa!}(\xi - \mathtt{a})^{\kappa}\right]\right)(\xi),$$

where $n = [\Re(\nu)] + 1$ for each $\nu \notin N_0$,$n = \nu$ for each $\nu \in N_0$.

In particular, where $0 < \Re(\nu) < 1$, we have

$$({}^{C}D_{\mathtt{a}}^{\nu}\Psi)(\xi) = \left(D_{\mathtt{a}}^{\nu}\left[\Psi(\xi) - \Psi(\mathtt{a})\right]\right)(\xi).$$

Theorem 1.25. *[29] Let $\Re(\nu) > 0$, $n = [\Re(\nu)] + 1$, the fractional derivative of Caputo of order ν of a function $\Psi \in AC^{n}(J)$.*

(1) If $\nu \notin N_0$

$$({}^{C}D^{\nu}\Psi)(\xi) = \frac{1}{\Gamma(n-\nu)}\int_{\mathtt{a}}^{\xi}\frac{\Psi^{n}(s)}{(\xi - s)^{1-n+\nu}}ds = (I^{n-\nu}D^{n}z)(\xi),$$

where $D = \frac{d}{d\xi}$.

In particulas, when $0 < \Re(\nu) < 1$ and $\Psi(\xi) \in AC(J)$,

$$({}^{C}D^{\nu}\Psi)(\xi) = \frac{1}{\Gamma(1-\nu)}\int_{\mathtt{a}}^{\xi}\frac{\Psi'(s)}{(\xi - s)^{\nu}}ds = (I^{1-\nu}Dy)(\xi).$$

(2) If $\nu = n \in N_0$

$$({}^{C}D^{\nu}\Psi)(\xi) = \Psi^{n}(\xi).$$

In particular,

$$({}^{C}D^{0}\Psi)(\xi) = \Psi(\xi).$$

Example 1.26. Let $\Psi(\xi) = (\xi - \mathtt{a})^{\nu_1}$ where $\xi > \mathtt{a}$ and $\nu_1 > -1$, then we have

$$({}^{C}D^{\nu}_{\mathtt{a}}\Psi)(\xi) = \frac{1}{\Gamma(n-\nu)} \int_{\mathtt{a}}^{\xi} (\xi - s)^{n-\nu-1} \frac{d^n}{ds^n}(s-\mathtt{a})^{\nu_1} ds.$$

Since

$$\frac{d^n}{ds^n}(s-\mathtt{a})^{\nu_1} = \frac{\Gamma(\nu_1+1)}{\Gamma(\nu_1-n+1)} \left((\xi-\mathtt{a})^{\nu_1-n}\right),$$

then

$$\begin{aligned}
{}^{C}D^{\nu}(\xi-\mathtt{a})^{\nu_1} &= \frac{\Gamma(\nu_1+1)}{\Gamma(n-\nu)\Gamma(\nu_1-n+1)} \int_{\mathtt{a}}^{\xi} (\xi-s)^{n-\nu-1}(s-\mathtt{a})^{\nu_1-n} ds \\
&= \frac{\Gamma(\nu_1+1)}{\Gamma(n-\nu)\Gamma(\nu_1-n+1)} \int_{0}^{1} (1-z)^{n-\nu-1} z^{\nu_1-n} dy \\
&= \frac{\Gamma(\nu_1+1)(\xi-\mathtt{a})^{\nu_1-\nu}}{\Gamma(n-\nu)\Gamma(\nu_1-n+1)} \mathrm{B}(n-\nu, \nu_1-n+1) \\
&= \frac{\Gamma(\nu_1+1)\Gamma(n-\nu)\Gamma(\nu_1-n+1}{\Gamma(\nu_1-\nu+1)\Gamma(n-\nu+\nu_1-n+1}(\xi-\mathtt{a})^{\nu_1-\nu}
\end{aligned}$$

we find

$${}^{C}D^{\nu}_{\mathtt{a}}(\xi-\mathtt{a})^{\nu_1} = \frac{\Gamma(\nu_1+1)(\xi-\mathtt{a})^{\nu_1-\nu}}{\Gamma(\nu_1-\nu+1)}.$$

Remark 1.27. If d is a constant, that the derivative of a function is zero

$${}^{C}D^{\nu}_{\mathtt{a}}d = 0.$$

Relation each by the fractional derivative of Caputo and that of Riemann-Liouville.

√ Let $\Re(\nu) \geq 0$, $n = [\Re(\nu)] + 1$, for $\nu \notin N_0$,

$$({}^{C}D^{\nu}_{\mathtt{a}}\Psi)(\xi) = ({}^{RL}D^{\nu}_{\mathtt{a}}\Psi)(\xi) - \sum_{\kappa=1}^{n} \frac{\Psi^{\kappa}(\mathtt{a})}{\Gamma(1+\kappa-\nu)}(\xi-\mathtt{a})^{\kappa-\nu}.$$

In particular, when $0 < \Re(\nu) < 1$, we have

$$({}^{C}D^{\nu}_{\mathtt{a}}\Psi)(\xi) = (D^{\nu}_{\mathtt{a}})\Psi(\xi) - \frac{1}{\Gamma(1-\nu)}\Psi(\mathtt{a})(\xi-\mathtt{a})^{-\nu}.$$

√ Let $\Re(\nu) > 0$, $n = [\Re(\nu)] + 1$, for $\nu \notin N_0$, $n = \nu$ for $\nu \in N_0$, if $\Psi(\xi) \in C^n(J)$ or $\Psi(\xi) \in AC^n(J)$, then

$$I^{\nu C}D^{\nu}\Psi(\xi) = \Psi(\xi) - \sum_{\kappa=1}^{n} \frac{\Psi^n(\mathtt{a})}{\kappa!}(\xi - \mathtt{a})^{\kappa}.$$

In particular if $0 < \Re(\nu) < 1$, then

$$I^{\nu C}D^{\nu}\Psi(\xi) = \Psi(\xi) - \Psi(\mathtt{a}).$$

Lemma 1.28. *[27] Let $\Re(\nu) > 0$, $n = [\Re(\nu)] + 1$, then the differential equation :*

$${}^{C}D^{\nu}\Psi(\xi) = 0,$$

has a solution

$$\Psi(\xi) = \sum_{j=0}^{n-1} c_j(\xi - a)^j,$$

where $c_j \in R, j = 0, ..., n-1$.

Lemma 1.29. *[27] Let $\Re(\nu) > 0$, $n = [\Re(\nu)] + 1$, then*

$$I^{\nu}\ {}^{C}D^{\nu}\Psi(\xi) = \Psi(\xi) + c_0 + c_1\xi + c_2\xi^2 + ... + c_{n-1}\xi^{n-1}$$

.

1.4.4 Fractional Integral and Derivative of Hadamard

This part, we present the definitions, some properties the inteqrals and fractional derivatives Hadamard-type.

Fractional Integral Hadamard

The Hadamard fractional integral was based by generalizing the following inteqral:

$$I_{\mathtt{a}}^{n}\Psi(\xi) = \int_{\mathtt{a}}^{\xi} \frac{d\xi_1}{\xi_1} \int_{\mathtt{a}}^{\xi_1} \frac{d\xi_2}{\xi_2} \int_{\mathtt{a}}^{\xi_2} \frac{d\xi_3}{\xi_3} ... \int_{\mathtt{a}}^{\xi_{n-1}} \Psi(\xi_n)\frac{d\xi_n}{\xi_n}.$$

Definition 1.30. [10] The Hadamard fractional integral of order $\Re(\nu) > 0$ of function Ψ be defined as :

$$({}_{H}I^{\nu}\Psi)(\xi) = \frac{1}{\Gamma(\nu)} \int_{\mathtt{a}}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} \Psi(s)\frac{ds}{s}.$$

Remark 1.31. From the fractional integral of Hadamard, we can see the following relations:

$\surd$ $({}^H I^\nu \Psi)(e^\xi) = 1\Gamma(\nu)_1^\xi(\xi - s)^{\nu-1}\Psi(s)ds = (I_1^r\Psi)(\xi)$.
$\surd$ $\xi\frac{d}{d\xi}({}^H I^{\nu+1}\Psi)(\xi) = ({}^H I^\nu \Psi)(\xi)$.

Example : we consider the function Ψ defined by: $\Psi(\xi) = (\log\frac{\xi}{\mathtt{a}})^{\nu_1}$, we have

$$({}_H I_{\mathtt{a}}^\nu \Psi)(\xi) = {}_H I_{\mathtt{a}}^\nu (\log\frac{\xi}{\mathtt{a}})^{\nu_1} = \frac{1}{\Gamma(\nu)}\int_{\mathtt{a}}^\xi (\log\frac{\xi}{s})^{\nu-1}(\log\frac{s}{\mathtt{a}})^{\nu_1}\frac{ds}{s},$$

using the change of variable, we have $z\left(\log\frac{\xi}{\mathtt{a}}\right) = \log\frac{s}{\mathtt{a}}$,

$$\begin{aligned} {}_H I_{\mathtt{a}}^\nu (\log\frac{\xi}{\mathtt{a}})^{\nu_1} &= \frac{(\log\frac{\xi}{\mathtt{a}})^{\nu+\nu_1}}{\Gamma(\nu)}\int_0^1 z^{\nu_1}(1-z)^{\nu-1}dz \\ &= \frac{\Gamma(\nu_1+1)}{\Gamma(\nu+\nu_1+1)}(\log\frac{\xi}{\mathtt{a}})^{\nu+\nu_1}. \end{aligned}$$

For $\nu = 1/2$ et $\nu_1 = 3/2$,the relation becomes

$$ {}_H I_{\mathtt{a}}^{\frac{1}{2}}(\log\frac{\xi}{\mathtt{a}})^{\frac{3}{2}} = \frac{3\sqrt{\pi}}{4}(\log\frac{\xi}{\mathtt{a}})^2.$$

Let $\Psi_1, \Psi_2 \in C(J, R)$, for $\nu \geq 0, \nu_1 \geq 0$ and $\alpha, \beta \in R$, we have

$${}_H I^\nu({}_H I^{\nu_1}\Psi_1)(\xi) = {}_H I^{\nu_1}({}_H I^\nu \Psi_1)(\xi) = ({}_H I^{\nu+\nu_1}\Psi_1)(\xi)$$

Proof: Let $\nu > 0, \nu_1 > 0$ and $\Psi_1 \in C(J, R)$, the proof is obtained by direct calculation using the Beta function. Indeed :

$$\begin{aligned} {}_H I^\nu({}_H I^{\nu_1}\Psi_1)(\xi) &= \frac{1}{\Gamma(\nu)}\int_{\mathtt{a}}^\xi (\xi s)^{\nu-1}(I^{\nu_1}\Psi_1)(s)\frac{ds}{s} \\ &= \frac{1}{\Gamma(\nu)\Gamma(\nu_1)}\int_{\mathtt{a}}^\xi\int_{\mathtt{a}}^s (\xi s)^{\nu-1}(s\mu)^{\nu_1-1}\Psi_1(\mu)\frac{d\mu}{\mu}\frac{ds}{s} \\ &= \frac{1}{\Gamma(\nu)\Gamma(\nu_1)}\int_{\mathtt{a}}^\xi \Psi_1(\mu)\int_\mu^\xi (\xi s)^{\nu-1}(s\mu)^{\nu_1-1}\frac{ds}{s}\frac{d\mu}{\mu}. \end{aligned}$$

$$= \frac{1}{\Gamma(\nu)\Gamma(\nu_1)}\int_a^\xi \Psi_1(\mu)\int_\mu^\xi (\xi s)^{\nu-1}(s\mu)^{\nu_1-1}\frac{ds}{s}\frac{d\mu}{\mu}. \tag{1.5}$$

By asking $x = \frac{s\mu}{\xi\mu}$, we obtain

$$\begin{aligned}\int_{\mu}^{\xi}(\xi s)^{\nu-1}(s\mu)^{\nu_1-1}\frac{ds}{s} &= (\log\frac{\xi}{\mu})^{\nu+\nu_1-1}\int_0^1(1-x)^{\nu-1}x^{\nu_1-1}dx\\ &= (\xi\mu)^{\nu+\nu_1-1}\beta(\nu,\nu_1)\\ &= (\xi\mu)^{\nu+\nu_1-1}\frac{\Gamma(\nu)\Gamma(\nu_1)}{\Gamma(\nu+\nu_1)}\end{aligned}$$

Be replacing this formula in (1.5), we have :

$$_HI^{\nu}(_HI^{\nu_1}\Psi_1)(\xi)=\frac{1}{\Gamma(\nu+\nu_1)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{\mu})^{\nu+\nu_1-1}\Psi_1(\mu)\frac{d\mu}{\mu}=I^{\nu+\nu_1}\Psi_1(\xi),$$

Now to show $_HI^{\nu}_HI^{\nu_1}\Psi_1(\xi)=_H I^{\nu_1}_HI^{\nu}\Psi_1(\xi)$, using the previons property.
Therefore

$$_HI^{\nu}(_HI^{\nu_1}\Psi_1)(\xi)=_H I^{\nu+\nu_1}\Psi_1(\xi)=_H I^{\nu_1+\nu}\Psi_1(\xi)=_H I^{\nu_1}(_HI^{\nu}\Psi_1)(\xi).$$

Fractional Derivative of Hadamard

Definition 1.32. [10] The Hadamard fractional derivative of order $\Re(\nu)>0$ of function $\Psi(\xi):J\to R$ is defined as:

$$\begin{aligned}(^HD^{\nu}\Psi)(\xi) &= \frac{1}{\Gamma(n-\nu)}\left(\xi\frac{d}{d\xi}\right)^n\int_{\mathtt{a}}^{\xi}\left(\log\frac{\xi}{s}\right)^{(n-\nu-1)}\frac{\Psi(s)}{s}ds\\ &= \delta^n(_HI^{n-\nu}\Psi(\xi)),\end{aligned}$$

where $\delta=\xi\frac{d}{d\xi}$, here $n=[\Re(\nu)]+1$ indicates the inteqer part of ν, $\log(.)=\log_e(.)$.

Example: we consider of function Ψ defined by :

$$\Psi(\xi)=(\log\frac{\xi}{\mathtt{a}})^{\nu_1}$$

we then have

$$_HD^{\nu}_{\mathtt{a}}(\log\frac{\xi}{\mathtt{a}})^{\nu_1}=\frac{1}{\Gamma(n-\nu)}(\xi\frac{d}{d\xi})^n\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu-1}(\log\frac{s}{\mathtt{a}})^{\nu_1}\frac{ds}{s},$$

by changing the variable $\mu=\frac{\log\frac{s}{\mathtt{a}}}{\log\frac{\xi}{\mathtt{a}}}$, we obtain

$$D^{\nu}_{\mathtt{a}}(\log\frac{\xi}{\mathtt{a}})^{\nu_1}=\frac{(\xi\frac{d}{d\xi})^n(\log\frac{\xi}{\mathtt{a}})^{n-\nu+\nu_1}}{\Gamma(n-\nu)}\int_0^1\mu^{\nu_1}(1-\mu)^{n-\nu-1}d\mu,$$

$$= \frac{\Gamma(\nu_1+1)}{\Gamma(n-\nu+\nu_1+1)}(\xi\frac{d}{d\xi})^n(\log\frac{x}{\mathtt{a}})^{n-\nu+\nu_1}$$

If we take $\nu_1 = 5/2$,$\nu = 1/2$

$$D_{\mathtt{a}}^{1/2}(\log\frac{x}{\mathtt{a}})^{5/2} = \frac{\Gamma(7/2)}{\Gamma(4)}(x\frac{d}{dx})^1(\log\frac{x}{\mathtt{a}})^3 = \frac{15\sqrt{\pi}}{64}(\log\frac{x}{\mathtt{a}})^2,$$

Let $\nu > \nu_1 > 0$, $\Psi \in C(J,R)$, then

$${}_HD_{\mathtt{a}}^{\nu_1}({}_HI_{\mathtt{a}}^{\nu}\Psi)(\xi) = {}_HI_{\mathtt{a}}^{\nu-\nu_1}\Psi(\xi).$$

In particular, if $\nu = \nu_1$, then

$${}_HD_{\mathtt{a}}^{\nu}({}_HI_{\mathtt{a}}^{\nu}\Psi)(\xi) = \Psi(\xi).$$

proof: Let $n-1 < \nu_1 \leq n,\ and n \in N$, if $\nu_1 = n$

$${}_HD^n\Psi(\xi) = \left(\xi\frac{d}{d\xi}\right)^n\Psi(\xi).$$

We apply the operator D^n to the function $I^\nu\Psi(\xi)$, we get

$$\begin{aligned}
{}_HD^n\ \left({}_HI_{\mathtt{a}}^{\nu}\Psi\right)(\xi) &= \left(\xi\frac{d}{d\xi}\right)^n\frac{1}{\Gamma(\nu)}\int_{\mathtt{a}}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s)\frac{ds}{s}\\
&= \left(\xi\frac{d}{d\xi}\right)^{n-1}\frac{\xi}{\Gamma(\nu)}\int_{\mathtt{a}}^{\xi}\frac{d}{d\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s)\frac{ds}{s}\\
&= \left(\xi\frac{d}{d\xi}\right)^{n-1}\frac{1}{\Gamma(\nu-1)}\int_{\mathtt{a}}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-2}\Psi(s)\frac{ds}{s}\\
&= {}_HD^{n-1}\ {}_HI_{\mathtt{a}}^{\nu-1},
\end{aligned}$$

we repeat this procedure $(1 < \kappa \leq n)$, we get

$${}_HD^n\ \left({}_HI_{\mathtt{a}}^{\nu}\Psi\right)(\xi) =_H D^{n-\kappa}\left({}_HI_{\mathtt{a}}^{\nu-\kappa}\Psi\right)(\xi).$$

So for $\kappa = n$, we have

$${}_HD^n\ {}_HI_{\mathtt{a}}^{\nu}\Psi(\xi) = {}_HI_{\mathtt{a}}^{\nu-n}\Psi(\xi).$$

and more

$${}_HD^n\ {}_HI^n\Psi(\xi) = \Psi(\xi).$$

Consequently

$${}_HD^{\nu_1}\ {}_HI^{\nu}\Psi(\xi) =_H D^n({}_HI_H^{n-\nu_1}I^\nu\Psi)(\xi)$$

$=\mathrm{D}^nI^n\left(I^{\nu-\nu_1}\Psi\right)(\xi) = I^{\nu-\nu_1}\Psi(\xi).$

Lemma 1.33. *[61] Let $\Re(\nu) > 0$, $n = [\Re(\nu)] + 1$, the equation*

$$D^{\nu}_{a}\Psi(\xi) = 0 \Leftrightarrow z(\xi) = \sum_{j=0}^{n-1} c_j \left(\log \frac{\xi}{a}\right)^{\nu-j}$$

where $c_j \in R$, $\kappa = 1, ..., n$.
In particular, if $0 < \Re(\nu) \leq 1$, we have

$$D^{\nu}_{a}\Psi(\xi) = 0 \Leftrightarrow z(\xi) = d\left(\frac{\xi}{a}\right)^{\nu-1}, \text{ for any } d \in R.$$

Lemma 1.34. *[61] Let $\Re(\nu) > 0$, $n = [\Re(\nu)] + 1$, the equation*

$$D^{\nu}_{a}\Psi(\xi) = 0 \Leftrightarrow z(\xi) = \sum_{j=0}^{n-1} c_j (\log \frac{\xi}{a})^{\nu-j}$$

where $c_j \in R$, $j = 0, ..., n-1$.
In particular, if $0 < \Re(\nu) \leq 1$, we have

$$D^{\nu}_{a}\Psi(\xi) = 0 \Leftrightarrow z(\xi) = d\left(\frac{\xi}{a}\right)^{\nu-1} \text{ for any } c \in R.$$

1.4.5 Fractional Derivative of Caputo-Hadamard

Definition 1.35. ([3]): The Caputo-Hadamard fractional derivative of order $\nu \geq 0$ of function Ψ be defined as :

$$\begin{aligned} {}^{C}_{H}D^{\nu}_{\mathtt{a}}\Psi(\xi) &= \frac{1}{\Gamma(n-\nu)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu-1}(s\frac{d}{ds})^{n}\Psi(s)\frac{ds}{s}^{*} \\ &= {}_{H}I^{n-\nu}_{\mathtt{a}}\delta^{n}\Psi(\xi). \end{aligned}$$

Where $\delta = \xi\frac{d}{d\xi}$, $n-1 < \nu < n$, $n \in N$.

Example: we consider the function Ψ defined by: $\Psi(\xi) = (\log\frac{\xi}{\mathtt{a}})^{\nu_1}$.
We have

$${}^{C}_{H}D^{\nu}_{\mathtt{a}}\Psi(\xi) = {}^{C}_{H} D^{\nu}_{\mathtt{a}}(\log\frac{\xi}{\mathtt{a}})^{\nu_1} = \frac{1}{\Gamma(n-\nu)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu-1}\delta^{n}(\log\frac{s}{\mathtt{a}})^{\nu_1}\frac{ds}{s},$$

if we take $\nu_1 = 3/2$,$\nu = 1/2$, and $n = 1$, then

$$\delta^{n}(\log\frac{s}{\mathtt{a}})^{\nu_1} = \delta^{1}(\log\frac{s}{\mathtt{a}})^{\frac{3}{2}} = \frac{3}{2}(\log\frac{s}{\mathtt{a}})^{\frac{1}{2}}$$

and

$$ {}_{H}^{C}D_{\mathtt{a}}^{\frac{1}{2}}(\log\frac{\xi}{\mathtt{a}})^{\frac{3}{2}} = \frac{\frac{3}{2}}{\Gamma(\frac{1}{2})}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{1-1/2-1}(\log\frac{s}{\mathtt{a}})^{\frac{1}{2}}\frac{ds}{s}, $$

by change of variable $\mu = \frac{\log\frac{s}{\mathtt{a}}}{\log\frac{\xi}{\mathtt{a}}}$, we obtain

$$ \begin{aligned} {}_{H}^{C}D_{\mathtt{a}}^{1/2}(\log\frac{\xi}{\mathtt{a}})^{3/2} &= \frac{3/2(\log\frac{\xi}{\mathtt{a}})}{\Gamma(1/2)}\int_{0}^{1}\mu^{3/2}(1-\mu)^{1/2}d\mu, \\ &= \frac{3/2\Gamma(3/2)(\log\frac{\xi}{\mathtt{a}})}{\Gamma(2)} = \frac{3\sqrt{\pi}}{4}(\log\frac{\xi}{\mathtt{a}}). \end{aligned} $$

1.4.6 Relation Connecting Derivative of Hadamard that of Caputo-Hadamard

Let $n-1<\nu<n$, $n\in N$ and $\Psi\in AC_{\delta}^{n}(J,R)$, the connection between the tow derivatives are defined by :

$$ {}_{H}^{C}D_{\mathtt{a}}^{\nu}\Psi(\xi) = {}_{H}D_{\mathtt{a}}^{\nu}\left[\Psi(\xi) - \sum_{\kappa=0}^{n-1}\frac{\delta^{(\kappa)}\Psi(\mathtt{a})}{\kappa!}\left(\log\frac{\xi}{\mathtt{a}}\right)^{\kappa}\right]. $$

where

$$ {}_{H}^{C}D_{\mathtt{a}}^{\nu}\Psi(\xi) = {}_{H}D_{\mathtt{a}}^{\nu}\Psi(\xi) - \sum_{\kappa=0}^{n-1}\frac{\delta^{(\kappa)}\Psi(\mathtt{a})}{\Gamma(\kappa-\nu+1)}\left(\log\frac{\xi}{\mathtt{a}}\right)^{\kappa-\nu}. $$

proof: we by definition

$$ {}_{H}D_{\mathtt{a}}^{\nu}\Psi(\xi) = (\xi\frac{d}{d\xi})^{n})I^{n-\nu}\Psi(\xi) $$

$$ \begin{aligned} {}_{H}D_{\mathtt{a}}^{\nu}\left[\Psi(\xi) - \sum_{\kappa=0}^{n-1}\frac{\Psi^{(\kappa)}(\mathtt{a})}{\kappa!}(\log\frac{\xi}{\mathtt{a}})^{\kappa}\right] &= (\xi\frac{d}{d\xi})^{n}\left({}_{H}I_{\mathtt{a}}^{n-\nu}[\Psi(\xi) - \sum_{\kappa=0}^{n-1}\frac{\partial^{\kappa}\Psi(\mathtt{a})}{\kappa!}(\log\frac{\xi}{\mathtt{a}})^{\kappa}]\right) \\ = (\xi\frac{d}{d\xi})^{n}\frac{1}{\Gamma(n-\nu)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu-1}&\left[\Psi(s) - \sum_{\kappa=0}^{n-1}\frac{\partial^{\kappa}\Psi(\mathtt{a})}{\kappa!}(\log\frac{s}{\mathtt{a}})^{\kappa}\right]\frac{ds}{s} \end{aligned} $$

we pose $g(\xi) = \Psi(\xi) - \sum_{\kappa=0}^{n-1}\frac{\partial^{\kappa}\Psi(\mathtt{a})}{\kappa!}(\log\frac{\xi}{\mathtt{a}})^{\kappa}$, and we integrate by parts

$$ u(\xi) = g(\xi), \quad u'(\xi) = g'(\xi) $$

$$ v'(\xi) = (\log\frac{\xi}{s})^{n-\nu-1}\frac{ds}{s}, \quad v(\xi) = \frac{-1}{n-\nu}(\log\frac{\xi}{s})^{n-\nu} $$

we obtain

$$
\begin{aligned}
(\xi\frac{d}{d\xi})^n\frac{1}{\Gamma(n-\nu)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu-1}g(s)\frac{ds}{s} &= (\xi\frac{d}{d\xi})^n\frac{1}{\Gamma(n-\nu)}\left[-\frac{1}{n-\nu}(\log\frac{\xi}{s})^{n-\nu}g(s)\right]_{\mathtt{a}}^{\xi} \\
&\quad +(\xi\frac{d}{d\xi})^n\frac{1}{\Gamma(n-\nu+1)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu}\left[\frac{d}{ds}g(s)\right]ds \\
&= \frac{g(\mathtt{a})}{\Gamma(n-\nu+1)}\left[(\xi\frac{d}{d\xi})^n(\log\frac{\xi}{s})^{n-\nu}\right] \\
&\quad +(\xi\frac{d}{d\xi})^n\frac{1}{\Gamma(n-\nu+1)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu}\left[\frac{sd}{ds}g(s)\right]\frac{ds}{s} \\
&= \frac{g(\mathtt{a})}{\Gamma(1-\nu)}(\log\frac{\xi}{\mathtt{a}})^{-\nu}+\delta^n I^{n-\nu+1}(\delta g)(\xi)
\end{aligned}
$$

and by integration by parts

$$u(\xi)=(\xi),\qquad u'(\xi)=\delta g'(\xi)$$

$$v'(\xi)=(\log\tfrac{\xi}{s})^{n-\nu}\tfrac{ds}{s},\ \ v(\xi)=\tfrac{-1}{n-\nu+1}(\log\tfrac{\xi}{s})^{n-\nu+1}$$

$$
\begin{aligned}
\delta^n I_{\mathtt{a}}^{n-\nu+1}(\delta g)(\xi) &= \left(\frac{td}{d\xi}\right)^n\frac{1}{\Gamma(n-\nu+1)}\int_{\mathtt{a}}^{\xi}\left(\log\frac{\xi}{s}\right)^{n-\nu}[\delta g(s)]\,ds \\
&= \left(\xi\frac{d}{d\xi}\right)^n\frac{1}{\Gamma(n-\nu+1)}\left[\frac{-\delta g(s)}{n-\nu+1}(\log\frac{\xi}{s})^{n-\nu+1}\right]_{\mathtt{a}}^{\xi} \\
&\quad +\left(\frac{td}{d\xi}\right)^n\left[\frac{-1}{\Gamma(n-\nu+2)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu+1}\delta\left(s\frac{d}{ds}\frac{g(s)}{s}\right)ds\right] \\
&= \frac{\delta g(\mathtt{a})}{\Gamma(2-\nu)}(\log\frac{\xi}{\mathtt{a}})^{-\nu+1}+\left(\frac{td}{d\xi}\right)^n\frac{1}{\Gamma(n-\nu+2)}\int_{\mathtt{a}}^{\xi}(\log\frac{\xi}{s})^{n-\nu+1}\delta^2 g(s)\frac{ds}{s}.
\end{aligned}
$$

$$
\begin{aligned}
\delta^n I_{\mathtt{a}}^{n-\nu}g(\xi) &= \frac{g(\mathtt{a})}{\Gamma(1-\nu)}(\log\frac{\xi}{\mathtt{a}})^{-\nu}++\frac{\delta g(\mathtt{a})}{\Gamma(2-\nu)}(\log\frac{\xi}{\mathtt{a}})^{-\nu+1}+\left(\frac{td}{d\xi}\right)^n I_{\mathtt{a}}^{n-\nu+2}\delta^2 g(\xi) \\
&= \sum_{\kappa=0}^{n-1}\frac{\delta^\kappa g(\mathtt{a})}{\Gamma(\kappa-\nu+1}\left(\frac{\xi}{\mathtt{a}}\right)^{\kappa-\nu}+\delta^n I_{\mathtt{a}}^{2n-\nu}\delta^n g(\xi) \\
&= \sum_{\kappa=0}^{n-1}\frac{\delta^\kappa g(\mathtt{a})}{\Gamma(\kappa-\nu+1}\left(\frac{\xi}{\mathtt{a}}\right)^{\kappa-\nu}+\delta^n I_{\mathtt{a}}^{n} I_{\mathtt{a}}^{n-\nu}\delta^n g(\xi) \\
&= \sum_{\kappa=0}^{n-1}\frac{\delta^\kappa g(\mathtt{a})}{\Gamma(\kappa-\nu+1}\left(\frac{\xi}{\mathtt{a}}\right)^{\kappa-\nu}+I_{\mathtt{a}}^{n-\nu}\delta^n g(\xi)
\end{aligned}
$$

In addition

$$
\begin{aligned}
I_{\mathsf{a}}^{n-\nu}\delta^n g(\xi) &= I_{\mathsf{a}}^{n-\nu}\delta^n\left[\Psi(\xi)-\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa\Psi(\mathsf{a})}{\kappa!}(\log\frac{\xi}{\mathsf{a}})^\kappa\right]\\
&= I_{\mathsf{a}}^{n-\nu}\delta^n\Psi(\xi)=^C D_{\mathsf{a}}^{\nu}\Psi(\xi)
\end{aligned}
$$

because $\delta^n\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa\Psi(\mathsf{a})}{\kappa!}(\log\frac{\xi}{\mathsf{a}})^\kappa=0$ and $D^\nu\left(\frac{\xi}{\mathsf{a}}\right)^\kappa=\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\nu+1)}\left(\log\frac{\xi}{\mathsf{a}}\right)^{\kappa-\nu}$, we have

$$
{}_HD_{\mathsf{a}}^{\nu}\left[\Psi(\xi)-\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa\Psi(\mathsf{a})}{\kappa!}(\log\frac{\xi}{\mathsf{a}})^\kappa\right]={}_H^CD_{\mathsf{a}}^{\nu}\Psi(\xi). \tag{1.6}
$$

this relation can also be written:

$$
{}_HD_{\mathsf{a}}^{\nu}\Psi(\xi)={}_H^C D_{\mathsf{a}}^{\nu}\Psi(\xi)+\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa\Psi(\mathsf{a})}{\Gamma(\kappa-\nu+1)}(\log\frac{\xi}{\mathsf{a}})^{\kappa-\nu}.
$$

Let $\nu>\nu_1>0$, si $\Psi\in C(J,R)$, then

$$
{}_H^CD_{\mathsf{a}}^{\nu_1}({}_HI_{\mathsf{a}}^{\nu}\Psi)(\xi)=I_{\mathsf{a}}^{\nu-\nu_1}\Psi(\xi).
$$

In particular, if $\nu=\nu_1$, then

$$
{}_H^CD_{\mathsf{a}}^{\nu}({}_HI_{\mathsf{a}}^{\nu}\Psi)(\xi)=\Psi(\xi).
$$

proof: The relation (1.6) allows to obtain the following result:

$$
\begin{aligned}
{}_H^CD_{\mathsf{a}}^{\nu_1}\left({}_HI_{\mathsf{a}}^{\nu}\Psi\right)(\xi) &= {}_HD_{\mathsf{a}}^{\nu_1}\left[{}_HI_{\mathsf{a}}^{\nu}\Psi(\xi)-\sum_{\kappa=0}^{n-1}\frac{\delta_H^\kappa I_{\mathsf{a}}^{\nu}\Psi(\mathsf{a})}{\kappa!}\left(\log\frac{\xi}{\mathsf{a}}\right)^\kappa\right]\\
&= {}_HD_{\mathsf{a}}^{\nu}({}_HI_{\mathsf{a}}^{\nu_1}\Psi)(\xi)-\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa I^\nu\Psi(\mathsf{a})}{\kappa!}\,{}_HD_{\mathsf{a}}^{\nu}\left(\log\frac{\xi}{\mathsf{a}}\right)^\kappa\\
&= {}_HI_{\mathsf{a}}^{\nu-\nu_1}\Psi(\xi)-\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\nu_1}\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa I^\nu\Psi(\mathsf{a})}{\kappa!}\,{}_H\left(\log\frac{\xi}{\mathsf{a}}\right)^{\kappa-\nu_1}\\
&= {}_HI_{\mathsf{a}}^{\nu-\nu_1}\Psi(\xi)-\sum_{\kappa=0}^{n-1}\frac{\delta^\kappa I^\nu\Psi(\mathsf{a})}{\Gamma(\kappa-\nu_1+1)}\left(\log\frac{\xi}{\mathsf{a}}\right)^{\kappa-\nu_1},
\end{aligned}
$$

and since $\kappa\le n-1<\nu$, for $\kappa=0,1,...,n-1$, then, the derivatives $\delta_H^\kappa I^\nu\Psi(\mathsf{a})=0$, which give

$$
{}_H^CD^{\nu_1}\left({}_HI^\nu\Psi\right)(\xi)=I^{\nu-\nu_1}\Psi(\xi).
$$

Lemma 1.36. *[3] Let $\Re(\nu) \geq 0$ and $n = [\Re(\nu)] + 1$. If $y(\xi) \in AC^n_\delta(J, R)$, then Caputo-Hadamard fractional differential equation*

$$^{C}D^{\nu}_{a}y(\xi) = 0$$

has a solution

$$y(\xi) = \sum_{i=0}^{n-1} c_i \left(\log \frac{\xi}{a}\right)^{\nu-i}.$$

where $c_i \in R$, $i = 0, 1, 2, ..., n-1$.

proof: Let $n-1 < \nu < n, n \in N^*$, and $\Psi \in C([\mathsf{a}, T])$, we have by definition:

$${}_HD^{\nu}_{\mathsf{a}}\Psi(\xi) = \delta^n \left(I^{n-\nu}_{\mathsf{a}}\Psi\right)(\xi) = 0$$

that is, the derivative of order n of $(I^{n-\nu}\Psi)$ is zero.
So we can write:

$$\left(I^{n-\nu}_{\mathsf{a}}\Psi\right)(\xi) = \sum_{\kappa=0}^{n-1} c_\kappa \left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa}, \quad c_\kappa \in R, \quad \kappa = 0, 1, \ldots, n-1.$$

Applying ${}_HI^{\nu}_{\mathsf{a}}$ to the two members of the previous identity, we get

$${}_HI^{\nu}_{\mathsf{a}}\left(I^{n-\nu}_{\mathsf{a}}\Psi\right)(\xi) =_H I^{\nu}_{\mathsf{a}}\left[\sum_{\kappa=0}^{n-1} c_\kappa \left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa}\right],$$

$$\begin{aligned}\left(I^{n-\nu+\nu}_{\mathsf{a}}\Psi\right)(\xi) = {}_HI^{n}_{\mathsf{a}}\Psi(\xi) &= \sum_{\kappa=0}^{n-1} c_\kappa \left[{}_HI^{\nu}_{\mathsf{a}}\left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa}\right] \\ &= \sum_{\kappa=0}^{n-1} c_\kappa \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1+\nu)}\left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa+\nu}.\end{aligned}$$

Then, we apply δ^n on the two members of the equality

$$\delta^n\left({}_HI^{n}_{\mathsf{a}}\Psi\right)(\xi) = \sum_{\kappa=0}^{n-1} c_\kappa \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1+\nu)}\delta^n\left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa+\nu},$$

Hence

$$\begin{aligned}\Psi(\xi) &= \sum_{\kappa=0}^{n-1} c_\kappa \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1+\nu)}\frac{\Gamma(\kappa+1+\nu)}{\Gamma(\kappa+1+\nu-n)}\left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa+\nu-n} \\ &= \sum_{\kappa=0}^{n-1} c_\kappa \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1+\nu-n)}\left(\log \frac{\xi}{\mathsf{a}}\right)^{\kappa+\nu-n}\end{aligned}$$

Finally, for $i = n - \kappa$, we get

$$\Psi(\xi) = \sum_{i=1}^{n} c_i \left(\log \frac{\xi}{\mathsf{a}}\right)^{\nu-i}, c_i, \quad i = 0, 1, \ldots n-1 \in R.$$

Lemma 1.37. *[3] Let $\Re(\nu) > 0, n = [\Re(\nu)] + 1$, and $\Psi \in C(J, R)$, the fractional differential equation*

$$ {}_H I^{\nu} \left({}_H D^{\nu} \Psi \right)(\xi) = \Psi(\xi) + \sum_{i=1}^{n} c_i \left(\log \frac{\xi}{a} \right)^{\nu - i}. $$

Lemma 1.38. *Let $\Re(\nu) > 0, n = [\Re(\nu)] + 1$ and $\Psi \in AC^n_\delta$, then*

$$ {}_H I^{\nu}_{a} \left({}^C_H D^{\nu}_{a} \Psi \right)(\xi) = \Psi(\xi) - \sum_{\kappa=0}^{n-1} \frac{\delta^{\kappa} \Psi(a)}{\kappa!} \left(\log \frac{\xi}{a} \right)^{\kappa} $$

such that δ^κ is the fractional Hadamard derivative of order κ, where $\kappa = 0, 1, \dots, n-1$.

proof: by Proposition (1.4.6), we have

$$ D^n I^n \Psi(\xi) = \Psi(\xi). $$

Using the identity (1.9), we find

$$ \begin{aligned} \Psi(\xi) = D^n I^n \Psi(\xi) &= I^n_{\mathsf{a}} \delta^n \Psi(\xi) + \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa}(\mathsf{a})}{\Gamma(\kappa+1)} \left(\log \frac{\xi}{\mathsf{a}} \right)^{\kappa} \\ &= I^n \delta^n \Psi(\xi) + \sum_{\kappa=0}^{n-1} \frac{\delta^{\kappa} \Psi(\mathsf{a})}{\kappa!} \left(\log \frac{\xi}{\mathsf{a}} \right)^{\kappa} \\ &= I^{\nu} I^{n-\nu} \delta^n \Psi(\xi) + \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa} \Psi(\mathsf{a})}{\kappa!} \left(\log \frac{\xi}{\mathsf{a}} \right)^{\kappa} \\ &= I^{\nu CH} D^{\nu}_{\mathsf{a}} \Psi(\xi) + \sum_{\kappa=0}^{n-1} \frac{\delta^{\kappa} \Psi(\mathsf{a})}{\kappa!} \left(\log \frac{\xi}{\mathsf{a}} \right)^{\kappa} \end{aligned} $$

as a result, we get

$$ I^{\nu} C_H D^{\nu}_{\mathsf{a}} \Psi(\xi) = \Psi(\xi) - \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa} \Psi(\mathsf{a})}{\kappa!} \left(\log \frac{\xi}{\mathsf{a}} \right)^{\kappa} $$

Lemma 1.39. *[61] Suppose that $\Psi(\xi, y)$ is continuous. Then the initial value problem (IVP)*

$$ {}^C_H D^{\nu} y(\xi) = \Psi(\xi, y), \ \ \xi > a > 0, \ \ 0 < \nu < 1, \tag{1.7} $$

$$ y(a) = y_a, \tag{1.8} $$

is equivalent to the following Volterra inteqral equation

$$y(\xi) = y_{a} + \frac{1}{\Gamma(\nu)} \int_{a}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \Psi(s, y(s)) \frac{ds}{s}. \tag{1.9}$$

Lemma 1.40. *Let $\nu > 0, n = [\nu] + 1$ and $h \in AC_{\delta}^{n}$, then*

$$_{H}I^{\nu} \left(\begin{matrix} c \\ H \end{matrix} \; D^{\nu} \Psi \right) (\xi) = \Psi(\xi) - \sum_{\kappa=0}^{n-1} \frac{\delta^{i} \Psi(a)}{\kappa!} \left(\log \frac{\xi}{a}\right)^{\kappa}$$

such that δ^{κ} is the fractional Hadamard derivative of order κ ,$\kappa = 0, 1, \ldots, n-1$.

proof: Let $D^{n} I^{n} \Psi(\xi) = \Psi(\xi)$.
Using the identity (1.9), we find

$$D^{n} I^{n} \Psi(z) = I^{n} \partial^{n} \Psi(\xi) + \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa)} \Psi(\mathtt{a})}{\Gamma(\kappa+1)} \left(\log \frac{\xi}{\mathtt{a}}\right)^{\kappa}$$

$$\begin{aligned} D^{n} I^{n} \Psi(z) &= I^{n} \delta^{n} \Psi(z) + \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa)} \Psi(\mathtt{a})}{\kappa!} \left(\log \frac{z}{\mathtt{a}}\right)^{\kappa} \\ &= I^{\nu} I^{n-\nu} \delta^{n} \Psi(z) + \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa)} \Psi(\mathtt{a})}{\kappa!} \left(\log \frac{z}{\mathtt{a}}\right)^{\kappa} \\ &= I^{\nu CH} D^{\nu} \Psi(\xi) + \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa)} \Psi(\mathtt{a})}{\kappa!} \left(\log \frac{z}{\mathtt{a}}\right)^{\kappa} \end{aligned}$$

as a result, we get

$$I^{\nu CH} D_{\mathtt{a}}^{\nu} \Psi(\xi) = \Psi(\xi) - \sum_{\kappa=0}^{n-1} \frac{\delta^{(\kappa)} \Psi(\mathtt{a})}{\kappa!} \left(\log \frac{z}{\mathtt{a}}\right)^{\kappa}.$$

1.5 Impulsive Differential Equations (IDE)

Differential equations involving impulse effects appear as an appropriate model for some evolutionary problems. It is the case of many real-world processes that are subject of abrupt of changes in certain moments of times and arising in a variety of disciplines, including biology, population dynamics, electric technology, control theory, engineering, etc.[14, 39].

This type of impulse is seen in lasers and when drugs are injected intravenously into the bloodstream.

Where a method is used to prove solutions of this type $J = [s_\kappa, \xi_{\kappa+1}]$, for $\kappa = 0, 1, ..., w$, in impulsive interval $(\xi_\kappa, s_\kappa]$. The points

$$\mathtt{a} = s_0 < \xi_1 < s_1 < ... < \xi_w < s_w < \xi_{w+1} = \mathtt{T}$$

The type of problems we are going to consider in demonstrating the existence and uniqueness of the solution on a bounded interval of type $J = [\mathtt{a}, \mathtt{T}]$, $(0 < \mathtt{a} < \mathtt{T} < +\infty)$, is for fixed impulsive point: $\xi = \xi_\kappa$, $\kappa = 0, 1, ..., w,$, such that:

$$\mathtt{a} = \xi_0 < \xi_1 < ... < \xi_w < \xi_{w+1} = \mathtt{T}$$

1.5.1 A Brief Description of (IDE)

An impulsive differential equation is described by three components, namely:

1. A continuous-time differential equation, which governs the state of the system between pulses.
2. An impulsive equation that represents an impulse jump defined by the jump function at the time of the pulse.
3. A jump criterion which defines a set of jump events in which the impulsive equation is active.

The simple form of the first order impulsive equation is given by:

$$\begin{cases} z'(\xi) = \Psi(\xi, z(\xi)), \quad \xi \in J := [\mathtt{0}, \mathtt{T}], \xi \neq \xi_\kappa, \, \kappa = 1, ..., w, \\ \\ \Delta z \mid_{\xi=\xi_\kappa} = I_\kappa(z(\xi_\kappa^-)), \qquad \kappa = 1, .., w, \end{cases},$$

where
$\Psi\ : J * R \to R$ is a given function, $\mathtt{0} = \xi_0 < \xi_1 < ... < \xi_w < \xi_{w+1} = \mathtt{T}$,
$I_\kappa : R\ \to R$ is a continuous function for $\kappa = 1, ..., w$, and $\Delta z|_{\xi=\xi_\kappa} = z(\xi_\kappa^+) - z(\xi_\kappa^-)$, $z(\xi_\kappa^+)$ and $z(\xi_\kappa^-)$ represent the right and left hand limits of $z(\xi)$ at $\xi = \xi_\kappa$.

1.6 Concept of Solutions for (IDE) in the Fractional Case

In the fractional case, there is mainly two different approaches defining the concepts of solutions for impulsive differential equations. This is due to the nonlocal character of the fractional order differential operators, as it is pointed out in the previous section.

In order to be more precise when presenting these two approaches, we specify the following impulsive initial value problem's Caputo type of order $0 < \nu \leq 1$:

$$^{C}D^{\nu}z(\xi) = \Psi(\xi, z(\xi)), \quad \xi \geq \xi_0,\ \xi \neq \xi_\kappa,\ \kappa = 1, 2, ... \tag{1.10}$$

$$\Delta z\ |_{\xi=\xi_\kappa} = I_\kappa(z(\xi_\kappa^-)), \qquad \kappa = 1, 2, .., \tag{1.11}$$

$$z(\xi_0) = z_0 \tag{1.12}$$

1.6.1 First Approach: Fractional Derivative with Fixed Lower Limit

This approach (noted respectively by V_2 in [5] and by A_1 in [6]) considers that the fractional equation in (1.10) is kept on any interval between two consecutive impulses: $]\xi_k, \xi_{k+1}[$, with only modified initial conditions. That is, we consider the following concept of solutions for (1.10)-(1.12).

Definition 1.41.
A function z is said to be a solution for (1.10)-(1.12), if z is a solution of (1.10) and moreover $z = Z_0\ |_{[\xi_0,\xi_1]}$, and for every $\kappa = 1, 2, ...,$ $z = Z_\kappa\ |_{]\xi_\kappa,\xi_{\kappa+1}]}$, where $Z_0 \in \mathcal{C}\left([\xi_0, \xi_1]\right)$ is the solution of

$$\begin{cases} ^{C}D^{\nu}_{\xi_0}z(\xi) = \Psi(\xi, z(\xi)), \quad \xi_0 < \xi \leq \xi_1 \\ \\ z(\xi_0) = z_0 \end{cases}$$

and for $\kappa = 1, 2, ...,$ $Z_\kappa \in \mathcal{C}\left([\xi_0, \xi_{\kappa+1}]\right)$ is the solution of

$$\begin{cases} ^{C}D^{\nu}_{\xi_0}z(\xi) = \Psi(\xi, z(\xi)), \qquad \xi \in [0, \xi_{\kappa+1}] \\ \\ z(\xi_\kappa^+) = z(\xi_\kappa^-) + I_\kappa\left(z\left(\xi_\kappa^-\right)\right), \\ \\ z(\xi_0) = z_0 \end{cases}$$

In this meaning, that is in the sens of Definition 1.41, the solutions of the problem (1.10)-(1.12) are the solutions of the following integral equation:

$$z(\xi) = \begin{cases} z_0 + \displaystyle\int_{\xi_0}^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)}\Psi(s, z(s))ds, & \xi \in [\xi_0, \xi_1] \\ z_0 + \displaystyle\int_{\xi_0}^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)}\Psi(s, z(s))ds + \sum_{i=1}^{k} I_i(z(\xi_i^-)), & \xi \in]\xi_k, \xi_{k+1}]\ : k = 1, 2, ... \end{cases} \tag{1.13}$$

1.6.2 Second Approach: Fractional Derivative with Varying Lower Limits

This approach (noted respectively by V_1 in [5] and by A_2 in [6]) neglects the lower limit of the fractional derivative at the initial time and moves it to each impulsive time. That is , we consider the following definition for solutions of (1.10)-(1.12).

Definition 1.42.

A function z is said to be a solution for (1.10)-(1.12), if the Caputo derivative of z on each sub-interval $J_\kappa :=]\xi_\kappa, \xi_{\kappa+1}]$ exists and for every ξ_κ, $\kappa = 1, 2; ...,$ z is a solution of the following problem:

$$\begin{cases} {}^C D^{\nu}_{\xi_\kappa} z(\xi) = \Psi(\xi, z(\xi)), & \xi \in J_\kappa \\ z(\xi_\kappa^+) = z(\xi_\kappa^-) + I_\kappa\left(z\left(\xi_\kappa^-\right)\right) \end{cases},$$

where:

$${}^C D^{\nu}_{\xi_\kappa} z(\xi) = \frac{1}{\gamma(1-\nu)} \int_{\xi_\kappa}^{\xi} (\xi - s)^{\nu - 1} \; z'(s) ds$$

In this meaning, that is in the sens of Definition 1.42, the solutions of the problem (1.10)-(1.12) are the solutions of the following integral equation:

$$z(\xi) = \begin{cases} z_0 + \displaystyle\int_{\xi_0}^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)} \Psi(s, z(s)) ds, & \xi \in [\xi_0, \xi_1] \\ z_0 + \displaystyle\sum_{i=1}^{k} \int_{\xi_{i-1}}^{\xi_i} \frac{(\xi_i - s)^{\nu-1}}{\Gamma(\nu)} \Psi(s, z(s)) ds + & \\ \displaystyle\int_{\xi_k}^{\xi} \frac{(\xi - s)^{\nu-1}}{\Gamma(\nu)} \Psi(s, z(s)) ds + \sum_{i=1}^{k} I_i(z(\xi_i^-)), & \xi \in]\xi_k, \xi_{k+1}] \; : k = 1, 2, ... \end{cases} \tag{1.14}$$

Remark 1.43.

In all the problems which will be considered in the forthcoming chapters, we will adopt the second approach.

Chapter 2

Impulsive Fractional Differential Equations with Caputo-Hadamard Derivative of order $0 < \nu \leq 1$

2.1 Introduction

This chapter, we concerned the existence of solutions for certain classes of impulsive fractional differential equations with Caputo-Hadamard derivative. First, we give results the existence, and uniqueness of solutions for initial value problem for then impulsive fractional differential equations. By applying some standard fixed point theorems. Following that, we extend the study of existence of solutions in Banach space. by employing the Mönch's fixed point theorem in conjunction then the technique of measures of non-compactness. Examples are provided the demonstrate our findings. Finally, we study the existence and uniqueness of solutions for initial value problem of impulsive fractional neutral differential equations with delay.

2.2 Fractional Differential Equations with Delay and Impulses

This section, we study of existence and uniqueness of solution for the following initial value problem (IVP for short) :

$$ {}^{C}_{H}D^{\nu} z(\xi) = \Psi(\xi, z_{\xi}), \quad \xi \in J := [\mathtt{a}, \mathtt{T}], \xi \neq \xi_{\kappa}, \, \kappa = 1, ..., w, \tag{2.1} $$

$$ \Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}(z(\xi_{\kappa}^{-})), \qquad \kappa = 1, .., w, \tag{2.2} $$

$$z(\xi) = \Phi(\xi), \quad \xi \in [\mathtt{a}-\tau, \mathtt{a}], \tag{2.3}$$

where ${}^{C}_{H}D^{\nu}$ the fractional Caputo-Hadamard derivative of order $0 < \nu \leq 1$, $\Psi : J * C([\mathtt{a}-\tau, \mathtt{a}], R) \to R$ is a given functions, $\Phi \in C([\mathtt{a}-\tau, \mathtt{a}], R)$, $\mathtt{a} > 0$,$\mathtt{a} = \xi_0 < \xi_1 < ... < \xi_w < \xi_{w+1} = \mathtt{T}$, $I_\kappa : R \to R$ is a continuous function of $\kappa = 1, ..., w$, $\Delta z|_{\xi=\xi_\kappa} = z(\xi_\kappa^+) - z(\xi_\kappa^-)$, $z(\xi_\kappa^+)$ and $z(\xi_\kappa^-)$ represent the right and left hand limits of $z(\xi)$ at $\xi = \xi_\kappa$.

For each function z defined in J' $(:= [\mathtt{a}-\tau, \mathtt{T}] \setminus \{\xi_1, ..., \xi_w\})$ and each $\xi \in [\mathtt{a}, \mathtt{T}]$, we indicate by z_ξ the element for C_τ $(:= C([\mathtt{a}-\tau, \mathtt{a}], R))$ defined by

$$z_\xi(s) = z(\xi + s) \ ,s \in [\mathtt{a}-\tau, \mathtt{a}].$$

Also C_τ is endowed with the norm

$$\|\phi\|_{C_\tau} = \sup\{|\phi(\theta)| : a - \tau \leq \theta \leq a\}.$$

We present the following Banach space:

$$AC' = \left\{ \begin{array}{l} z : J \to \ R, \ z \in AC_\delta((\xi_\kappa, \xi_{\kappa+1}], \ R), \ \kappa = 1, \ldots, w, \text{ and there exist} \\ z(\xi_\kappa^+) \text{ and } z(\xi_\kappa^-), \ \kappa = 1, \ldots, w, \text{ with } z(\xi_\kappa^-) = z(\xi_\kappa) \end{array} \right\}.$$

with the norm

$$\|z\|_{AC'} = \sup\{|z(\xi)|_\infty : \mathtt{a} \leq \xi \leq \mathtt{T}\}.$$

Set

$$\mathrm{B} = \{z : (\mathtt{a}-\tau, \mathtt{T}] \to \ R \setminus \ z \in AC' \cap C_\tau\}.$$

2.2.1 Equivalent Integral Equation

Definition 2.1.

A function $z \in \mathrm{B}$ is said to be a solution of the problem (2.1)-(2.3), if z satisfies the fractional differential equation ${}^{C}_{H}D^{\nu} z(\xi) = \Psi(\xi, z_\xi)$ in J' and the following conditions (2.2)-(2.3).

Lemma 2.2.

Assume that $\vartheta \in AC(J', R)$, *let* $0 < \nu \leq 1$. *A function* z *is a solution of the fractional integral equation*

$$z(\xi) = \begin{cases} \overline{z} + \frac{1}{\Gamma(\nu)} \int_{\mathfrak{a}}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \vartheta(s) \frac{ds}{s}, & \text{if } \xi \in [\mathfrak{a}, \xi_1], \\ \overline{z} + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{\kappa} \int_{\xi_{i-1}}^{\xi_i} \left(\log \frac{\xi_i}{s}\right)^{\nu-1} \vartheta(s) \frac{ds}{s} \\ + \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \vartheta(s) \frac{ds}{s} \\ + \sum_{i=1}^{\kappa} I_i(z(\xi_i^-)), & \text{if } \xi \in (\xi_\kappa, \xi_{\kappa+1}], \end{cases} \tag{2.4}$$

with $\kappa = 1, ..., w$.

If and only if z *is a solution of the fractional impulsive (IVP) :*

$${}_{H}^{C}D^{\nu} z(\xi) = \vartheta(\xi), \text{ for each } \xi \in J', \tag{2.5}$$

$$\Delta z|_{\xi=\xi_\kappa} = I_\kappa(z(\xi_\kappa^-)), \quad \kappa = 1, ..., w, \tag{2.6}$$

$$z(\mathfrak{a}) = \overline{z}. \tag{2.7}$$

Proof:
Assume z satisfies (2.5)-(2.7), if $\xi \in [\mathsf{a}, \xi_1]$, then

$${}_{H}^{C}D^{\nu} z(\xi) = \vartheta(\xi)$$

Lemma 1.39 implies

$$z(\xi) = \overline{z} + \frac{1}{\Gamma(\nu)} \int_{\mathsf{a}}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \vartheta(s) \frac{ds}{s}.$$

If $\xi \in (\xi_1, \xi_2]$, then according to the Lemma 1.39, we have

$$
\begin{aligned}
z(\xi) &= z(\xi_1^+) + \frac{1}{\Gamma(\nu)}\int_{\xi_1}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} \\
&= \Delta z|_{\xi=\xi_1} + z(\xi_1^-) + \frac{1}{\Gamma(\nu)}\int_{\xi_1}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} \\
&= \overline{z} + I_1(z(\xi_1^-)) + \frac{1}{\Gamma(\nu)}\int_{\mathtt{a}}^{\xi_1}\left(\log\frac{\xi_1}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} + \frac{1}{\Gamma(\nu)}\int_{\xi_1}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s}.
\end{aligned}
$$

If $\xi \in (\xi_2, \xi_3]$ then aqain Lemma 3.2, we get

$$
\begin{aligned}
z(\xi) &= z(\xi_2^+) + \frac{1}{\Gamma(\nu)}\int_{\xi_2}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} \\
&= \Delta z|_{\xi=\xi_2} + z(\xi_2^-) + \frac{1}{\Gamma(\nu)}\int_{\xi_2}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} \\
&= \overline{z} + I_2(z(\xi_2^-)) + I_1(z(\xi_1^-)) + \frac{1}{\Gamma(\nu)}\int_{\mathtt{a}}^{\xi_1}\left(\log\frac{\xi_1}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_{\xi_1}^{\xi_2}\left(\log\frac{\xi_2}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} + \frac{1}{\Gamma(\nu)}\int_{\xi_2}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s}.
\end{aligned}
$$

If $\xi \in (\xi_\kappa, \xi_{\kappa+1}]$

$$
\begin{aligned}
z(\xi) &= \overline{z} + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\vartheta(s)\frac{ds}{s} + \sum_{i=1}^{\kappa} I_i(z(\xi_i^-))
\end{aligned}
$$

Conversely, we assume that z is a solution of impulsive feactional inteqral equation (2.4). By a direct account, it followes that the solution presented by (2.4) satisfies.

The following lemma can be deduced from the previous one.

Lemma 2.3.

The formula of solutions for the impulsive problem (2.1)-(2.3), should be written as follows

$$z(\xi)=\begin{cases}\Phi(\xi), \quad if \quad \xi\in[a-\tau,a],\\[2ex] \Phi(a)+\frac{1}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}\\[2ex] +\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}\\[2ex] +\sum_{a<\xi_\kappa<\xi}I_\kappa(z(\xi_\kappa^-)),\ if\ \xi\in[a,T],\ \ where\ \kappa=1,...,w.\end{cases}$$

2.2.2 First type: Problems with real valued data-functions

In the following parts, we prove of results existence and uniqueness for solutions to the initial value problem (2.1)-(2.3), using different of fixed point theorems.

We will show that problem (2.1)-(2.3) has a unique solution using Banach fixed point theorem

.

We will need to introduce the following hypotheses :

(A1) There exists a constant $l_1>0$ such that

$$|\Psi(\xi,u)-\Psi(\xi,v)|\leq l_1\|u-v\|,\ \text{ for each } \xi\in J \text{ and } u,v\in C_\tau.$$

(A2) There exists a constant $l_2>0$ such that

$$|I_\kappa(x)-I_\kappa(z)|\leq l_2\|x-z\|,\ \text{ for each } x,z\in R \text{ and } \kappa=1,...,w.$$

Theorem 2.4.
Assume that conditions (A1) and (A2) hold, if

$$\left[\frac{l_1(w+1)\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}+wl_2\right]<1, \tag{2.8}$$

then the problem (2.1)-(2.3) has a unique solution on $[a-\tau,T]$.

Proof: Transform the problem (2.1)-(2.3) into a fixed point problem. Let the operator $N : \mathrm{B} \to \mathrm{B}$ be defined by :

$$(Nz)(\xi) = \begin{cases} \Phi(\xi), \quad \text{if} \quad \xi \in [\mathtt{a}-\tau, \mathtt{a}], \\ \\ \Phi(\mathtt{a}) + \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \Psi(s, z_s) \frac{ds}{s} \\ \\ +\frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \Psi(s, z_s) \frac{ds}{s} \\ \\ +\sum_{\mathtt{a}<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)), \text{ if } \xi \in [\mathtt{a}, \mathtt{T}], \end{cases} \tag{2.9}$$

where $\kappa = 1, ..., w$. Clearly, of lemma 3.2 the fixed point of N is a solution to problem (2.1)-(2.3), we prove N is a contraction will show that N has a fixed point. Let $x, z \in \mathrm{B}$. Then, for each $\xi \in J$, we have

$$\begin{aligned} |(Nx)(\xi) - (Nz)(\xi)| &= |\frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \Psi(s, x_s) - \Psi(s, z_s) \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \Psi(s, x_s) - \Psi(s, z_s) \frac{ds}{s} \\ &+ \sum_{\mathtt{a}<\xi_\kappa<\xi} I_\kappa(x(\xi_\kappa^-) - I_\kappa(z(\xi_\kappa^-)| \\ &\leq \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\ &+ \sum_{\mathtt{a}<\xi_\kappa<\xi} |I_\kappa(x(\xi_\kappa^-) - I_\kappa(z(\xi_\kappa^-)| \\ &\leq \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} l_1 |x_s - z_s|_{C_\tau} \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} l_1 |x_s - z_s|_{C_\tau} \frac{ds}{s} + \sum_{\mathtt{a}<\xi_\kappa<\xi} l_2 |x(\xi_\kappa^-) - z(\xi_\kappa^-)| \\ &\leq \frac{l_1 \sup_{\xi \in J} |x_\xi - z_\xi|}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\mathtt{a}<\xi_\kappa<\xi} l_2 \sup_{\xi_\kappa \in J} |x(\xi_\kappa^-) - z(\xi_\kappa^-)| \\
&+ \frac{l_1 \sup_{\xi \in J} |x_\xi - z_\xi|}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \frac{ds}{s} \\
&\leq \frac{l_1 \|x-z\|_\infty}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \frac{ds}{s} + wl_2 \|x-z\|_\infty \\
&+ \frac{l_1 \|x-z\|_\infty}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \frac{ds}{s} \\
&\leq \frac{l_1 w \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} \|x-z\|_\infty + \frac{l_1 \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} \|x-z\|_\infty + wl_2 ||x-z||_\infty \\
&\leq \left[\frac{l_1 (w+1) \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + wl_2 \right] \|x-z\|_\infty .
\end{aligned}
$$

Thus

$$
\|N(x) - N(z)\|_\infty \leq \left[\frac{l_1 (w+1) \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + wl_2 \right] \|x-z\|_\infty .
$$

Consequently by (2.8), the N is a contraction a conclusion of Banach fixed point theorem, we deduce that N has a unique fixed point which is solution to problem (2.1)-(2.3).

The proven the existence of a solution for problem (2.1)-(2.3) by using Schaefer fixed point theorem.

Theorem 2.5.

Assume that the following conditions hold :

(A3) The function $\Psi : J * C_\tau \to R$ is continuous.

(A4) There exists a constant $M > 0$ such that

$$
|\Psi(\xi, u)| \leq M, \quad \textit{for each } \xi \in J \quad \textit{and each } u \in C_\tau .
$$

(A5) The functions $I_\kappa : R \to R$ is continuous and there exists a constant $M^* > 0$ such that

$$
|I_\kappa(u)| \leq M^*, \quad \textit{for each} u \in R, \ \kappa = 1, \ldots, w,
$$

then problem (2.1)-(2.3) has at least one solution on $[\mathtt{a} - \tau, \mathtt{T}]$.

Proof: We consider the operator $N : \mathrm{B} \to \mathrm{B}$ defined by (2.9). The proof will be given in several steps.

Step 1: *N is continuous.*
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B, we have

$$\begin{aligned}|(Nz_n)(\xi) - (Nz)(\xi)| &\leq \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} (\log \frac{\xi_\kappa}{s})^{\nu-1} |\Psi(s, z_{ns}) - \Psi(s, z_s)| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} (\log \frac{\xi}{s})^{\nu-1} |\Psi(s, z_{ns}) - \Psi(s, z_s)| \frac{ds}{s} \\ &\quad + \sum_{\mathtt{a}<\xi_\kappa<\xi} |I_\kappa(z_n(\xi_\kappa^-)) - I_\kappa(z(\xi_\kappa^-))|.\end{aligned}$$

Since Ψ and $I_\kappa,\ \kappa = 1, \ldots, w$, are continuous functions, then we have

$$\|Nz_n - Nz\|_\infty \to 0 \ as \ n \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in* B.

Indeed, it is enough to show that for any η^*, there exists a positive constant l such that for each $z \in \Omega_{\eta^*} = \{z \in \mathrm{B} : \|z\|_\infty \leq \eta^*\}$, we have $\|N(z)\|_\infty \leq l$.
By (A4) and (A5), for each $\xi \in J$, we have

$$\begin{aligned}\|(Nz)(\xi)\| &\leq |\Phi(\mathtt{a})| + \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} |\Psi(s, z_s)| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} |\Psi(s, z_s)| \frac{ds}{s} + \sum_{\mathtt{a}<\xi_\kappa<\xi} |I_\kappa(z(\xi_\kappa^-)| \\ &\leq |\Phi(\mathtt{a})| + \frac{M}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \frac{ds}{s} \\ &+ \frac{M}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \frac{ds}{s} + \sum_{\mathtt{a}<\xi_\kappa<\xi} M^* |(z(\xi_\kappa^-)| \\ &\leq |\Phi(\mathtt{a})| + \frac{wM \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + \frac{M \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + wM^* \\ &\leq \left[\|\Phi(\mathtt{a})\| + \frac{M(w+1)\left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + wM^*\right] := l.\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets in B.
Let $z \in \Omega_{\eta^*}$ be a bounded set of B as in Step 2, then for any $\lambda_1, \lambda_2 \in J, \lambda_1 < \lambda_2$, we

have

$$
\begin{aligned}
|(Nz)(\lambda_2)-(Nz)(\lambda_1)| &\leq \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\lambda_1}\left[\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}-\left(\log\frac{\lambda_1}{s}\right)^{\nu-1}\right]|\Psi(s,z_s)|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\nu)}\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\\
&\leq \frac{M}{\Gamma(\nu)}\left[\int_{\xi_\kappa}^{\lambda_1}\left(\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}-\left(\log\frac{\lambda_1}{s}\right)^{\nu-1}\right)\frac{ds}{s}+\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}\frac{ds}{s}\right]\\
&\leq \frac{M}{\Gamma(\nu+1)}\left[2\left(\log\frac{\lambda_2}{\lambda_1}\right)+\left(\log\frac{\lambda_2}{\mathtt{a}}\right)-\left(\log\frac{\lambda_1}{\mathtt{a}}\right)\right].
\end{aligned}
$$

As $\lambda_1 \longrightarrow \lambda_2$, the right-hand side of the above inequality tends to zero.
As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem 1.10, we can conclude that $N : \mathrm{B} \to \mathrm{B}$ is completely continuous.

Step 4: *A priori bounds.*
Now it remains to show that the set

$$\varepsilon = \{z \in \mathrm{B} : z = \lambda N(z)\, for\ \ some\ \ 0 < \lambda < 1\},\ \text{is bounded.}$$

Let $z \in \varepsilon$, then $z = \lambda N(z)$ for some $0 < \lambda < 1$. Thus, for each $\xi \in J$, we have

$$z(\xi) = \lambda\Phi(\mathtt{a}) + \frac{\lambda}{\Gamma(\nu)}\sum_{\mathtt{a}<\xi_\kappa<\xi}\int_{\kappa-1}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} + \lambda\sum_{\mathtt{a}<\xi_\kappa<\xi}|I_\kappa(z(\xi_\kappa^-))|$$

$$+\frac{\lambda}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}(\log\frac{\xi}{s})^{\nu-1}\Psi(s,z_s)\frac{ds}{s}.$$

Then, by (A4)-(A5) and (as in Step 2), it follows that, for each $\xi \in J$, we have

$$|(Nz)(\xi)| \leq |\Phi(\mathtt{a})| + \frac{wM\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + \frac{M\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + wM^*.$$

Thus for every $z \in \varepsilon$, we have

$$\|Nz\|_\infty \leq \left[\|\Phi(\mathtt{a})\| + \frac{wM\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + \frac{M\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)} + wM^*\right] := \eta^*.$$

This shows that the set ε is bounded. Consequently, we deduce by the Schaefer fixed point theorem that N has a fixed point which is a solution of the problem (2.1)-(2.3).

The next result prove the existence of solution for problem (2.1)-(2.3) is based on Leray-Schauder nonlinear alternative type

.

Theorem 2.6. *Assume that (A3) and the following conditions hold :*

(A6) There exist $\phi_\Psi \in C(J, R^+)$ and $\psi : [0,\infty) \to [0,\infty)$ continuous and nondecreasing such that

$$|\Psi(\xi, u)| \leq \phi_\Psi(\xi)\psi(\|u\|) \ \ for \ \ each \ \ \xi \in J, \ \ u \in C_\tau.$$

(A7) There exists $\psi^* : [0,\infty) \to [0,\infty)$ continuous and nondecreasing such that

$$|I_\kappa(u)| \leq \psi^*(\|u\|) \ \ for \ \ all \quad u \in R, \ \ \kappa = 1, \ldots, w.$$

(A8) There exists a number $\overline{M} > 0$ such that

$$\frac{\overline{M}}{\|\Phi(a)\| + \overline{\Phi}_\Psi\psi(\overline{M})\frac{w\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + \overline{\Phi}_\Psi\psi(\overline{M})\frac{\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + w\psi^*(\overline{M})} > 1,$$

where $\overline{\Phi}_\Psi = \sup\{\phi_\Psi(\xi) : \xi \in J\}$.

Then problem (2.1)-(2.3) has at least one solution on $[a - \tau, T]$.

Proof: We prove that the operator N : B $\to$ B defined by (2.9) is continuous and completely continuous.

Step 1: *Continuity of N.*

Let $\{z_n\}_{n\in N}$ be a sequence such that $z_n \to z$ in B, then

$$\begin{aligned}
|(Nz_n)(\xi) - (Nz)(\xi)| \ \ &\leq \ \ \frac{1}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}(\log\frac{\xi_\kappa}{s})^{\nu-1}|\Psi(s, z_{ns}) - \Psi(s, z_s)|\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}(\log\frac{\xi}{s})^{\nu-1}|\Psi(s, z_{ns}) - \Psi(s, z_s)|\frac{ds}{s} \\
&\quad + \sum_{a<\xi_\kappa<\xi}|I_\kappa(z_n(\xi_\kappa^-)) - I_\kappa(z(\xi_\kappa^-))|.
\end{aligned}$$

The continuity of the functions Ψ and I_κ, $\kappa = 1, \ldots, w$, yield to:

$$\|Nz_n - Nz\|_\infty \to 0 \ as \ n \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in* B.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant l^* such that for each $z \in \Omega_{\eta^*}$, $\|N(z)\|_\infty \leq l^*$. By (A6) and (A7), for each $\xi \in J$, we have

$$\begin{aligned}
|(Nz)(\xi)| &\leq |\Phi(a)| + \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} |\Psi(s, z_s)| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} |\Psi(s, z_s)| \frac{ds}{s} + \sum_{a<\xi_\kappa<\xi} |I_\kappa(z(\xi_\kappa^-)| \\
&\leq |\Phi(a)| + \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \phi_\Psi(s)\psi(\|z_s\|) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \phi_\Psi(s)\psi(\|z_s\|) \frac{ds}{s} + \sum_{a<\xi_\kappa<\xi} \psi^*(\|(z(\xi_\kappa^-)\|) \\
&\leq |\Phi(a)| + \frac{\overline{\Phi}_\Psi \psi(\|z\|)}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \frac{ds}{s} \\
&+ \frac{\overline{\Phi}_\Psi \psi(\|z\|)}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \frac{ds}{s} + \sum_{a<\xi_\kappa<\xi} \psi(\|z\|) \\
&\leq |\Phi(a)| + \frac{w\overline{\Phi}_\Psi \psi(\|z\|) \left(\log \frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + \frac{\overline{\Phi}_\Psi \psi(\|z\|) \left(\log \frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + w\psi(\|z\|) \\
&\leq \left[\|\Phi(a)\| + \frac{(w+1)\overline{\Phi}_\Psi \psi(\|z\|)}{\Gamma(\nu+1)} \left(\log \frac{T}{a}\right)^\nu + w\psi(\|z\|)\right] := l^*.
\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets in B.
Let $z \in \Omega_{\eta^*}$ be a bounded set of B as in Step 2, then for any $\lambda_1, \lambda_2 \in J, \lambda_1 < \lambda_2$, we have

$$\begin{aligned}
|(Nz)(\lambda_2) - (Nz)(\lambda_1)| &\leq \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\lambda_1} \left[\left(\log \frac{\lambda_2}{s}\right)^{\nu-1} - \left(\log \frac{\lambda_1}{s}\right)^{\nu-1}\right] |\Psi(s, z_s)| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \int_{\lambda_1}^{\lambda_2} \left(\log \frac{\lambda_2}{s}\right)^{\nu-1} |\Psi(s, z_s)| \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\lambda_1} \left[\left(\log \frac{\lambda_2}{s}\right)^{\nu-1} - \left(\log \frac{\lambda_1}{s}\right)^{\nu-1}\right] \phi_\Psi(s)\psi(\|z_s\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \int_{\lambda_1}^{\lambda_2} \left(\log \frac{\lambda_2}{s}\right)^{\nu-1} \phi_\Psi(s)\psi(\|z_s\| \frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\overline{\Phi}_{\Psi}\psi(\|z\|)}{\Gamma(\nu)}\int_{\xi_\kappa}^{\lambda_1}\left[\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}-\left(\log\frac{\lambda_1}{s}\right)^{\nu-1}\right]\frac{ds}{s}\\
&\quad+\frac{\overline{\Phi}_{\Psi}\psi(\|z\|)}{\Gamma(\nu)}\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}\frac{ds}{s}\\
&\leq \frac{\overline{\Phi}_{\Psi}\psi(\|z\|)}{\Gamma(\nu+1)}\left[2\left(\log\frac{\lambda_2}{\lambda_1}\right)+\left(\log\frac{\lambda_2}{\mathtt{a}}\right)-\left(\log\frac{\lambda_1}{\mathtt{a}}\right)\right].
\end{aligned}
$$

As $\lambda_1 \longrightarrow \lambda_2$, the right-hand side of the above inequality tends to zero.
As a consequence of Steps 1 to 3, it follows by the Arzela-Ascoli theorem 1.10 that $N : \mathrm{B} \to \mathrm{B}$ is continuous and completely continuous.

Step 4: *N We show that there exists an open sets in $U \subset \mathrm{B}$ with $z(\xi) \neq \lambda(Nz)(\xi)$ for $\lambda \in (0,1)$ and $0 < \lambda < 1$ and $z \in \partial U$.*
Let $z \in \mathrm{B}$, be such that $z(\xi) = \lambda(Nz)(\xi)$, for each $\xi \in \mathrm{J}$, defined by (2.2.2), we have

$$
\begin{aligned}
\|(Nz)(\xi)\| &\leq |\Phi(\mathtt{a})| + \frac{1}{\Gamma(\nu)}\sum_{\mathtt{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\\
&+ \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}+\sum_{\mathtt{a}<\xi_\kappa<\xi}|I_\kappa(z(\xi_\kappa^-)|\\
&\leq \|\Phi(\mathtt{a})\| + \frac{1}{\Gamma(\nu)}\sum_{\mathtt{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}(\log\frac{\xi_\kappa}{s})^{\nu-1}\phi_\Psi(s)\psi(\|z_s\|)\frac{ds}{s}\\
&+ \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}(\log\frac{\xi}{s})^{\nu-1}\phi_\Psi(s)\psi(\|z_s\|)\frac{ds}{s}+\sum_{\mathtt{a}<\xi_\kappa<\xi}\psi^*(\|z(\xi_\kappa^-)\|)\\
&\leq \|\Phi(\mathtt{a})\| + \overline{\Phi}_\Psi\psi(\|z\|_\infty)\frac{w\left(\log\frac{T}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)}+\overline{\Phi}_\Psi\psi(\|z\|_\infty)\frac{\left(\log\frac{T}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)}+w\psi^*(\|z\|_\infty).
\end{aligned}
$$

Thus

$$
\frac{\|z\|_\infty}{\|\Phi(\mathtt{a})\| + \overline{\Phi}_\Psi\psi(\|z\|_\infty)\frac{w\left(\log\frac{T}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)}+\overline{\Phi}_\Psi\psi(\|z\|_\infty)\frac{\left(\log\frac{T}{\mathtt{a}}\right)^\nu}{\Gamma(\nu+1)}+w\psi^*(\|z\|_\infty)}\leq 1.
$$

Then by condition (A8), there exists $\overline{M}$ such that $\|z\|_\infty \neq \overline{M}$.
Let

$$
U = \{z \in \mathrm{B} : \|z\|_\infty < \overline{M}\}.
$$

The operator $N : \overline{U} \to \mathrm{B}$ is continuous and completely continuous. From the choice of U, there is no $z \in \partial U$ such that $z = \lambda N(z)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that N has a fixed point $z \in \overline{U}$ which is a solution of the problem (2.1)-(2.3).

2.2.3 Example

Example 1

We consider the followinq equation :

$$ {}^{C}_{H}D^{\frac{1}{2}} z(\xi) = \frac{|z_\xi|}{100(1+e^\xi)(1-|z_\xi|)}, \; for \;\; \xi \in J = [\frac{1}{2}, 2], \xi \neq \frac{7}{4}, \tag{2.10} $$

$$ \Delta z|_{\xi = \frac{7}{4}} = \frac{1}{8} z(\frac{7}{4}), \tag{2.11} $$

$$ z(\xi) = \Phi(\xi), \;\; \xi \in [\frac{1}{2} - \tau, \frac{1}{2}], \;\; \tau > 0, \tag{2.12} $$

where

$$ \Psi(\xi, z) = \frac{z}{100(1+e^\xi)(1-z)}, \;\; (\xi, z) \in J * C([\frac{1}{2} - \tau, \frac{1}{2}], R), $$

$$ I_\kappa(v) = \frac{1}{8} v, \;\; v \in R, $$

with $\mathtt{T} = 2$, $\nu = \frac{1}{2}$, $\mathtt{a} = \frac{1}{2}$ *and* $w = 1$.
Let $x, z \in C([\frac{1}{2} - \tau, \frac{1}{2}], R)$ *and* $\xi \in J$. *Then we have*

$$ \begin{aligned} |\Psi(\xi, x) - \Psi(\xi, z)| &= \frac{1}{100(1+e^\xi)} \left| \frac{x}{1+x} - \frac{z}{1+z} \right| \\ &= \frac{1}{100(1+e^\xi)} \frac{|x-z|}{|(1+x)(1+z)|} \\ &\leq \frac{|x-z|}{100(9+e^\xi)} \leq \frac{1}{200}|x-z|. \end{aligned} $$

Hence the condition (H1) holds with $l_1 = \frac{1}{200}$.
Let $x, z \in R$, *then we have*

$$ |I_\kappa(x) - I_\kappa(z)| = \left| \frac{1}{8x} - \frac{1}{8y} \right| \leq \frac{1}{8}|x-z|. $$

Hence the condition (H2) holds with $l_2 = \frac{1}{8}$, *we shall check that condition (2.8) is satisfied . Indeed,*

$$\left[\frac{l_1(w+1)}{\Gamma(\nu+1)}\left(\log\frac{T}{a}\right)^{\nu} + wl_2\right] = \left[\frac{\frac{1}{200}(1+1)}{\Gamma(\frac{3}{2})}(\log 4)^{\frac{1}{2}} + \frac{1}{8}\right]$$

$$= 0,13446421 < 1,$$

then by theorem (2.4) the problem (3.11)-(3.13) has a unique solution on $[\frac{1}{2} - \tau, 2]$.

Example 2

We consider the following functional impulsive fractional differential equation :

$$ {}^{C}_{H}D^{\frac{1}{2}} z(\xi) = \frac{1}{5}\sin(z(\xi - \frac{\pi}{2})),\ \xi \in J = [\frac{\pi}{4}, \frac{5\pi}{4}], \xi \neq \frac{\kappa\pi}{4},\ \kappa = 1,2,3, \tag{2.13}$$

$$\Delta z|_{\frac{\kappa\pi}{4}} = \frac{1}{5}z(\frac{\kappa\pi}{4}),\quad \xi = \frac{\kappa\pi}{4},\ \kappa = 1,2,3, \tag{2.14}$$

$$z(\xi) = \sin(\xi),\ \ \xi \in [-\frac{\pi}{2}, \frac{\pi}{4}]. \tag{2.15}$$

where $T = \frac{5\pi}{4}$, $w = 3$, $\nu = \frac{1}{2}$, *and*

$$\Psi(\xi, z) = \frac{1}{5}\sin(z(\xi - \frac{\pi}{2})),\ \ (\xi, z) \in J * C([-\frac{\pi}{2}, \frac{\pi}{4}], R),$$

$$I_\kappa(v) = \frac{1}{5}v,\ \ v \in R.$$

Here $|\Psi(\xi, z)| \le \frac{1}{5}$, *and* $|I_\kappa(z)| \le \frac{1}{5}$.

Hence, assumptions (H4) and (H5) are satisfied with $M = M^* = \frac{1}{5}$.
Then by theorem (2.5), the problem (2.13)-(2.15) has a solutions on $[-\frac{\pi}{2}, \frac{5\pi}{4}]$.

2.2.4 Second type: Problems with valued data-functions in Banach spaces

This part is devoted to the study of the existence of solutions for problem (2.1)-(2.3), in which the function is given by $\Psi : J \to C([a - \tau, a], \chi) \to \chi$, $I_\kappa : \chi \to \chi$, *where the existence findings are presented for the problem using a measure of noncompactness and a Mönch's fixed point theorem.*

We present the following Banach space:

$$AC' = \left\{ \begin{array}{l} z\,:J \to\ \chi,\ z \in AC_\delta((\xi_\kappa, \xi_{\kappa+1}],\ \chi),\ \kappa = 1, \ldots, w,\ \textit{and there exist} \\ z(\xi_\kappa^+)\ \textit{and}\ z(\xi_\kappa^-),\ \kappa = 1, \ldots, w,\ \textit{with}\ z(\xi_\kappa^-) = z(\xi_\kappa) \end{array} \right\}.$$

with the norm

$$\|z\|_{AC'} = \sup\{|z(\xi)| : a \le \xi \le T\}.$$

Set

$$\overline{B} = \{z : (a - \tau, T] \to\ \chi \setminus\ z \in AC' \cap C_\tau\}.$$

Before starting and verifying the results, we introduce the following the assumptions.

(H1) $\Psi : J * C_\tau \to \chi$ *is a Carathéodory function.*

(H2) $\exists\, p \in C(J, R_+)$:

$$\|\Psi(\xi, z)\| \le p(\xi)\|z\|_{C_\tau}, \quad \textit{for each } \xi \in J \quad \textit{and } z \in C_\tau.$$

(H3) $\exists\, c > 0$ *such that*

$$\|I_\kappa(x)\| \le c\|x\|, \quad \textit{for each } x \in \chi.$$

(H4) $\forall\, M \subset \chi$ *bounded, we have*

$$\mu(I_\kappa(M)) \le c\mu(M), \quad \textit{for each } \kappa = 1, 2, ..., w.$$

(H5) $\forall\, \xi \in J$ *and each bounded set* $M \subset \chi$*, we have*

$$\mu(\Psi(\xi, M)) \le p(\xi)\mu(M),$$

where μ *is a measure of noncompactness on* χ*.*

Theorem 2.7. *Suppose that conditions (H1)-(H5) hold, if*

$$\left[P_\Psi \frac{(w+1)}{\Gamma(\nu+1)} \left(\log \frac{T}{a}\right)^\nu + wc\right] < 1, \tag{2.16}$$

where

$$p_\Psi = \sup_{\xi \in J} p(\xi)$$

then the problem IVP (2.1)-(2.3) has at least one solution on $[a - \tau, T]$*.*

Proof : *Consider the operator* $F : \overline{B} \to \overline{B}$ *defined as :*

$$(Fz)(\xi) = \begin{cases} \Phi(\xi), \quad \text{if} \quad \xi \in [a-\tau, a], \\ \\ \Phi(a) + \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \Psi(s, z_s) \frac{ds}{s} \\ \\ + \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \Psi(s, z_s) \frac{ds}{s} \\ \\ + \sum_{a<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)), \ \text{if} \ \xi \in [a, T], \end{cases} \tag{2.17}$$

where $\kappa = 1, ..., w$.

Let us fix

$$\eta \geq \frac{\|\phi(a)\|}{1 - \frac{P_f(m+1)\left(\log \frac{T}{a}\right)^r}{\Gamma(r+1)} - mc}.$$

and consider the set

$$D_\eta = \{y \in \overline{B} : \|y\| \leq \eta\}$$

Obviously, the subset D_η *is closed, bounded and convex. Clearly, that fixed points of operator* F *are solutions of the problem (2.1)-(2.3). the proof will be given in three steps.* **Step 1:** F is continuous.

let $\{z_n\}_{n\in N}$ *be a sequence such that* $z_n \to z$ *in* B. *If* $\xi \in [a-\tau, a]$, *we get*

$$\|F(z_n)(\xi) - F(z)(\xi)\| = \|\Phi(\xi) - \Phi(\xi)\| = 0$$

For $\xi \in [a, T]$, *we get*

$$\begin{aligned} \|F(z_n)(\xi) - F(z)(\xi)\| &\leq \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} (\log \frac{\xi}{s})^{\nu-1} \|\Psi(s, z_n(s)) - \Psi(s, z(s))\| \frac{ds}{s} \\ &+ \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} (\log \frac{\xi_\kappa}{s})^{\nu-1} \|\Psi(s, z_n(s)) - \Psi(s, z(s))\| \frac{ds}{s} \\ &+ \sum_{a<\xi_\kappa<\xi} \|I_\kappa(z_n(\xi_\kappa^-)) - I_\kappa(z(\xi_\kappa^-))\| \end{aligned}$$

.

Since $z_n \to z$ as $n \to \infty$, and I_κ is continuous and Ψ is of Caratheodory, then by using the Lebesgus dominated convergence theorem, we have

$$\|F(z_n)(\xi) - F(z)(\xi)\| \to 0 \ as \ n \to \infty.$$

Step 2: F maps D_η into itself.

If $\xi \in [\mathtt{a} - \tau, \mathtt{a}]$, then $\|F(z)(\xi)\| \leq \|\Phi(\xi)\|_{C_\tau} \leq \|\Phi\| \leq \eta$.
Let $z \in D_\eta$. By (H2)-(H3), show that $F(z) \in D_\eta$, for each $\xi \in [\mathtt{a}, \mathtt{T}]$, we have

$$\begin{aligned}
\|F(z)(\xi)\| &\leq \|\Phi(\mathtt{a})\| + \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \|\Psi(s, z_s)\| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \|\Psi(s, z_s)\| \frac{ds}{s} + \sum_{\mathtt{a}<\xi_\kappa<\xi} \|I_\kappa(z(\xi_\kappa^-)\| \\
&\leq \|\Phi(\mathtt{a})\|_{C_\tau} + \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} P(s)\|z_s\|_{C_\tau} \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} P(s)\|z_s\|_{C_\tau} \frac{ds}{s} + \sum_{\mathtt{a}<\xi_\kappa<\xi} c\|z(\xi_\kappa^-)\| \\
&\leq \|\Phi(\mathtt{a})\|_{C_\tau} + \frac{P_\Psi\|z\|}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \frac{ds}{s} \\
&+ \frac{P_\Psi\|z\|}{\Gamma(\nu)} \sum_{\mathtt{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \frac{ds}{s} + wc\|z\| \\
&\leq \|\Phi(\mathtt{a})\| + \left[P_\Psi \frac{(w+1)}{\Gamma(\nu+1)} \left(\log \frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + wc\right] \|z\| \\
&\leq \eta
\end{aligned}$$

from which follows that for each $\xi \in [\mathtt{a} - \tau, \mathtt{T}]$, which implies that $\|Fz\| \leq \eta$.
Since $F(D_\eta) \subset D_\eta$, then $F(D_\eta)$ is bounded.

Step 3: $F(D_\eta)$ is bounded and equicontinuous.

By step 2, it is obvious that it $F(D_\eta) = \{F(z) : z \in D_\eta\} \subset D_\eta$ is bounded.
Let $\tau_1, \tau_2 \in [\mathtt{a} - \tau, \mathtt{T}]$, with $\tau_1 < \tau_2$, if $\mathtt{a} - \tau \leq \tau_1 < \tau_2 \leq \mathtt{a}$, then

$$\|(Fz)(\tau_2) - (Fz)(\tau_1)\| = \|\Phi(\tau_2) - \Phi(\tau_1)\| = 0 \to 0, \ as \ \tau_2 \to \tau_1.$$

For $\mathtt{a} \leq \tau_1 < \tau_2 \leq \mathtt{T}$, we have

$$\|(Fz)(\tau_2) - (Fz)(\tau_1)\| \ \leq \ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\tau_1} (\left(\log \frac{\tau_2}{s}\right)^{\nu-1} - \left(\log \frac{\tau_1}{s}\right)^{\nu-1}) \|\Psi(s, z_s)\| \frac{ds}{s}$$

$$\begin{aligned}
&+ \frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}\|\Psi(s,z_s)\|\frac{ds}{s}\\
&\leq \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1}-\left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]p(s)\|z_s\|_{C_\tau}\frac{ds}{s}\\
&+ \frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}p(s)\|z_s\|_{C_\tau}\frac{ds}{s}\\
&\leq \frac{p_\Psi\|z\|}{\Gamma(\nu)}\int_{\xi_\kappa}^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1}-\left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]\frac{ds}{s}\\
&+ \frac{p_\Psi\|z\|}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}\frac{ds}{s}\\
&\leq \frac{p_\Psi\|z\|}{\Gamma(\nu+1)}\left[2\left(\log\frac{\tau_2}{\tau_1}\right)^{\nu}+\left(\log\frac{\tau_2}{\mathtt{a}}\right)^{\nu}-\left(\log\frac{\tau_1}{\mathtt{a}}\right)^{\nu}\right],\ \textit{as}\ \tau_1\to\tau_2.
\end{aligned}$$

Hence, for $\xi \in [\mathtt{a}-\tau, \mathtt{T}]$, *we get*

$$\|F(z)(\tau_2)-F(z)(\tau_1)\| = 0 \to 0,\ \textit{as}\ \tau_2\to\tau_1.$$

Hence $F(D_\eta)$ *is bounded and equicontinuous.*
Finally, we need to prove the implication, we have $F : D_\eta \longrightarrow D_\eta$ *bounded and continuous.*

$$\boldsymbol{V} = \overline{conv}F(\boldsymbol{V})\ \textit{or}\ \boldsymbol{V} = F(\boldsymbol{V})\cup\{0\} \Rightarrow \mu(\boldsymbol{V}) = 0$$

Let $\boldsymbol{V}\subset D_\eta$ *such that* $\boldsymbol{V}\subset\overline{conv}(F(\boldsymbol{V})\cup\{0\})$. $\boldsymbol{V}$ *is bounded and equicontinuous and therefore the function* $\xi\to\upsilon(\xi)=\mu(\boldsymbol{V}(\xi))$ *is continuous on* $[\mathtt{a}-\tau,\mathtt{T}]$.
If $\xi\in[\mathtt{a}-\tau,\mathtt{a}]$, *we have*

$$\upsilon(\xi) = \mu(F\boldsymbol{V})(\xi)) = \mu\left(\Phi(\xi)\right) = 0$$

If $\xi\in[\mathtt{a},\mathtt{T}]$, *using lemma 1.10, the properties (1.2.1) of the measure* μ, *we have*

$$\mu\left((F\boldsymbol{V})(\xi)\cup\{0\}\right) = \max\{\mu\left((F\boldsymbol{V})(\xi)\right),\mu\left(\{0\}\right)\}$$

$$\begin{aligned}
\upsilon(\xi) &= \mu\left((F\boldsymbol{V})(\xi)\right)\\
&\leq \mu\left((F\boldsymbol{V})(\xi)\cup\{0\}\right)\\
&\leq \mu\left(F(\boldsymbol{V})(\xi)\right)
\end{aligned}$$

where $\mu\left(x_0+A\right)=\mu\left(A\right)$, *for any* $x_0\in\chi$, *we find that*

$$
\begin{aligned}
\mu\left(F(\boldsymbol{V})(\xi)\right) &= \mu\{\Phi(\mathfrak{a})+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)}\sum_{\mathfrak{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} \\
&+ \sum_{\mathfrak{a}<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)),\ z\in\boldsymbol{V}\} \\
&= \mu\{\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{\mathfrak{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} \\
&+ \sum_{\mathfrak{a}<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)),\ z\in\boldsymbol{V}\}
\end{aligned}
$$

In view $\mu\left((A_1+A_2\right)=\mu\left(A_1\right)+\mu\left(A_2\right)$, *we obtain*

$$
\begin{aligned}
\mu\left(F(\boldsymbol{V})(\xi)\right) &\leq \mu\left\{\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}, z\in\boldsymbol{V}\right\} \\
&+ \mu\left\{\frac{1}{\Gamma(\nu)}\sum_{\mathfrak{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}, z\in\boldsymbol{V}\right\} \\
&+ \mu\left\{\sum_{\mathfrak{a}<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)), z\in\boldsymbol{V}\right\}
\end{aligned}
$$

On the other hand, the condition (H4) and (H5), we have

$$
\begin{aligned}
\mu\left(F(\boldsymbol{V})(\xi)\right) &\leq \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\mu\left(\Psi(s,\boldsymbol{V}_s)\right)\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)}\sum_{\mathfrak{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\mu\left(\Psi(s,\boldsymbol{V}_s)\right)\frac{ds}{s} \\
&+ \sum_{\mathfrak{a}<\xi_\kappa<\xi}\mu\left(I_\kappa(\boldsymbol{V}(\xi_\kappa^-))\right) \\
&\leq \frac{1}{\Gamma\left(\nu\right)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}p(s)\mu(\boldsymbol{V}_s)\frac{ds}{s}+\sum_{\mathfrak{a}<\xi_\kappa<\xi}c\mu(\boldsymbol{V}(\xi_\kappa^-)) \\
&+ \frac{1}{\Gamma\left(\nu\right)}\sum_{\mathfrak{a}<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}p(s)\mu(\boldsymbol{V}_s)\frac{ds}{s}
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}p(s)\upsilon_s\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}p(s)\upsilon_s\frac{ds}{s}\\
&+\sum_{a<\xi_\kappa<\xi}c\upsilon(\xi_\kappa^-)\\
&\leq \frac{P_\Psi\|\upsilon\|}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\frac{ds}{s}+\frac{p_\Psi\|\upsilon\|}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\frac{ds}{s}+wc\|\upsilon\|\\
&\leq \|\upsilon\|\left[p_\Psi\frac{(w+1)}{\Gamma(\nu+1)}\left(\frac{T}{a}\right)^{\nu}+wc\right].
\end{aligned}$$

Hence

$$\|\upsilon\|\left[1-\left(p_\Psi\frac{(w+1)}{\Gamma(\nu+1)}\left(\log\frac{T}{a}\right)^{\nu}+wc\right)\right]\leq 0$$

From (2.16), we get $\|\upsilon\|=0$, that is $\upsilon(\xi)=\mu(\boldsymbol{V}(\xi))=0$ for each $\xi\in J$ and then $\boldsymbol{V}(\xi)$ is relatively compact in χ, together with the Ascoli-Arzela theorem, $\boldsymbol{V}$ is relatively compact in D_η. Using theorem 1.14, we conclude that F has a fixed point which is a solution to the problem (2.1)-(2.3).

2.2.5 Example

We consider the following functional impulsive fractional differential equations of the form :

$$ {}_H^C D^{\frac{1}{2}} z(\xi)=\frac{1}{e^{\xi+2}}\|z_\xi\|,\quad \xi\in J=[1,e],\xi_1\neq\frac{1}{2}, \tag{2.18}$$

$$\Delta z|_{\frac{1}{2}}=\frac{1}{5}z\left(\frac{1^-}{2}\right)\ ,\quad \xi_1=\frac{1}{2}, \tag{2.19}$$

$$z(\xi)=\Phi(\xi),\ \ \xi\in[1-\tau,1]. \tag{2.20}$$

Set

$$\chi=l^1=\{z=(z_1,z_2,...,z_n,...):\sum_{n=1}^{\infty}|z_n|<\infty\},$$

with the norm

$$\|z_n\|_\chi=\sum_{n=1}^{\infty}|z_n|,$$

Here

$$\Psi_n(\xi, z_\xi) = \frac{1}{e^{\xi+2}}|z_n| \ \ , \ \ (\xi, z) \in J * C([a - \tau, a], \chi),$$

and

$$I_\kappa(z) = \frac{1}{5}|z_n|, \qquad z \in \chi.$$

Where $z = (z_1, z_2, ..., z_n, ...), \ \Psi = (\Psi_1, \Psi_2, ..., \Psi_n, ...)$.

Clearly, condition (H1) holds and (H2)-(H3) are satisfied with

$$\|\Psi(\xi, z)\| \leq p(\xi)\|z\|,$$

where

$$p(\xi) = \frac{1}{e^{\xi+2}}.$$

$p_\Psi = e^{-2}$ *and* $c = \frac{1}{5}$, $T = e$, $\nu = \frac{1}{2}$, $a = 1$ *and* $w = 1$.

We shall show that condition (2.16) is satisfied,Indeed

$$\left[P_\Psi \frac{(w+1)}{\Gamma(\nu+1)}\left(\log\frac{T}{a}\right)^\nu + wc\right] = \left[e^{-2}\frac{2}{\Gamma(\frac{3}{2})}\left(\log e\right)^{\frac{1}{2}} + \frac{1}{5}\right]$$
$$= 0,505419028 < 1.$$

Therefore, we conclude from the conclusion of theorem 2.7 the problem (2.18)-(2.20) has at least one solution on $[1, e]$.

2.3 Impulsive Fractional Neutral Differential Equations with Delay

In this section, we study the existence and uniqueness of solutions to the following impulsive fractional neutral differential equations with delay :

$${}^C_H D^\nu \left[z\left(\xi\right) - g\left(\xi, z_\xi\right)\right] = \Psi\left(\xi, z_\xi\right), \qquad \xi \in J := \left[a, T\right], \xi \neq \xi_\kappa, 0 < \xi \leq 1 \qquad (2.21)$$

$$\Delta z \mid_{\xi=\xi_\kappa} = I_\kappa\left(z\left(\xi_\kappa^-\right)\right), \qquad \kappa = 1, .., w, \qquad (2.22)$$

$$z(\xi) = \Phi(\xi), \quad \xi \in (a - \tau, a], \qquad (2.23)$$

where Ψ, Φ, *and* I_κ *are as in problem (2.1)-(2.3), and* $g : J * C([\mathfrak{a}-\tau, \mathfrak{a}], R) \to R$ *is a given function.*

We shall present two existence and uniqueness results for the problem (2.21)-(2.23). The first result is based on Banach fixed point theorem, while the other is based upon for the nonlinear alternative of Leray-Schauder type.

2.3.1 Existence and Uniquenss Results

Definition 2.8. A function $z \in \mathrm{B}$ is said to be a solution of (2.21)-(2.23), if z satisfies the equation ${}^{C}_{H}D^{\nu}\left[z\left(\xi\right) - g\left(\xi, z_{\xi}\right)\right] = \Psi(\xi, z_{\xi})$ on J' and satisfies conditions (2.22)-(2.23).

Our first exstence result for the (2.21)-(2.23) is based on the Banach fixed piont theorem .

Theorem 2.9. *Assume that conditions (A1)-(A2) and following hypothesi hold*

(M1) There exists a constant $l_g > 0$ *such that*

$$|g(\xi, u) - g(\xi, v)| \leq l_g \|u - v\|_{C_\tau} \ \textit{for each } \xi \in J \ \textit{ and } u, v \in C_\tau.$$

If

$$\left[l_g + \frac{l_1(w+1)\left(\log \frac{T}{\mathfrak{a}}\right)^{\nu}}{\Gamma(\nu+1)} + wl_2\right] < 1, \tag{2.24}$$

then the problem (2.21)-(2.23) has a unique solution on $[\mathfrak{a}-\tau, T]$.

Proof: *Consider the operator* $N_1 : \mathrm{B} \to \mathrm{B}$ *defined by :*

$$N_1 z\left(\xi\right) = \begin{cases} \Phi(\xi), \quad \textit{if} \quad \xi \in (\mathfrak{a}-\tau, \mathfrak{a}], \\[2ex] \Phi(\mathfrak{a}) - g(\mathfrak{a}, \Phi(\mathfrak{a})) + g(\xi, z_\xi) + \frac{1}{\Gamma(\nu)} \sum\limits_{\mathfrak{a}<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log \frac{\xi_\kappa}{s}\right)^{\nu-1} \Psi(s, z_s) \frac{ds}{s} \\[2ex] + \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \Psi(s, z_s) \frac{ds}{s} + \sum\limits_{\mathfrak{a}<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)), \ \textit{if } \xi \in [\mathfrak{a}, T], \ \kappa = 1, ..., w. \end{cases} \tag{2.25}$$

To show the operator N_1 is a contraction, let $x, z \in \mathrm{B}$. Then, for each $\xi \in J$ we have

$$\begin{aligned}
|N_1(x)(\xi) - N_1(z)(\xi)| &\leq |g(\xi, x_\xi) - g(\xi, z_\xi)| \\
&+ \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\
&\quad + \sum_{a<\xi_\kappa<\xi} |I_\kappa(x(\xi_\kappa^-)) - I_\kappa(z(\xi_\kappa^-))| \\
&\leq l_g \|x_\xi - z_\xi\|_{C_\tau} + \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1} l_1 \|x_s - z_s\|_{C_\tau} \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} l_1 \|x_s - z_s\|_{C_\tau} \frac{ds}{s} + \sum_{a<\xi_\kappa<\xi} l_2 \|x(\xi_\kappa^-) - z(\xi_\kappa^-)\| \\
&\leq l_g \|x - z\|_\infty + \frac{l_1 \|x - z\|_\infty}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi} \int_{\xi_{\kappa-1}}^{\xi_\kappa} \left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1} \frac{ds}{s} \\
&+ \frac{l_1 \|x - z\|_\infty}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} \frac{ds}{s} + w l_2 \|x - z\|_\infty \\
&\leq l_g \|x - z\|_\infty + \frac{l_1 w \left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} \|x - z\|_\infty + \frac{l_1 \left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} \|x - z\|_\infty \\
&+ w l_2 ||x - z||_\infty \\
&\leq \left[l_g + \frac{l_1 (w+1) \left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + w l_2 \right] \|x - z\|_\infty.
\end{aligned}$$

Thus

$$\|N_1(x) - N_1(z)\|_\infty \leq \left[l_g + \frac{l_1 (w+1) \left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + w l_2 \right] \|x - z\|_\infty.$$

Consequently by (2.24), N_1 is a contraction. As a consequence of the Banach fixed point theorem, we deduce that N_1 has a unique fixed point which is the solution of the problem (2.21)-(2.23).

The second existence result for thr problem (2.21)-(2.23) based on the nonlinear alternative of Leray-Schauder

Theorem 2.10. *Assume that (H3),(H6)-(H7) and the following conditions hold:*

(M2) *The function $g : J * C_\tau \to R$ is continuous.*

(M3) *There exist $\Phi_g \in C(J, R^+)$ and $\psi_1 : [0,\infty) \to [0,\infty)$ continuous and non-decreasing such that*

$$|g(\xi,u)| \leq \Phi_g(\xi)\psi_1(\|u\|_{C_\tau}) \ \ for \ \ all \ \ \xi \in J, \ \ u \in C_\tau.$$

(M4) *There exists a number $\overline{M_1} \geq 0$ such that*

$$\frac{\overline{M_1}}{\|\Phi(a)\| + \|g(a,\Phi(a))\| + \Phi_g\psi_1(\overline{M_1}) + \psi(\overline{M_1})\frac{(w+1)\phi_\Psi\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + w\psi^*(\overline{M_1})} > 1,$$

where $\Phi_g = \sup\{\phi_g(\xi) : \xi \in J\}$,

then(2.21)-(2.23) has at least one solution on $[a-\tau, T]$.

Proof: *Consider the operator N_1 : B $\to$ B defined in 2.25. It can be easily shown that N_1 is continuous and completely continuous.*
For $\lambda \in [0,1]$, let z be such that, for each $\xi \in J$, we have $z(\xi) = \lambda(N_1 z)(\xi)$.

$$\begin{aligned}(z)(\xi) &= \lambda\Phi(a) - \lambda g(a,\Phi(a)) + \lambda g(\xi,z_\xi) + \frac{\lambda}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} \\ &+ \frac{\lambda}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} + \lambda\sum_{a<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-))\end{aligned}$$

Then from (A6)-(A7), and (M2)-(M3) for each $\xi \in J$, we have

$$\begin{aligned}|z(\xi)| &\leq |\Phi(a)| + |g(a,\Phi(a))| + |g(\xi,z_\xi)| \\ &+ \frac{1}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}+\sum_{a<\xi_\kappa<\xi}|I_\kappa(z(\xi_\kappa^-))|\\
\leq\ & |\Phi(a)|+|g(a,\Phi(a))|+\Phi_g(\xi)\psi_1(\|z_\xi\|)\\
+\ & \frac{1}{\Gamma(\nu)}\sum_{a<\xi_\kappa<\xi}\int_{\xi_{\kappa-1}}^{\xi_\kappa}(\log\frac{\xi_\kappa}{s})^{\nu-1}\phi_\Psi(s)\psi(\|z_s\|)\frac{ds}{s}\\
+\ & \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}(\log\frac{\xi}{s})^{\nu-1}\phi_\Psi(s)\psi(\|z_s\|)\frac{ds}{s}+\sum_{a<\xi_\kappa<\xi}\psi^*(\|z\|)\\
\leq\ & \|\Phi(a)\|+\|g(a,\Phi(a))\|+\Phi_g\psi_1(\|z\|_\infty)+\phi_\Psi\psi(\|z\|_\infty)\frac{w\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}\\
+\ & \phi_\Psi\psi(\|z\|_\infty)\frac{\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}+w\psi^*(\|z\|_\infty)\\
\leq\ & \|\Phi(a)\|+\|g(a,\Phi(a))\|+\Phi_g\psi_1(\|z\|_\infty)\\
&+\phi_\Psi\psi(\|z\|_\infty)\frac{(w+1)\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}+w\psi^*(\|z\|_\infty).
\end{aligned}
$$

Thus

$$
\frac{\|z\|_\infty}{\|\Phi(a)\|+\|g(a,\Phi(a))\|+\Phi_g\psi_1\|z\|_\infty+\phi_\Psi\psi(\|z\|_\infty)\frac{(w+1)\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}+w\psi^*(\|z\|_\infty)}\leq 1.
$$

Then by condition (M4), there exists $\overline{M}$ *such that* $\|z\|_\infty\neq\overline{M}$. *Let*

$$
U=\{z\in \mathrm{B}:\|z\|_\infty<\overline{M}\}.
$$

The operator $N_1:\overline{U}\to \mathrm{B}$ *is continuous and completely continuous. From the choice of* U, *there is no* $z\in\partial U$ *such that* $z=\lambda N_1(z)$ *for some* $\lambda\in(0,1)$. *As a consequence of the nonlinear alternative of Leray-Schauder, we deduce that* N_1 *has a fixed point* $z\in\overline{U}$ *which is a solution of the problem (2.21)-(2.23).*

Chapter 3

Impulsive Fractional Differential Equations with Caputo-Hadamard Derivative of order $1 < \nu \leq 2$

3.1 Introduction

In this chapter, we concentrate on the existence and uniqueness of solutions for a classes of functional impulsive fractional differential equations with Caputo-Hadamard derivative.Here results are discussed, the based on fixed point theorems the Banach, Schaufer's and Leary-Schauder

Consider the space

$$AC'(J,R) = \left\{ \begin{array}{l} z : J \to R,\ z \in AC^2_\delta((\xi_\kappa, \xi_{\kappa+1}], R) \ \textit{and there exist } z(\xi_\kappa^+) \\ \textit{and } \ z(\xi_\kappa^-),\ \ \kappa = 1, ..., w,\ \textit{with},\ \ z(\xi_\kappa^-) = z(\xi_\kappa) \end{array} \right\},$$

endowed with the norm

$$\|z\|_{AC'} = \sup\{|z(\xi)| : \mathtt{a} \leq \xi \leq \mathtt{T}\}.$$

Let B^* be set defined by

$$B^* = \{z : (\mathtt{a} - \tau, \mathtt{T}] \to R \ \setminus \ z \in AC'(J,R) \cap C_\tau\},$$

with the norm

$$\|z\|_{B^*} = \sup\{|z(\xi)| : \xi \in [\mathtt{a} - \tau, \mathtt{T}]\}.$$

3.2 Initial Value Problem (IVP) with Delay

In this section, we study the existence and uniqueness of solutions for the following initial value problem, for impulsive fractional differential equations with delay and Caputo-Hadamard derivative of the form :

$$ {}_{H}^{C}D^{\nu}z(\xi) = \Psi(\xi, z_{\xi}), \quad \xi \in J = [a, T], \xi \neq \xi_{\kappa}, a > 0,\ 1 < \nu \leq 2. \tag{3.1}$$

$$\Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \tag{3.2}$$

$$\Delta z' \mid_{\xi=\xi_{\kappa}} = \overline{I}_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi = \xi_{\kappa}, \quad \kappa = 1, .., w, \tag{3.3}$$

$$z(\xi) = \Phi(\xi), \quad \xi \in (a - \tau, a],\ z'(a) = \gamma, \tag{3.4}$$

where ${}_{H}^{C}D^{\nu}$ *is the Caputo-Hadamard fractional derivative of order* $1 < \nu \leq 2$, $a > 0$, $\Psi : J{*}C_{\tau} \to R$ *, and* $\gamma \in R$ *are given continuous functions,* $\Phi \in C([a-\tau, a], R)$ *and* $I_{\kappa}, \overline{I_{\kappa}} \in C(R, R)$, $\kappa = 1, 2, ..., w$, $a = \xi_0 < \xi_1 < ... < \xi_w < \xi_{w+1} = T$. *For any continuous functions z defined on* $J' = J \setminus \{\xi_1, ..., \xi_w\}$, $\Delta z|_{\xi=\xi_{\kappa}} = z(\xi_{\kappa}^{+}) - z(\xi_{\kappa}^{-})$ *and* $\Delta z'|_{\xi=\xi_{\kappa}} = z'(\xi_{\kappa}^{+}) - z'(\xi_{\kappa}^{-})$, $z(\xi_{\kappa}^{+})$, $z(\xi_{\kappa}^{-})$ *represent the right and the left limits of* $z(\xi)$ *at* $\xi = \xi_{\kappa}$.

3.2.1 Equivalent Integral Equation

Definition 3.1. A function $z \in B^*$ is said to be a solution of the problem (3.1)-(3.4), if z satisfies the fractional differential equation ${}_{H}^{C}D^{\nu}z(\xi) = \Psi(\xi, z_{\xi})$ on J' and the following conditions (3.2)-(3.4).

Lemma 3.2. *Let* $\rho \in AC(J, R)$. *A function z is a solution of the IVP:*

$$ {}_{H}^{C}D^{\nu}z(\xi) = \rho(\xi), \quad \xi \in J = [a, T], \xi \neq \xi_{\kappa}, a > 0,\ 1 < \nu \leq 2. \tag{3.5}$$

$$\Delta z \mid_{\xi=\xi_{\kappa}} = I_{\kappa}(z(\xi_{\kappa}^{-})), \quad ,\xi = \xi_{\kappa}, \quad \kappa = 1, .., w \tag{3.6}$$

$$\Delta z\prime \mid_{\xi=\xi_{\kappa}} = \overline{I}_{\kappa}(z(\xi_{\kappa}^{-})), \quad ,\xi = \xi_{\kappa}, \quad \kappa = 1, .., w \tag{3.7}$$

$$z(a) = z_{a}, \quad z\prime(a) = \gamma, \tag{3.8}$$

has if z is a solution of the fractional integral equation

$$z(\xi)=\begin{cases} z_{a}+a\gamma(\log\frac{\xi}{a})+\frac{1}{\Gamma(\nu)}\int_{a}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s}, \quad \text{if } \xi\in[a,\xi_1] \\ z_{a}+a\gamma(\log\frac{\xi}{a})+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s} \\ +\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s}+\sum_{i=1}^{\kappa}I_i\left(z\left(\xi_i^-\right)\right) \\ +\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\rho(s)\frac{ds}{s} \\ +\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-)), \ \text{if } \xi\in(\xi_\kappa,\xi_{\kappa+1}],\ \kappa=1,...,w. \end{cases} \tag{3.9}$$

Proof: *Let z be the solution of (3.5)-(3.8), if $\xi\in J_0=[a,\xi_1]$, by Lemma (1.39), we get*

$$\begin{aligned} z(\xi) &= I_1^{\nu}\rho(\xi)+c_0+c_1(\log\frac{\xi}{a}) \\ &= \frac{1}{\Gamma(\nu)}\int_{a}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s}+c_0+c_1(\log\frac{\xi}{a}), \quad c_0,c_1\in R. \end{aligned}$$

$$z'(\xi)=\frac{1}{\xi\Gamma(\nu-1)}\int_{a}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-2}\rho(s)\frac{ds}{s}+\frac{c_1}{\xi}$$

Appliyinq the condition

$$z(a)=z_{a}, \quad z'(a)=\gamma,$$

we deduce that

$$c_0=z_{a}, \quad c_1=a\gamma,$$

we obtain

$$z(\xi)=z_{a}+a\gamma(\log\frac{\xi}{a})+\frac{1}{\Gamma(\nu)}\int_{a}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s}$$

$$z'(\xi)=\frac{a\gamma}{\xi}+\frac{1}{\xi\Gamma(\nu-1)}\int_{a}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-2}\rho(s)\frac{ds}{s}$$

If $\xi \in (\xi_1, \xi_2]$, *then*

$$z(\xi) = \frac{1}{\Gamma(\nu)} \int_{\xi_1}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-1} \rho(s) \frac{ds}{s} + c_0 + c_1 (\log \frac{\xi}{\xi_1})$$

and

$$z'(\xi) = \frac{1}{\xi \Gamma(\nu - 1)} \int_{\xi_1}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-2} \rho(s) \frac{ds}{s} + \frac{c_1}{\xi}$$

Usinq the impulses conditions

$$\Delta z \mid_{\xi=\xi_1} = z(\xi_1^+) - z(\xi_1^-) = I_1(z(\xi_1^-))$$

$$\Delta z' \mid_{\xi=\xi_1} = z'(\xi_1^+) - z'(\xi_1^-) = \bar{I}_1(z(\xi_1^-)),$$

we obtain

$$c_0 = I_1(z(\xi_1^-)) + \gamma + \mathfrak{a}\gamma (\log \frac{\xi_1}{\mathfrak{a}}) + \frac{1}{\Gamma(\nu)} \int_{\mathfrak{a}}^{\xi_1} \left(\log \frac{\xi_1}{s}\right)^{\nu-1} \rho(s) \frac{ds}{s},$$

and

$$\frac{c_1}{\xi_1} = \bar{I}_1(z(\xi_1^-)) + \frac{\mathfrak{a}\gamma}{\xi_1} + \frac{1}{\xi_1 \Gamma(\nu - 1)} \int_{\mathfrak{a}}^{\xi_1} \left(\log \frac{\xi_1}{s}\right)^{\nu-2} \rho(s) \frac{ds}{s}$$

$$c_1 = \xi_1 \bar{I}_1(z(\xi_1^-)) + \mathfrak{a}\gamma + \frac{1}{\Gamma\nu - 1)} \int_{\mathfrak{a}}^{\xi_1} \left(\log \frac{\xi_1}{s}\right)^{\nu-2} \rho(s) \frac{ds}{s}$$

For $\xi \in (\xi_1, \xi_2]$, *we have*

$$\begin{aligned} z(\xi) &= z_{\mathfrak{a}} + \mathfrak{a}\gamma (\log \tfrac{\xi}{\mathfrak{a}}) + \tfrac{1}{\Gamma(\nu)} \textstyle\int_{\xi_1}^{\xi} \left(\log \tfrac{\xi}{s}\right)^{\nu-1} \rho(s) \tfrac{ds}{s} \\ &+ \tfrac{1}{\Gamma(\nu)} \textstyle\int_{\mathfrak{a}}^{\xi_1} \left(\log \tfrac{\xi_1}{s}\right)^{\nu-1} \rho(s) \tfrac{ds}{s} + \tfrac{(\log \frac{\xi}{\xi_1})}{\Gamma(\nu-1)} \textstyle\int_{\mathfrak{a}}^{\xi_1} \left(\log \tfrac{\xi_1}{s}\right)^{\nu-2} \rho(s) \tfrac{ds}{s} \\ &+ \xi_1 (\log \tfrac{\xi}{\xi_1}) \bar{I}_1(z(\xi_1^-)) + I_1(z(\xi_1^-)). \end{aligned}$$

$$\begin{aligned} z'(\xi) &= \frac{\mathfrak{a}\gamma}{\xi} + \frac{1}{\xi \Gamma(\nu - 1)} \int_{\xi_1}^{\xi} \left(\log \frac{\xi}{s}\right)^{\nu-2} \rho(s) \frac{ds}{s} \\ &+ \frac{(1}{\xi \Gamma((\nu - 1)} \int_{\mathfrak{a}}^{\xi_1} \left(\log \frac{\xi_1}{s}\right)^{\nu-2} \rho(s) \frac{ds}{s} + \frac{\xi_1}{\xi} \bar{I}_1(z(\xi_1^-)) \end{aligned}$$

Repeatinq the process in this ways, the solution $z(\xi)$ for $\xi_\kappa \in [\xi_\kappa, \xi_{\kappa+1}]$ where $\kappa = 1, ..., w$, can be written as

$$\begin{aligned} z(\xi) &= z_{\mathtt{a}} + \mathtt{a}\gamma(\log\frac{\xi}{\mathtt{a}}) + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s} \\ &+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\rho(s)\frac{ds}{s} + \sum_{i=1}^{\kappa} I_i(z(\xi_i^-)) \\ &+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\rho(s)\frac{ds}{s} + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-)) \end{aligned}$$

3.2.2 Existence and Uniqueness Results

First result: Existence and uniqueness

The first result, stated in the following theorem is based on Banach fixed point theorem.

Theorem 3.3.

Assume that :

(A1) There exists constant $l > 0$ such that

$$|\Psi(\xi, y) - \Psi(\xi, z)| \le l|y - z|, \ \ \textit{for each } \xi \in J, \ \ \textit{and } y, z \in C_\tau.$$

(A2) There exists constant $l_1 > 0$, such that

$$|I_\kappa(x) - I_\kappa(r)| \le l_1|x - r|, \ \ , \ \ \textit{for each} \ \ x, r \in R, \kappa = 1, 2, ..., w.$$

(A3) There exists constant $l_2 > 0$, such that

$$|\overline{I}_\kappa(x) - \overline{I}_\kappa(r)| \le l_2|x - r|, \ \ \ \textit{for each } \xi \in J, \ \ \textit{and all } x, r \in R, \kappa = 1, 2, ..., w.$$

If the condition

$$\left[l\left(\frac{1+w}{\Gamma(\nu+1)} + \frac{w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + wl_1 + wT(\log\frac{\mathtt{T}}{\mathtt{a}})l_2\right] < 1, \tag{3.10}$$

then the problem (3.1)-(3.4) has a unique solution on $[\mathtt{a} - \tau, \mathtt{T}]$.

Proof: *Transform the problem (3.1)-(3.4) into a fixed point problem. Consider the operator $F : B^* \to B^*$ defined by:*

$$(Fz)(\xi) = \begin{cases} \phi(\xi) \qquad , \qquad if \;\; \xi \in (a-\tau, a]. \\ \phi(a) + a\gamma(\log\frac{\xi}{a}) + \frac{1}{\Gamma(\nu)} \sum_{a<\xi_\kappa<\xi_{\xi_{\kappa-1}}}^{\xi_\kappa} \left(\log\frac{\xi_\kappa}{s}\right)^{\nu-1} \Psi(s, z_s)\frac{ds}{s} \\ + \frac{1}{\Gamma(\nu)} \int_a^{\xi_\kappa} \left(\log\frac{\xi}{s}\right)^{\nu-1} \Psi(s, z_s)\frac{ds}{s} + \sum_{a<\xi_\kappa<\xi} I_\kappa(z(\xi_\kappa^-)) \\ + \sum_{i=1}^{\kappa} \frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)} \int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-2} \Psi(s, z_s)\frac{ds}{s} \\ + \sum_{i=1}^{\kappa} \xi_i \left(\log\frac{\xi}{\xi_i}\right) \bar{I}_i(z(\xi_i^-)), \qquad if\; \xi \in [a, T]. \end{cases}$$

Clearly, of lemma 3.2 the fixed point of F is a solution to problem (3.1)-(3.4). Let $x, z \in B^*$, if $\xi \in [a-\tau, a]$ we have

$$\|(Fx)(\xi) - (Fz)(\xi)\| = \|\phi(\xi) - \phi(\xi)\| = 0$$

If $\xi \in [a, T]$, $\kappa = 1, ..., w$,we have

$$\begin{aligned} |(Fx)(\xi) - (Fz)(\xi)| \;\; &\leq \;\; \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{\kappa} \int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-1} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\ &\quad + \sum_{i=1}^{\kappa} \frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)} \int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-2} |\Psi(s, x_s) - \Psi(s, z_s)| \frac{ds}{s} \\ &\quad + \sum_{i=1}^{\kappa} |I_i(x(\xi_i^-)) - I_i(z(\xi_i^-))| + \sum_{i=1}^{\kappa} \xi_i \left(\log\frac{\xi}{\xi_i}\right) |\bar{I}_i(x(\xi_i^-)) - \bar{I}_i(z(\xi_i^-))| \\ &\leq \;\; \frac{1}{\Gamma(\nu)} \int_{\xi_\kappa}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} l\|x - z\| \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{\kappa} \int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-1} l\|x - z\| \frac{ds}{s} \\ &\quad + \sum_{i=1}^{\kappa} \frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)} \int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-2} l\|x - z\| \frac{ds}{s} \end{aligned}$$

$$+\sum_{i=1}^{\kappa} l_1|(x(\xi_i^-)) - (z(\xi_i^-))| + \sum_{i=1}^{\kappa} \xi_i \left(\log \frac{\xi}{\xi_i}\right) l_2|(x(\xi_i^-)) - (z(\xi_i^-))|$$

$$\leq \left[l\frac{(1+w)}{\Gamma(\nu+1)} \left(\log \frac{T}{a}\right)^{\nu} + l\frac{w}{\Gamma(\nu)} \left(\log \frac{T}{a}\right)^{\nu} + wl_1 + wTl_2 \left(\log \frac{T}{a}\right)\right] \|x - z\|.$$

Thus, we have

$$\|Fx - Fz\| \leq \left[l \left(\frac{1+w}{\Gamma(\nu+1)} + \frac{w}{\Gamma(\nu)} \right) \left(\log \frac{T}{a}\right)^{\nu} + ml_1 + mT(\log \frac{T}{a})l_2 \right] \|x - z\|.$$

Consequently by (3.10), the F is a contraction a conclusion of Banach fixed point theorem, we deduce that F has a unique fixed point which is solution to problem (3.1)-(3.4).

Second result

Our second result deals with the existence of solutions for problem (3.1)-(3.4) by applying Scheafer fixed point theorem.

Theorem 3.4. *Assume that the followinq conditions hold :*

(A4) *The function $\Psi : J * C_\tau \to R$ is continuous.*

(A5) *The functions $I_\kappa,\ \overline{I}_\kappa : R \to R$ are continuous .*

(A6) *There exists constant $M > 0$ such that $|\Psi(\xi, x)| \leq M$ for each $\xi \in J$ and each $x \in C_\tau$.*

(A7) *There exist a constants $N > 0, N^* > 0$ such that $|I_\kappa(x)| \leq N,\ |\overline{I}_\kappa(x)| \leq N^*$ for each $x \in R,\ \kappa = 1, \dots, w,$*

then the problem (3.1)-(3.4) has at least one solution in $[a - \tau, T]$.

Proof: We will show use Scheafer fixed point theorem to prove that F has a fixed point.
This will be achieved in the following series of steps.
Step 1: *F is continuous.*
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B^*.
If $\xi \in [\mathtt{a} - \tau, \mathtt{a}]$, then

$$\|F(z_n)(\xi) - F(z)(\xi)\| = 0.$$

If $\xi \in [\mathtt{a}, \mathtt{T}]$, we have

$$\begin{aligned}
|F(z_n)(\xi) - F(z)(\xi)| \leq\ & \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_{ns})-\Psi(s,z_s)|\frac{ds}{s} \\
& + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s,z_{ns})-\Psi(s,z_s)|\frac{ds}{s} \\
& + \sum_{\kappa=1}^{n}|I_\kappa(z_n(\xi_\kappa^-)) - I_\kappa(z(\xi_\kappa^-))| \\
& + \sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_{ns})-\Psi(s,z_s)|\frac{ds}{s} \\
& + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}(z_n(\xi_i^-)) - \bar{I}_i(z(\xi_i^-))|
\end{aligned}$$

Since $\Psi,\ g$ and $I_\kappa, \bar{I}_\kappa,\quad \kappa = 1,\ldots,w$, are continuous functions, we have

$$\|F(z_n) - F(z)\|_{B^*} \to 0 \ as\ n \to \infty.$$

Step 2: *F : maps bounded sets into bounded sets in B^*.*
Indeed, it is enough to show that for any η^*, there exists a positive constant l such that for each $z \in B_{\eta^*} = \{z \in B^* : \|z\|_{B^*} \leq \eta^*\}$, we have $\|F(z)\|_{B^*} \leq l$. By (A6)and (A7), we have for each $\xi \in J$,

$$\begin{aligned}
|F(z)(\xi)| \leq\ & |\phi(a)| + |a\gamma|(\log\frac{\xi}{a}) + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s} \\
& + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s} + \sum_{i=1}^{\kappa}|I_i(z(\xi_i^-))| \\
& + \sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}_i(z(\xi_i^-)| \\
\leq\ & |\phi(a)| + |a\gamma|(\log\frac{T}{a}) + \frac{M\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} + \frac{Mw\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)} \\
& + wN + \frac{Mw\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu)} + wT\left(\log Ta\right)N^* \\
\leq\ & |\phi(a)| + |a\gamma|(\log\frac{T}{a}) + M\left(\frac{1+w}{\Gamma(\nu+1)} + \frac{w}{\Gamma(\nu)}\right)\left(\log\frac{T}{a}\right)^\nu + wN + wT(\log\frac{T}{a})N^* := \eta^*.
\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets of B^* .
Let $\lambda_1, \lambda_2 \in J, \lambda_1 < \lambda_2,\ B_{\eta^*}$ be a bounded set of B^* as in Step 2,and let $z \in B^*$.

Then

$$\begin{aligned}
|F(z)(\lambda_2)-F(z)(\lambda_1)| &\leq \mathtt{a}\gamma(\log\frac{\lambda_2}{\lambda_1}) \\
&+\frac{1}{\Gamma(\nu)}\int_{\mathtt{a}}^{\lambda_1}(\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}-\left(\log\frac{\lambda_1}{s}\right)^{\nu-1})|\Psi(s,z_s)|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\nu)}\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s} \\
&+\sum_{\mathtt{a}<\xi_\kappa<\lambda_2-\lambda_1}\frac{\left(\log\frac{\lambda_2}{\lambda_1}\right)}{\Gamma(\nu-1)}\int_{\xi_{\kappa-1}}^{\xi_\kappa}\left(\log\frac{\xi_\kappa}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} \\
&+\sum_{\mathtt{a}<\xi_\kappa<\lambda_2-\lambda_1}\xi_\kappa\left(\log\frac{\lambda_2}{\lambda_1}\right)\bar{I}_\kappa(z(\xi_\kappa^-).
\end{aligned}$$

As $\lambda_1 \longrightarrow \lambda_2$, the right-hand side of the above ingality tends to zero.
As a consequence of steps 1 to 3, it follows by the Arzela-Ascoli theorem 1.10 that F is completly continuous.
Step 4: *A priori bounds.*
Let set

$$\varepsilon=\{z\in B^*: z=\lambda F(z)\, for\;\; some\;\; 0<\lambda<1\}$$

is bounded. Let $z\in\varepsilon$, then $z=\lambda F(z)$ for some $0<\lambda<1$. Thus, for each $\xi\in J$ we have

$$\begin{aligned}
z(\xi) &= \lambda\phi(\mathtt{a})+\mathtt{a}\lambda\gamma(\log\frac{\xi}{\mathtt{a}})+\frac{\lambda}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s} \\
&+\frac{\lambda}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}+\lambda\sum_{i=1}^{\kappa}I_i(z(\xi_i^-)) \\
&+\sum_{i=1}^{\kappa}\frac{\lambda\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s,z_s)\frac{ds}{s}+\lambda\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-))
\end{aligned}$$

This implies by (A6)-(A7), we obtain

$$\begin{aligned}
|z(\xi)| &\leq |\phi(\mathtt{a})|+|\mathtt{a}\gamma|\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)+\frac{M(1+w)}{\Gamma(\nu+1)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} \\
&+wN+\frac{wM}{\Gamma(\nu)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu}+mT\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)N^*.
\end{aligned}$$

Thus

$$\|z\|_\infty\leq\left[|z_{\mathtt{a}}|+|\mathtt{a}\gamma|\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)+M\left(\frac{1+w}{\Gamma(\nu+1)}+\frac{w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu}+wN+wT\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)N^*\right]:=R.$$

This shows that the set ε is bounded. Consequence we deduce be F has a fixed point which is a solution of the problem (3.1)-(3.4).

Third result

Our third result is obtained by applying the Leray-Schauder nonlinear alternative type.

Theorem 3.5. *Assume that (A4) and (A5) and the following conditions hold:*

(A8) There exist $\phi_\Psi \in C(J, R^+)$ and $\psi : [0,\infty) \to [0,\infty)$ continuous and non-decreasing such that

$$|\Psi(\xi, y)| \leq \phi_\Psi(\xi)\psi(|y|) \ \ for \ \ all \ \ \xi \in J, \ \ y \in C_\tau.$$

(A9) There exist ψ^*, $\bar{\psi}^* : [0,\infty) \to [0,\infty)$ continuous and nondecreasing such that

$$|I_\kappa(x)| \leq \psi^*(|x|) \ \ for \ \ all \ \ x \in R, \ \ \kappa = 1, \ldots, w.$$

$$|\bar{I}_\kappa(x)| \leq \bar{\psi}^*(|x|) \ \ for \ \ all \ \ x \in R, \ \ \kappa = 1, \ldots, w.$$

(A10) There exists $\mathtt{a}$ number $\overline{M} > 0$, let $\phi^* = \sup\{\phi_\Psi(\xi) : \xi \in J\}$ such that

$$\frac{\overline{M}}{|\phi(\mathtt{a})| + \phi^*\psi(\overline{M})\left(\frac{1+w}{\Gamma(\nu+1)} + \frac{w}{\Gamma(\nu)}\right)\left(\log\frac{T}{\mathtt{a}}\right)^\nu + w\psi^*(\overline{M}) + \left(|\mathtt{a}\gamma| + wT\bar{\psi}^*(\overline{M})\right)\left(\log\frac{T}{\mathtt{a}}\right)} \geq 1,$$

then the problem (3.1)-(3.4) has at least one solution on $[\mathtt{a} - \tau, T]$.

Proof: we consider the operator F defined by (3.11) . It can be easily shown the operator F is continuous and completely continuous. Let $\lambda \in [0,1]$, and $z(\xi) = \lambda(Fy)(\xi)$. For $\xi \in J$, we have

$$\begin{aligned}
|F(z)(\xi)| \ \leq\ & |\phi(\mathtt{a})| + |\mathtt{a}\gamma|(\log\frac{\xi}{\mathtt{a}}) + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s, z_s)|\frac{ds}{s} \\
& + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s, z_s)|\frac{ds}{s} + \sum_{i=1}^{\kappa}|I_i(z(\xi_i^-))| \\
& + \sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s, z_s)|\frac{ds}{s} + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}_i(z(\xi_i^-))| \\
\leq\ & |\phi(\mathtt{a})| + |\mathtt{a}\gamma|\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right) + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z(s)|)\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z(s)|)\frac{ds}{s}+\sum_{i=1}^{\kappa}\psi^*(|z|) \\
& +\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z(s)|)\frac{ds}{s}+\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{\psi}^*(|z|) \\
\leq\ & |\phi(\mathtt{a})|+|\mathtt{a}\gamma|(\log\frac{\mathtt{T}}{\mathtt{a}})+\phi^*\psi(|z|)\frac{(\log\frac{\mathtt{T}}{\mathtt{a}})^\nu}{\Gamma(\nu+1)}+\phi^*\psi(|z|)\frac{w(\log\frac{\mathtt{T}}{\mathtt{a}})^\nu}{\Gamma(\nu+1)} \\
& +w\psi^*(|z|)+\phi^*\psi(|z|)\frac{w(\log\frac{\mathtt{T}}{\mathtt{a}})^\nu}{\Gamma(\nu)}+wT(\log\frac{\mathtt{T}}{\mathtt{a}})\bar{\psi}^*(|z|) \\
\leq\ & |\phi(\mathtt{a})|+\phi^*\psi(|z|)\left(\frac{1+w}{\Gamma(\nu+1)}+\frac{w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu+w\psi^*(|z|) \\
& +\left(|\mathtt{a}\gamma|+wT\bar{\psi}^*(|z|)\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right).
\end{aligned}
$$

Also, the inequality

$$
\frac{\|z\|}{|\phi(\mathtt{a})|+\phi^*\psi(\|z\|)\left(\frac{1+w}{\Gamma(\nu+1)}+\frac{w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu+w\psi^*(\|z\|)+\left(|ay_1|+mT\bar{\psi}^*(\|z\|)\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)}\leq 1,
$$

which condition (A10), there exist $\overline{M}$ such that $\|z\| \neq \overline{M}$. Let us set

$$
U=\{z\in B^*:\|z\|\leq\overline{M}\}.
$$

Note, that the operator $F:\overline{U}\to B^*$ is continuous and completely continuous. From the choice of U, there is no $z\in\partial U$ such that $z=\lambda F(z)$ for some $\lambda\in(0,1)$. Consequently by the nonlinear alternative of Leray-Schauder theorem, F has a fixed point that is a solution of the problem (3.1)-(3.4).

3.2.3 Example

$$
{}^C_H D^{\frac{3}{2}} z(\xi)=\frac{1}{(e^\xi+4)^2}\frac{|z(\xi)|}{(1+|z(\xi)|)} \quad ,\xi\in J=[\frac{1}{6},2],\xi\neq\frac{1}{3}, \tag{3.11}
$$

$$
\Delta z(\frac{1}{3})=\frac{|z(\frac{1}{3}^-)|}{15+|z(\frac{1}{3}^-)|}, \tag{3.12}
$$

$$
\Delta z'(\frac{1}{3})=\frac{|z(\frac{1}{3}^-)|}{17+|z(\frac{1}{3}^-)|}, \tag{3.13}
$$

$$z(\xi) = \phi(\xi), \ \ \xi \in [\frac{1}{6} - \tau, \frac{1}{6}], \qquad z'(\frac{1}{6}) = 3. \tag{3.14}$$

Set

$$\Psi(\xi, z) = \frac{1}{(e^{\xi}+4)^2} \frac{z}{(1+z)}, \qquad (\xi, z) \in J * C([\frac{1}{6} - \tau, \frac{1}{6}], R).$$

$$I_\kappa(z) = \frac{z}{15+z}, \qquad z \in R.$$

$$\overline{I_\kappa}(z) = \frac{z}{17+z}, \qquad z \in R.$$

Let $x, z \in C([\frac{1}{6} - \tau, 2], R)$ and $\xi \in J$. Then we have

$$\begin{aligned} |\Psi(\xi, x) - \Psi(\xi, z)| &= \frac{1}{(e^{\xi}+4)^2} |\frac{x}{1+x} - \frac{z}{1+z}| \\ &\leq \frac{1}{(e^{\xi}+4)^2} \frac{|x-z|}{|(1+x)(1+z)|} \\ &\leq \frac{|x-z|}{(e^{\xi}+4)^2} \leq \frac{1}{36}|x-z|. \end{aligned}$$

Let $x, z \in R$, then we have

$$\begin{aligned} |I_\kappa(x) - I_\kappa(z)| &= |\frac{x}{17+x} - \frac{Y}{17+y}| \\ &\leq \frac{1}{17}|x-z| \end{aligned}$$

and

$$\begin{aligned} |I_\kappa(x) - I_\kappa(z)| &= |\frac{x}{17+x} - \frac{Y}{17+y}| \\ &\leq \frac{1}{17}|x-z|. \end{aligned}$$

Obviously, $l = \frac{1}{25}, l_1 = \frac{1}{15}, l_2 = \frac{1}{17}$, where $\nu = \frac{3}{2}, \mathtt{T} = 2, w = 1$.
Further,

$$\begin{aligned} & \left[l \left(\frac{1+w}{\Gamma(\nu+1)} + \frac{w}{\Gamma(\nu)} \right) \left(\log \frac{\mathtt{T}}{\mathtt{a}} \right)^{\nu} + wl_1 + wT(\log \frac{\mathtt{T}}{\mathtt{a}}) l_2 \right] \\ = & \left[\frac{1}{25} \left(\frac{2}{\Gamma(\frac{5}{2})} + \frac{1}{\Gamma(\frac{3}{2}}) \right) (\log 12)^{(\frac{3}{2})} + \frac{1}{15} + \frac{2(\log 12)}{17} \right] \\ = & \ 0,3132361029 < 1. \end{aligned}$$

Then all hypotheses of theorem (3.3) are fulfilled, and consequently the boundary value problem (3.1)-(3.4) has a unique solution on $[\frac{1}{6} - \tau, 2]$.

3.3 Boundary Value Problem (BVP) with Delay and Integral Condition

Our main objective in this section is to discuss the existence and uniqueness of solutions for the following boundary value problem (BVP for short), for impulsive fractional differential equations with delay .

$$ {}^{C}_{H}D^{\nu}z(\xi)=\Psi(\xi,z_{\xi}), \quad \xi\in J=[\mathtt{a},\mathtt{T}], \xi\neq\xi_{\kappa}, \quad \kappa=1,..,w, \tag{3.15}$$

$$\Delta z\mid_{\xi=\xi_{\kappa}}=I_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi=\xi_{\kappa}, \quad \kappa=1,..,w, \tag{3.16}$$

$$\Delta z'\mid_{\xi=\xi_{\kappa}}=\overline{I}_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi=\xi_{\kappa}, \quad \kappa=1,..,w, \tag{3.17}$$

$$z(\xi)=\Phi(\xi), \quad \xi\in[\mathtt{a}-\tau,\mathtt{a}], \quad z'(\mathtt{T})=\int_{\mathtt{a}}^{\mathtt{T}}\hbar(s,z(s))ds \tag{3.18}$$

where Ψ,Φ and I_{κ}, $\overline{I_{\kappa}}$ ere as in problem (3.1)-(3.4), and $\hbar : J*R\rightarrow R$ are given continuous functions.
We shall present the existence and uniqueness results for the problem (3.15)-(3.18) which rely Banach fixed point theorem. Then, results of the existence of solutions are proved using on the fixed point theorem of Schaefer and the nonlinear alternative of Leary-Schauder, we give illustrative example.

3.4 Equivalent Integral Equation

Definition 3.6. A function $z\in B^{*}$ is said to be a solution of problem (3.15)-(3.18) if z satisfies the equation ${}^{C}_{H}D^{\nu}z(\xi)=\Psi(\xi,z_{\xi})$ on J' and the conditions (3.16)-(3.18).

We need of the following lemma to which is important to obtain the integral solution of (3.15)-(3.18).

Lemma 3.7. *Assume that $\varrho,\overline{\varrho}\in AC^{2}(J,R)$, let $1<\nu\leq 2$, a function z is a solution of BVP :*

$$ {}^{H}_{C}D^{\nu}z(\xi)=\varrho(\xi), \quad \xi\in J=[a,T], \xi\neq\xi_{\kappa}, \tag{3.19}$$

$$\Delta z\mid_{\xi=\xi_{\kappa}}=I_{\kappa}(z(\xi_{\kappa}^{-})), \quad \xi=\xi_{\kappa}, \quad \kappa=1,...,w, \tag{3.20}$$

$$\Delta z'\mid_{\xi=\xi_{\kappa}}=\overline{I}_{\kappa}(z(\xi_{\kappa}^{-})) \ , \quad \xi=\xi_{\kappa}, \quad \kappa=1,..,w, \tag{3.21}$$

$$z(a)=\overline{z},\quad z'(T)=\int_a^T \overline{\varrho}(s)ds, \tag{3.22}$$

has is a solution of the fractional inteqral equation :

$$z(\xi)=\begin{cases} \overline{z}+T\left(\log\frac{\xi}{a}\right)\int_a^T\overline{\varrho}(s)ds-\left(\log\frac{\xi}{a}\right)[\frac{1}{\Gamma(\nu-1)}\int_{\xi_w}^T\left(\log\frac{T}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s} \\ +\sum_{i=1}^{w}\frac{1}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{w}(\xi_i)\bar{I}_i(z(\xi_i^-)] \\ +\frac{1}{\Gamma(\nu)}\int_a^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s}, \qquad\qquad if\ \ \xi\in[a,\xi_1] \\ \overline{z}+T\left(\log\frac{\xi}{a}\right)\int_a^T\overline{\varrho}(s)ds-\left(\log\frac{\xi}{a}\right)[\frac{1}{\Gamma(\nu-1)}\int_{\xi_w}^T\left(\log\frac{T}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s} \\ +\sum_{i=1}^{w}\frac{1}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{w}(\xi_i)\bar{I}_i(z(\xi_i^-)] \\ +\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s} \\ +\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{\kappa}I_i(z(\xi_i^-)) \\ +\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-)),\ \ if\quad \xi\in(\xi_\kappa,\xi_{\kappa+1}],\ \ \kappa=1,...,w. \end{cases} \tag{3.23}$$

Proof: By Lemma 1.39, let $1<\nu\leq 2$, for each $\xi\in[a,\xi_1]$, we get

$$z(\xi)={}_H I^{\nu}\varrho(\xi)+c_1+c_2\left(\log\frac{\xi}{a}\right)$$

$$=\frac{1}{\Gamma(\nu)}\int_a^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s}+c_1+c_2\left(\log\frac{\xi}{a}\right),\quad c_1,c_2\in R. \tag{3.24}$$

So applying the condition $z(a)=\overline{z}$, we deduce that $c_1=\overline{z}$, we have

$$z(\xi)=\overline{z}+\frac{1}{\Gamma(\nu)}\int_a^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s}+c_2\left(\log\frac{\xi}{a}\right).$$

Then, we obtain

$$z'(\xi) = \frac{c_2}{\xi} + \frac{1}{\xi\Gamma(\nu-1)} \int_{\mathsf{a}}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-2} \varrho(s)\frac{ds}{s}.$$

If $\xi \in (\xi_1, \xi_2]$, then

$$z(\xi) = \frac{1}{\Gamma(\nu)} \int_{\xi_1}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} \varrho(s)\frac{ds}{s} + d_1 + d_2\left(\log\frac{\xi}{\xi_1}\right),\ d_1,\ d_2 \in R. \tag{3.25}$$

$$z'(\xi) = \frac{1}{\xi\Gamma(\nu-1)} \int_{\xi_1}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-2} \varrho(s)\frac{ds}{s} + \frac{d_2}{\xi}.$$

Using the impulses conditions

$$\begin{cases} \Delta z\mid_{\xi=\xi_1} = z(\xi_1^+) - z(\xi_1^-) = I_1(z(\xi_1^-)) \\ \Delta z'\mid_{\xi=\xi_1} = z'(\xi_1^+) - z'(\xi_1^-) = \bar{I}_1(z(\xi_1^-)) \end{cases}$$

we obtain

$$\begin{aligned} z(\xi_1^+) &= d_1 = I_1(z(\xi_1^-)) + z(\xi_1^-) \\ &= I_1(z(\xi_1^-)) + \overline{z} + c_2\left(\log\frac{\xi_1}{\mathsf{a}}\right) + \frac{1}{\Gamma(\nu)} \int_{\mathsf{a}}^{\xi_1} \left(\log\frac{\xi_1}{s}\right)^{\nu-1} \varrho(s)\frac{ds}{s} \end{aligned}$$

$$\begin{aligned} z'(\xi_1^+) &= \frac{d_2}{\xi_1} = \bar{I}_1(z(\xi_1^-)) + z'(\xi_1^-) \\ &= \bar{I}_1(z(\xi_1^-)) + \frac{c_2}{\xi_1} + \frac{1}{\xi_1\Gamma(\nu-1)} \int_{\mathsf{a}}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-2} \varrho(s)\frac{ds}{s} \end{aligned}$$

$$d_2 = \xi_1\bar{I}_1(z(\xi_1^-)) + c_2 + \frac{1}{\Gamma(\nu-1)} \int_{\mathsf{a}}^{\xi_1} \left(\log\frac{\xi_1}{s}\right)^{\nu-2} \varrho(s)\frac{ds}{s}$$

The integro-differential equation (3.25), equivalent to :

$$\begin{aligned} z(\xi) &= \overline{z} + c_2\left(\log\frac{\xi}{\mathsf{a}}\right) + \frac{1}{\Gamma(\nu)} \int_{\xi_1}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} \varrho(s)\frac{ds}{s} + \frac{1}{\Gamma(\nu)} \int_{\mathsf{a}}^{\xi_1} \left(\log\frac{\xi_1}{s}\right)^{\nu-1} \varrho(s)\frac{ds}{s} \\ &+ \frac{\left(\log\frac{\xi}{\xi_1}\right)}{\Gamma(\nu-1)} \int_{\mathsf{a}}^{\xi_1} \left(\log\frac{\xi_1}{s}\right)^{\nu-2} \varrho(s)\frac{ds}{s} + \xi_1\left(\log\frac{\xi}{\xi_1}\right)\bar{I}_1(z(\xi_1^-)) + I_1(z(\xi_1^-)). \end{aligned}$$

Repeatinq the process in this ways, the solution $z(\xi)$ for $\xi_\kappa \in [\xi_\kappa, \xi_{\kappa+1}]$ where $\kappa = 1, ..., w$, can be written as

$$z(\xi) = \overline{z}+c_2\left(\log\frac{\xi}{\mathtt{a}}\right)+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\sigma(s)\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\sigma(s)\frac{ds}{s} \tag{3.26}$$

$$+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\sigma(s)\frac{ds}{s}+\sum_{i=1}^{\kappa}I_i(z(\xi_i^-))+\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-))$$

Continuing in the same manner, we obtain for $\xi \in (\xi_w, \mathtt{T}]$,

$$\begin{aligned} z(\xi) &= \overline{z}+c_2\left(\log\frac{\xi}{\mathtt{a}}\right)+\frac{1}{\Gamma(\nu)}\int_{\xi_w}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\varrho(s)\frac{ds}{s} \\ &\quad +\sum_{i=1}^{w}I_i(z(\xi_i^-))+\sum_{i=1}^{w}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{w}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-)) \end{aligned}$$

and

$$\begin{aligned} z'(\xi) &= \frac{c_2}{\xi}+\frac{1}{\xi\Gamma(\nu-1)}\int_{\xi_w}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{w}\frac{1}{\xi\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s} \\ &\quad +\sum_{i=1}^{w}\left(\frac{\xi_i}{\xi}\right)\bar{I}_i(z(\xi_i^-) \end{aligned}$$

By the application of the boundary condition $z'(\mathtt{T}) = \int_{\mathtt{a}}^{\mathtt{T}}\overline{\varrho}(s)ds$, we obtain

$$\begin{aligned} z'(\mathtt{T}) &= \frac{c_2}{\mathtt{T}}+\frac{1}{\mathtt{T}\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{w}\frac{1}{\mathtt{T}\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s} \\ &\quad +\sum_{i=1}^{w}\left(\frac{\xi_i}{\mathtt{T}}\right)\bar{I}_i(z(\xi_i^-)) \end{aligned}$$

We obtain the reguired value of the constant c_2 as

$$\begin{aligned} c_2 &= \mathtt{T}\int_{\mathtt{a}}^{\mathtt{T}}\overline{\varrho}(s)ds-[\frac{1}{\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s}+\sum_{i=1}^{w}\frac{1}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\varrho(s)\frac{ds}{s} \\ &\quad +\sum_{i=1}^{w}(\xi_i)\bar{I}_i(z(\xi_i^-\xi))], \end{aligned}$$

replacing the values of c_2 int(3.24) and (3.26), we get (3.23).

3.5 Existence Results

Our first result is based then the problem (3.15)-(3.18) has a unique of solution using Banach fixed point theorem.

Theorem 3.8. *Assume that :*

(A1) There exists a constant $L_1 > 0$ such that

$$|\Psi(\xi, x) - \Psi(\xi, y)| \leq L_1 \|x - y\|_{C_\tau}, \ \text{for each } \xi \in J, \ \ x, y \in C_\tau.$$

(A2) There exists a constant $L_2 > 0$ such that

$$|\hbar(\xi, x) - \hbar(\xi, z)| \leq L_2 |x - z|, \ \text{for each } \xi \in J \text{ and } x, z \in R.$$

(A3) For each $\kappa = 1, 2, ..., w$, there exist $l, l^* > 0$ such that

$$|I_\kappa(y) - I_\kappa(z)| \leq l|y - z|, \ \ \text{for each} \ \ y, z \in R.$$
$$|\overline{I}_\kappa(y) - \overline{I}_\kappa(z)| \leq l^*|y - z|, \ \ \text{for each} \ \ y, z \in R.$$

if the condition

$$\left[L_1\left(\frac{w+1}{\Gamma(\nu+1)} + \frac{2w+1}{\Gamma(\nu)}\right)\left(\log\frac{T}{a}\right)^\nu + L_2 T(T - a)\left(\log\frac{T}{a}\right) + wl + 2wl^* T\left(\log\frac{T}{a}\right)\right] < 1, \tag{3.27}$$

then the boundary value problem (3.15)-(3.18) has a unique solution in $[a - \tau, T]$.

Proof: Transform the problem (3.15)-(3.18) into a fixed point problem. Consider the operator $F : B^* \to B^*$ defined by:

$$(Fz)(\xi) = \begin{cases} \Phi(\xi), & \text{if} \quad \xi \in (\mathtt{a} - \tau, \mathtt{a}] \\ \Phi(\mathtt{a}) + \mathtt{T}\left(\log\frac{\xi}{\mathtt{a}}\right)\int_{\mathtt{a}}^{\mathtt{T}} \hbar(s, z(s))ds - \left(\log\frac{\xi}{\mathtt{a}}\right)\left[\frac{1}{\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2}\Psi(s, z_s)\frac{ds}{s}\right. \\ \left. +\frac{1}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s, z_s)\frac{ds}{s} + \sum_{i=1}^{w}\xi_i \bar{I}_i(z(\xi_i^-))\right] + \sum_{i=1}^{\kappa} I_i(z(\xi_i^-)) \\ +\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s, z_s)\frac{ds}{s} + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\Psi(s, z_s)\frac{ds}{s} \\ +\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s, z_s)\frac{ds}{s} + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(z(\xi_i^-)), & \text{if } \xi \in [\xi_\kappa, \xi_{\kappa+1}], \end{cases} \tag{3.28}$$

where $\kappa = 1, ..., w$
Clearly, of lemma 3.2 the fixed point of F is a solution to problem (3.15)-(3.18).
Let $x, z \in B^*$, if $\xi \in [\mathtt{a} - \tau, \mathtt{a}]$, we have

$$|(Fx)(\xi) - (Fz)(\xi)| = |\Phi(\xi) - \Phi(\xi)| = 0$$

If $\xi \in [\mathtt{a}, \mathtt{T}]$, by (A1)-(A3), we have

$$\begin{aligned}
|(Fx)(\xi) - (Fz)(\xi)| &= |\mathtt{T}\left(\log\frac{\xi}{\mathtt{a}}\right)\int_{\mathtt{a}}^{\mathtt{T}}\hbar(s, x(s))ds + \frac{\left(\log\frac{\xi}{\mathtt{a}}\right)}{\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2}\Psi(s, x_s)\frac{ds}{s} \\
&+\frac{\left(\log\frac{\xi}{\mathtt{a}}\right)}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s, x_s)\frac{ds}{s} + \left(\log\frac{\xi}{\mathtt{a}}\right)\sum_{i=1}^{w}\xi_i(\bar{I}_i(x(\xi_i^-))) \\
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s, x_s)\frac{ds}{s} + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\Psi(s, x_s)\frac{ds}{s} \\
&+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s, x_s)\frac{ds}{s} + \sum_{i=1}^{\kappa}I_i(x(\xi_i^-)) + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{I}_i(x(\xi_i^-))| \\
\leq\ & \mathtt{T}|\left(\log\frac{\xi}{\mathtt{a}}\right)|\int_{\mathtt{a}}^{\mathtt{T}}|\hbar(s, x(s)) - \hbar(s, z(s))|ds \\
&+\frac{|\left(\log\frac{\xi}{\mathtt{a}}\right)|}{\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2}|\Psi(s, x_s) - \Psi(s, z_s)|\frac{ds}{s} \\
&+\frac{|\left(\log\frac{\xi}{\mathtt{a}}\right)|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s, x_s) - \Psi(s, z_s)|\frac{ds}{s} \\
&+|\left(\log\frac{\xi}{\mathtt{a}}\right)|\sum_{i=1}^{w}\xi_i|\bar{I}_i(x(\xi_i^-)) - \bar{I}_i(z(\xi_i^-))| \\
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s, x_s) - \Psi(s, z_s)|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s, x_s) - \Psi(s, z_s)|\frac{ds}{s} \\
&+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s, x_s) - \Psi(s, z_s)|\frac{ds}{s} \\
&+\sum_{i=1}^{\kappa}|I_i(x(\xi_i^-)) - I_i(z(\xi_i^-))| + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}_i(x(\xi_i^-)) - \bar{I}_i(z(\xi_i^-))|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \mathtt{T}\left(\log\frac{\xi}{\mathtt{a}}\right)\int_{\mathtt{a}}^{\mathtt{T}} L_2|x(s)-z(s)|ds + \frac{|\left(\log\frac{\xi}{\mathtt{a}}\right)|}{\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2} L_1|x_s-z_s|_{C_\tau}\frac{ds}{s} \\
& + \frac{|\left(\log\frac{\xi}{\mathtt{a}}\right)|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2} L_1|x_s-z_s|_{C_\tau}\frac{ds}{s} + |\left(\log\frac{\xi}{\mathtt{a}}\right)|\sum_{i=1}^{w}\xi_i l^*|x(\xi_i^-)-z(\xi_i^-)| \\
& + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1} L_1|x_s-z_s|_{C_\tau}\frac{ds}{s} + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1} L_1|x_s-z_s|_{C_\tau}\frac{ds}{s} \\
& + \sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2} L_1|x_s-z_s|_{C_\tau}\frac{ds}{s} + \sum_{i=1}^{\kappa} l|x(\xi_i^-)-z(\xi_i^-)| \\
& + \sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right) l^*|x(\xi_i^-)-z(\xi_i^-)|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \mathtt{T}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)(\mathtt{T}-\mathtt{a})L_2\|x-z\| + \left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)\frac{L_1\|x-z\|}{\Gamma(\nu-1)}\int_{\xi_w}^{\mathtt{T}}\left(\log\frac{\mathtt{T}}{s}\right)^{\nu-2}\frac{ds}{s} \\
& + \left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)\frac{L_1\|x-z\|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s} + w\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)Tl^*\|x-z\| \\
& + \frac{L_1\|x-z\|}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\frac{ds}{s} + \frac{L_1\|x-z\|}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\frac{ds}{s} \\
& + \sum_{i=1}^{\kappa}\frac{L_1\|x-z\|\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s} + wl\|x-z\| + wT\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)l^*\|x-z\| \\
\leq\ & \mathtt{T}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)(\mathtt{T}-\mathtt{a})L_2\|x-z\| + \frac{L_1\|x-z\|}{\Gamma(\nu)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + \frac{wL_1\|x-z\|}{\Gamma(\nu)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} \\
& + wTl^*\|x-z\| + \frac{L_1\|x-z\|}{\Gamma(\nu+1)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + \frac{wL_1\|x-z\|}{\Gamma(\nu+1)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + \frac{wL_1\|x-z\|}{\Gamma(\nu)}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} \\
& + wl\|x-z\| + wTl^*\|x-z\|\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right) \\
\leq\ & \left[L_1\left(\frac{w+1}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + L_2(\mathtt{T}-\mathtt{a})\mathtt{T}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right) + wl + 2wT\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)l^*\right]\|x-z\|
\end{aligned}
$$

Thus, we have
$\|Fx-Fz\| \leq \left[L_1\left(\frac{w+1}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^{\nu} + L_2\mathtt{T}(\mathtt{T}-\mathtt{a})\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right) + wl + 2wl^*\mathtt{T}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)\right]\|x-$

$z\|$.
Consequently by (3.27*), the F is a contraction a conclusion of Banach fixed point theorem, we deduce that F has a unique fixed point which is solution to problem (3.15)-(3.18).*

Our second reslut deals the existence of solution for problem (3.15)-(3.18) by applying Scheafer fixed point theorem.

Theorem 3.9. *Assume that the followinq conditions hold :*

(A4) The function $\Psi : J * C_\tau \to R$ is continuous.

(A5) The function $\hbar : J * R \to R$ is continuous.

(A6) The functions $I_\kappa,\ \ \overline{I}_\kappa : R \to R$ are continuous.

(A7) There exists a constant $N > 0$ such that

$$|\Psi(\xi, z)| \leq N, \quad \text{for each } \ \xi \in J, \text{ and } z \in C_\tau.$$

(A8) There exists a constant $N^* > 0$ such that

$$|\hbar(\xi, y)| \leq N^* \ \text{ for each } \ \xi \in J,\, y \in R.$$

(A9) There exist tow constants $N_1 > 0, N_2 > 0$ such that

$$|I_\kappa(y)| \leq N_1, \ \ |\overline{I}_\kappa(y)| \leq N_2 \ \text{ for each, } \ y \in R, \ \kappa = 1, \ldots, w.$$

then the boundary value problem (3.15)-(3.18) has at least one solution in $[a - \tau, T]$.

Proof: *we shall use Scheafer fixed point theorem to prove that F has a fixed point. The proof will be qiven in several steps.*
Step 1: F is continuous .
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in B^*.
If $\xi \in [a - \tau, a]$, then

$$\|F(z_n)(\xi) - F(z)(\xi)\| = 0.$$

For $\xi \in [a, T]$, we have

$$\begin{aligned} |F(z_n)(\xi) - F(z)(\xi)| &\leq T\left(\log \frac{\xi}{a}\right) \int_a^T |\hbar(s, z_n(s)) - \hbar(s, z(s))| ds \\ &\quad + \frac{(\log \frac{\xi}{a})}{\Gamma(\nu - 1)} \int_{\xi_w}^T \left(\log \frac{T}{s}\right)^{\nu - 2} |\Psi(s, z_{ns}) - \Psi(s, z_s))| \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\log\frac{\xi}{a})}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_{ns})-\Psi(s,z_s))|\frac{ds}{s}\\
&+\left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w}\xi_i|\bar{I}(z_n(\xi_i^-))-\bar{I}_i(z(\xi_i^-))|\\
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_{ns})-\Psi(s,z_s)|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s,z_{ns})-\Psi(s,z_s)|\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_{ns})-\Psi(s,z_s)|\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}|I_i(z_n(\xi_i^-))-I_i(z(\xi_i^-))|+\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}(z_n(\xi_i^-))-\bar{I}_i(z(\xi_i^-))|
\end{aligned}
$$

Since $\Psi,\ \hbar,$ and $I_\kappa, \bar{I}_\kappa,\quad \kappa=1,\dots,w,$ are continuous functions, we give

$$\|F(z_n)-F(z)\|_{B^*}0 \text{ as } n\to\infty.$$

Step 2 : F maps bounded sets into bounded sets in B^*.
Indeed, it is enough to show that for any $\eta^*>0$, there exists constant $\ell^*>0$, such each $z\in D_{\eta^*}=\{z\in B^*:\|z\|\le\eta^*\}$, we have $\|F(z)\|\le\ell^*$.
If $\xi\in[a-\tau,a]$, we have

$$|(Fz)(\xi)|\le|\Phi(\xi)|\le\|\Phi\|$$

. And if $\xi\in J$, by (A7)-(A9), then

$$
\begin{aligned}
|F(z)(\xi)| \;\le\;& |\Phi(a)|+T\left(\log\frac{\xi}{a}\right)\int_a^T|\hbar(s,z(s))|ds+\frac{|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\frac{|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}+|\left(\log\frac{\xi}{a}\right)|\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))|\\
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}\frac{|\left(\log\frac{\xi}{\xi_i}\right)|}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}+\sum_{i=1}^{\kappa}|I_i(z(\xi_i^-))|\\
&+\sum_{i=1}^{\kappa}\xi_i|\left(\log\frac{\xi}{\xi_i}\right)||\bar{I}_i(z(\xi_\kappa^-))|
\end{aligned}
$$

$$
\begin{aligned}
&\leq |\Phi(a)| + N^* T\left(\log\frac{\xi}{a}\right)\int_a^T 1ds + \frac{N|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\int_{\xi_w}^T\left(\log\frac{T}{s}\right)^{\nu-2}\frac{ds}{s} \\
&+\frac{N|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s} + w\left(\log\frac{T}{a}\right)TN_2 \\
&+\frac{N}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\frac{ds}{s} + \frac{N}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\frac{ds}{s} \\
&+\sum_{i=1}^{\kappa}\frac{N|\left(\log\frac{\xi}{\xi_i}\right)|}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s} + wN_1 + wT(\log\frac{T}{a})N_2 \\
&\leq \|\Phi\| + (T-a)T\left(\log\frac{T}{a}\right)N^* + N\frac{(1+2w)\left(\log\frac{T}{a}\right)^{\nu}}{\Gamma(\nu)} \\
&+N\frac{(1+w)\left(\log\frac{T}{a}\right)^{\nu}}{\Gamma(\nu+1)} + wN_1 + 2wT\left(\log\frac{T}{a}\right)N_2.
\end{aligned}
$$

Where

$$
\|Fz\| \leq \|\Phi\| + N\left[\frac{1+w}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right]\left(\log\frac{T}{a}\right)^{\nu} + wN_1 + [(T-a)N^* + 2wN_2]\, T\left(\log\frac{T}{a}\right) := Q.
$$

Thus

$$
\|Fz\| \leq \{\max Q, \|\Phi\|\} := \ell^*.
$$

Step 3: F maps bounded sets into equicontinuous sets of B^* .
Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$, D_{η^*} *be a bounded set of* B^* *as in Step 2, and let* $z \in D_{\eta^*}$. *Then*

$$
\begin{aligned}
|F(z)(\tau_2) - F(z)(\tau_1)| &\leq T\left(\log\frac{\tau_2}{\tau_1}\right)\int_a^T |\hbar(s,z(s))|ds + \frac{(\log\frac{\tau_2}{\tau_1})}{\Gamma(\nu-1)}\int_{\xi_w}^T\left(\log\frac{T}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} \\
&+\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\sum_{i=i}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} + \left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))| \\
&+\frac{1}{\Gamma(\nu)}\int_a^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1} - \left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]|\Psi(s,z_s)|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s} \\
&+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} + \left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\log\frac{\tau_2}{\tau_1}\right) T(T-a)N^* + \frac{N(\log\frac{\tau_1}{\tau_2})}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\frac{ds}{s}\\
&+\frac{N\left(\log\frac{\tau_1}{\tau_2}\right)}{\Gamma(\nu-1)}\sum_{i=i}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s}+wN_2\left(\log\frac{\tau_2}{\tau_1}\right)\\
&+\frac{N}{\Gamma(\nu)}\int_{a}^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1}-\left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]\frac{ds}{s}+\frac{N}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}\frac{ds}{s}\\
&+\frac{N\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s}+wN_2\left(\log\frac{\tau_2}{\tau_1}\right)\\
&\leq N^* T(T-a)\left(\log\frac{\tau_2}{\tau_1}\right)+\frac{N}{\Gamma(\nu)}\left(\log\frac{\tau_1}{\tau_2}\right)^{\nu}+\frac{2wN}{\Gamma(\nu)}\left(\log\frac{\tau_1}{\tau_2}\right)^{\nu}\\
&+2N_2 Tw\left(\log\frac{\tau_2}{\tau_1}\right)+\frac{N}{\Gamma(\nu+1)}\left[2\left(\log\frac{\tau_2}{\tau_1}\right)^{\nu}+\left(\log\frac{\tau_2}{a}\right)-\left(\log\frac{\tau_1}{a}\right)\right]\\
&\leq \frac{N}{\Gamma(\nu+1)}\left[2\left(\log\frac{\tau_2}{\tau_1}\right)^{\nu}+\left(\log\frac{\tau_2}{a}\right)-\left(\log\frac{\tau_1}{a}\right)\right]+\frac{(1+2w)N}{\Gamma(\nu)}\left(\log\frac{\tau_1}{\tau_2}\right)^{\nu}\\
&+T\left[N^*(T-a)+2wN_2\right]\left(\log\frac{\tau_2}{\tau_1}\right)
\end{aligned}
$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above ingualily tends to zero.
As a consequence of steps 1 to 3, it follows by the Arzela-Ascoli theorem 1.10 that F is completly continuous.

Step 4: A priori bounds.
Let set $\varepsilon = \{z \in B^* : z = \lambda F(z)\, for\ \ some\ \ 0 < \lambda < 1\}$ is bounded, and $z \in \varepsilon$, then $z = \lambda F(z)$ for some $0 < \lambda < 1$. Then, for each $\xi \in J$,

$$
\begin{aligned}
(Fz)(\xi) \;=\;& \lambda\Phi(a)+\lambda T\left(\log\frac{\xi}{a}\right)\int_{a}^{T}\hbar(s,z(s))ds-\frac{\lambda\left(\log\frac{\xi}{a}\right)}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\Psi(s,z_s)\frac{ds}{s}\\
&-\frac{\lambda\left(\log\frac{\xi}{a}\right)}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s,z_s)\frac{ds}{s}-\lambda\left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w}\xi_i\bar{I}_i(z(\xi_i^-))\\
&+\frac{\lambda}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}+\frac{\lambda}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\Psi(s,z_s)\frac{ds}{s}\\
&+\lambda\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\Psi(s,z_s)\frac{ds}{s}+\lambda\sum_{i=1}^{\kappa}I_i(z(\xi_i^-))
\end{aligned}
$$

$$+\lambda \sum_{i=1}^{\kappa} \xi_i \left(\log \frac{\xi}{\xi_i} \right) \bar{I}_i(z(\xi_i^-))$$

This implies by (A7)-(A9), we obtain

$$\begin{aligned} \|Fz\| &\leq \|\Phi\| + T(T-a)\left(\log \frac{T}{a}\right) N^* + N\left[\frac{1+w}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right]\left(\log \frac{T}{a}\right)^{\nu} \\ &+ wN_1 + 2wT\left(\log \frac{T}{a}\right) N_2. \end{aligned}$$

This shows that the set ε is bounded. Consequence we deduce be F has a fixed point which is a solution of the problem (3.15)-(3.18).

The three result is based by applying the Leray-Schauder nonlinear alternative type.

Theorem 3.10. *Assume that (A4)-(A6) and the following conditions hold :*

(A10) *There exist $\phi_\Psi \in C(J, R^+)$ and $\psi : [0, \infty) \to [0, \infty)$ continuous, and non-decreasing such that*

$$|\Psi(\xi, y)| \leq \phi_\Psi(\xi)\psi(|y|), \ \ for \ \ all \ \ \xi \in J, \ \ y \in C_\tau.$$

(A11) *There exist $\phi_\hbar \in L(J, R^+)$ and $\psi^* : [0, \infty) \to [0, \infty)$ continuous, and non-decreasing such that*

$$|\hbar(\xi, v)| \leq \phi_\hbar(\xi)\psi^*(|v|), \ \ for \ \ all \ \ \xi \in J, \ \ v \in R.$$

(A12) *There exist $\bar{\psi}^*$, $\bar{\psi}^{**} : [0, \infty) \to [0, \infty)$ continuous, and non-decreasing such that*

$$|I_\kappa(v)| \leq \bar{\psi}^*(|v|), \quad |\bar{I}_\kappa(v)| \leq \bar{\psi}^{**}(|v|), \ \ for \ \ all \ \ v \in R, \ \ \kappa = 1, \ldots, w.$$

(A13) *There exists a nomber $\overline{M} > 0$, let $\Phi = \sup\{\phi_\Psi(\xi) : \xi \in J\}$, such that*

$$\frac{\overline{M}}{\|\Phi\| + T\left(\log \frac{T}{a}\right)\psi^*(\|z\|)\|\phi_\hbar\|_{L^1} + \Phi\psi(\overline{M})\left(\frac{1+w}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right)\left(\log \frac{T}{a}\right)^{\nu} + w\bar{\psi}^*(\overline{M}) + 2wT\left(\log \frac{T}{a}\right)\bar{\psi}^{**}(\overline{M})} \geq 1,$$

then (3.15)-(3.18) has at least one solution in $[a - \tau, T]$.

Proof: *we consider the operator $F : B^* \to B^*$ defined by (**??**). We shall show that F is continuous and completely continuous.*
Step 1: F is continuous.

Let $\{z_n\}$ *be a sequence such that* $z_n \to z$ *in* B^*. *Let* $\gamma > 0$ *such that* $\|z_n\| \leq \gamma$, *and* $\xi \in [a, T]$, *than*

$$\begin{aligned}
|F(z_n)(\xi) - F(z)(\xi)| \;\; \leq \;\; & T\left(\log\frac{\xi}{a}\right)\int_a^T |\hbar(s, z_n(s)) - \hbar(s, z(s))|ds \\
& + \frac{\left(\log\frac{\xi}{a}\right)}{\Gamma(\nu-1)}\int_{\xi_w}^T \left(\log\frac{T}{s}\right)^{\nu-2} |\Psi(s, z_{ns}) - \Psi(s, z_s))|\frac{ds}{s} \\
& + \frac{\left(\log\frac{\xi}{a}\right)}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-2} |\Psi(s, z_{ns}) - \Psi(s, z_s))|\frac{ds}{s} \\
& + \left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w} \xi_i|\bar{I}(z_n(\xi_i^-)) - \bar{I}_i(z(\xi_i^-))| \\
& + \frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi} \left(\log\frac{\xi}{s}\right)^{\nu-1} |\Psi(s, z_{ns}) - \Psi(s, z_s)|\frac{ds}{s} \\
& + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-1} |\Psi(s, z_{ns}) - \Psi(s, z_s)|\frac{ds}{s} \\
& + \sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i} \left(\log\frac{\xi_i}{s}\right)^{\nu-2} |\Psi(s, z_{ns}) - \Psi(s, z_s)|\frac{ds}{s} \\
& + \sum_{i=1}^{\kappa} \xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}(z_n(\xi_i^-)) - \bar{I}_i(z(\xi_i^-))| \\
& + \sum_{i=1}^{\kappa} |I_i(z_n(\xi_i^-)) - I_i(z(\xi_\kappa^-))|
\end{aligned}$$

Since Ψ, $\hbar$ *and* $I_\kappa, \bar{I}_\kappa$, $\kappa = 1, \ldots, w$, *are continuous functions.*
Hence

$$\|F(z_n) - F(z)\|_{B^*}0, \text{ as } n \to \infty.$$

Consequently, F *is continuous.*
Step 2: F maps bounded sets into bounded sets in B^*.
Define the set

$$D_{\eta^*} = \{z \in B^* : \|z\| \leq \eta^*\}.$$

Let $z \in D_{\eta^*}$, *there exists a constant* $L^* > 0$, *such that* $\|F(z)\| \leq L^*$.
By (A10), (A11) and (A12), for each $\xi \in J$, *we have*

$$|F(z)(\xi)| \;\; \leq \;\; |\Phi(a)| + T\left(\log\frac{\xi}{a}\right)\int_a^T |\hbar(s, z(s))|ds$$

$$
\begin{aligned}
&+\frac{|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\frac{|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}\\
&+|\left(\log\frac{\xi}{a}\right)|\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))|\\
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}\frac{|\left(\log\frac{\xi}{\xi_i}\right)|}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}+\sum_{i=1}^{\kappa}|I_i(z(\xi_i^-))|\\
&+\sum_{i=1}^{\kappa}\xi_i|\left(\log\frac{\xi}{\xi_i}\right)||\bar{I}_i(z(\xi_\kappa^-))|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & |\Phi(a)|+T\left(\log\frac{\xi}{a}\right)\int_a^T\phi_\hbar(s)\psi(|z(s)|)ds\\
&+\frac{|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\frac{|\left(\log\frac{\xi}{a}\right)|}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|)\\
&+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}\frac{|\left(\log\frac{\xi}{\xi_i}\right)|}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\sum_{i=1}^{w}\bar{\psi}^{*}(|z(\xi_i^-)|)+\left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & |\Phi(a)| + T\left(\log\frac{\xi}{a}\right)(T-a)\|\phi_\hbar\|_{L^1}\psi(\|z\|) + \frac{\Phi\psi(\|z\|)\left(\log\frac{\xi}{a}\right)}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\frac{ds}{s} \\
& + \frac{\Phi\psi(\|z\|)\left(\log\frac{\xi}{a}\right)}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s} + w\left(\log\frac{T}{a}\right)T\bar{\psi}^{**}(\|z\|) \\
& + \frac{\Phi\psi(\|z\|)}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\frac{ds}{s} + \frac{\Phi\psi(\|z\|)}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\frac{ds}{s} \\
& + \sum_{i=1}^{\kappa}\frac{\Phi\psi(\|z\|)\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s} + w\bar{\psi}^{*}(\|z\|) + wT(\log\frac{T}{a})\bar{\psi}^{**}(\|z\|) \\
\leq\ & \|\Phi\| + (T-a)T\left(\log\frac{T}{a}\right)\|\phi_\hbar\|_{L_1}\psi(\|z\|) + \Phi\psi(\|z\|)\frac{(1+2w)\left(\log\frac{T}{a}\right)^{\nu}}{\Gamma(\nu)} \\
& + \Phi\psi(\|z\|)\frac{(1+w)\left(\log\frac{T}{a}\right)^{\nu}}{\Gamma(\nu+1)} + w\bar{\psi}^{*}(\|z\|) + 2wT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\|z\|).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|Fz\| \ \leq\ & \|\Phi\| + \Phi\psi(\|\eta^*\|)\left[\frac{1+w}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right]\left(\log\frac{T}{a}\right)^{\nu} + w\bar{\psi}^{*}(\|\eta^*\|) \\
& + T\left[(T-a)\|\phi_\hbar\|_{L_1}\psi(\|\eta^*\|) + 2w\bar{\psi}^{**}(\|\eta^*\|)\right]\left(\log\frac{T}{a}\right)
\end{aligned}
$$

Step 3: F maps bounded sets into equicontinuous sets of B^* .
Let $z \in D_{\eta^*}$ *be a bounded set of* B^* *as in Step 2, and* $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$*, we have*

$$
\begin{aligned}
|F(z)(\tau_2) - F(z)(\tau_1)| \ \leq\ & T\left(\log\frac{\tau_2}{\tau_1}\right)\int_a^T|\hbar(s,z(s))|ds + \frac{(\log\frac{\tau_2}{\tau_1})}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} \\
& + \frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\sum_{i=i}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} \\
& + \left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))| \\
& + \frac{1}{\Gamma(\nu)}\int_a^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1} - \left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]|\Psi(s,z_s)|\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}\\
&+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))|\\
\leq\ & T\left(\log\frac{\tau_2}{\tau_1}\right)\int_a^T\phi_\hbar(s)\psi(|z(s)|)ds\\
&+\frac{(\log\frac{\tau_1}{\tau_2})}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\frac{\left(\log\frac{\tau_1}{\tau_2}\right)}{\Gamma(\nu-1)}\sum_{i=i}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\nu)}\int_a^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1}-\left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|)\\
&+\frac{1}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}\\
&+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & T\left(\log\frac{\tau_2}{\tau_1}\right)\|\phi_\hbar\|_{L^1}\psi(\|z\|)(T-a)+\frac{\Phi\psi(\|z\|)(\log\frac{\tau_1}{\tau_2})}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\frac{ds}{s}\\
&+\frac{\Phi\psi(\|z\|)\left(\log\frac{\tau_1}{\tau_2}\right)}{\Gamma(\nu-1)}\sum_{i=i}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s}+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|)\\
&+\frac{\Phi\psi(\|z\|)}{\Gamma(\nu)}\int_a^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\nu-1}-\left(\log\frac{\tau_1}{s}\right)^{\nu-1}\right]\frac{ds}{s}\\
&+\frac{\Phi\psi(\|z\|)}{\Gamma(\nu)}\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\nu-1}\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\Phi\psi(\|z\|)\left(\log\frac{\tau_2}{\tau_1}\right)}{\Gamma(\nu-1)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\frac{ds}{s}+\left(\log\frac{\tau_2}{\tau_1}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|) \\
\leq\; & \|\phi_\hbar\|_{L^1}\psi(\|z\|)T(T-a)\left(\log\frac{\tau_2}{\tau_1}\right)+\frac{\Phi\psi(\|z\|)}{\Gamma(\nu)}(\log\frac{\tau_1}{\tau_2})^{\nu}+2w\frac{\Phi\psi(\|z\|)}{\Gamma(\nu)}\left(\log\frac{\tau_1}{\tau_2}\right)^{\nu} \\
& +2wT\bar{\psi}^{**}(\|z\|)\left(\log\frac{\tau_2}{\tau_1}\right)+\frac{\Phi\psi(\|z\|)}{\Gamma(\nu+1)}\left[2((\log\frac{\tau_2}{\tau_1})^{\nu}+(\log\tau_2)-(\log\tau_1)\right] \\
\leq\; & \frac{(1+2w)\Phi\psi(\|z\|)}{\Gamma(\nu)}\left(\log\frac{\tau_1}{\tau_2}\right)^{\nu}+T\left[\|\phi_\hbar\|_{L^1}\psi(\|z\|)(T-a)+2w\bar{\psi}^{**}(\|z\|)\right]\left(\log\frac{\tau_2}{\tau_1}\right) \\
& +\frac{\Phi\psi(\|z\|)}{\Gamma(\nu+1)}\left[2((\log\frac{\tau_2}{\tau_1})^{\nu}+(\log\tau_2)-(\log\tau_1)\right].
\end{aligned}
$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero.

In consequence of Steps 1-3, it follows by the Arzela-Ascoli theorem 1.10, we conclude that $F : B^* \to B^*$ is continuous and completely continuous.

Step 4: *We show that there exists an open set $U \subseteq B^*$ with $z \neq \lambda F(z)$ for $\lambda \in [0,1]$ and $z \in \partial U$. Let $z \in B^*$ and $z = \lambda F(z)$ for some $0 < \lambda < 1$. Thus, for each $\xi \in J$, defined by (3.28), be (A10)-(A12), we have*

$$
\begin{aligned}
|(Fz)(\xi)| \;\leq\; & |\Phi(a)|+T\left(\log\frac{\xi}{a}\right)\int_a^T|\hbar(s,z(s))|ds+\frac{(\log\frac{\xi}{a})}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s} \\
& +\frac{(\log\frac{\xi}{a})}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}+\left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w}\xi_i|\bar{I}_i(z(\xi_i^-))| \\
& +\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}|\Psi(s,z_s)|\frac{ds}{s} \\
& +\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}|\Psi(s,z_s)|\frac{ds}{s}+\sum_{i=1}^{\kappa}|I_i(z(\xi_i^-))| \\
& +\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)|\bar{I}_i(z(\xi_i^-))| \\
\leq\; & |\Phi(a)|+T\left(\log\frac{\xi}{a}\right)\int_a^T\phi_\hbar(s)\psi^*(|z(s)|)ds+\frac{(\log\frac{\xi}{a})}{\Gamma(\nu-1)}\int_{\xi_w}^{T}\left(\log\frac{T}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s} \\
& +\frac{(\log\frac{\xi}{a})}{\Gamma(\nu-1)}\sum_{i=1}^{w}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}+\left(\log\frac{\xi}{a}\right)\sum_{i=1}^{w}\xi_i\bar{\psi}^{**}(|z(\xi_i^-)|)
\end{aligned}
$$

$$+\frac{1}{\Gamma(\nu)}\int_{\xi_\kappa}^{\xi}\left(\log\frac{\xi}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}+\frac{1}{\Gamma(\nu)}\sum_{i=1}^{\kappa}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-1}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}$$

$$+\sum_{i=1}^{\kappa}\frac{\left(\log\frac{\xi}{\xi_i}\right)}{\Gamma(\nu-1)}\int_{\xi_{i-1}}^{\xi_i}\left(\log\frac{\xi_i}{s}\right)^{\nu-2}\phi_\Psi(s)\psi(|z_s|)\frac{ds}{s}+\sum_{i=1}^{\kappa}\bar{\psi}^*(|z(\xi_i^-)|)$$

$$+\sum_{i=1}^{\kappa}\xi_i\left(\log\frac{\xi}{\xi_i}\right)\bar{\psi}^{**}(|z(\xi_i^-)|)$$

$$\begin{aligned}\leq\ & \|\Phi\|+T\left(\log\frac{T}{a}\right)\psi^*(\|z\|)\int_a^T\phi_\hbar(s)ds+\frac{\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu)}\Phi\psi(\|z\|)+\frac{w\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu)}\Phi\psi(\|z\|)\\ & +wT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\|z\|)+\frac{\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}\Phi\psi(\|z\|)+\frac{w\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu+1)}\Phi\psi(\|z\|)\\ & +\frac{w\left(\log\frac{T}{a}\right)^\nu}{\Gamma(\nu)}\Phi\psi(\|z\|)+w\bar{\psi}^*(\|z\|)+wT\left(\log\frac{\xi}{a}\right)\bar{\psi}^{**}(\|z\|)\\ & \leq\|\Phi\|+T\left(\log\frac{T}{a}\right)\psi^*(\|z\|)\|\Phi_h\|_{L^1}+\Phi\psi(\|z\|)\left(\frac{(1+w)}{\Gamma(\nu+1)}+\frac{(1+2w)}{\Gamma(\nu)}\right)\left(\log\frac{T}{a}\right)^\nu\\ & +w\bar{\psi}^*(\|z\|)+2wT\left(\log\frac{T}{a}\right)\bar{\psi}^{**}(\|z\|)\end{aligned}$$

Also, the inequality

$$\frac{\|z\|}{\|\Phi\|+\Phi\psi(\|z\|)(\frac{1+w}{\Gamma(\nu+1)}+\frac{1+2w}{\Gamma(\nu)})\left(\log\frac{T}{a}\right)^\nu+w\bar{\psi}^*(\|z\|)+[\psi^*(\|z\|)\|\phi_\hbar\|_{L^1}+2w\ \bar{\psi}^{**}(\|z\|)]T\left(\log\frac{T}{a}\right)}\leq 1.$$

which condition (A13), there exist $\overline{M}$ *such that* $\|z\|\neq\overline{M}$. *Let us set*

$$U=\{z\in B^*:\|z\|\leq\overline{M}\}.$$

Note, that the operator $F:\overline{U}\to B^*$ *is continuous and completely continuous. From the choice of* U, *there is no* $z\in\partial U$ *such that* $z=\lambda F(z)$ *for some* $\lambda\in(0,1)$. *Consequently by the nonlinear alternative of Leray-Schauder theorem,* F *has a fixed point that is a solution of the problem (3.15)-(3.18).*

3.6 Example

Let consider the following problem :

$$^{C}_{H}D^{\frac{3}{2}}z(\xi)=\frac{e^\xi}{(e^\xi+5)^2}\frac{|z_\xi|}{(1+|z_\xi|)}\qquad,\xi\in[1,2],\xi\neq\frac{4}{3},\tag{3.29}$$

$$\Delta z(\frac{4}{3}) = \frac{|z(\frac{4}{3}^-)|}{15 + |z(\frac{4}{3}^-)|}, \tag{3.30}$$

$$\Delta z'(\frac{4}{3}) = \frac{|z(\frac{4}{3}^-)|}{17 + |z(\frac{4}{3}^-)|}, \tag{3.31}$$

$$z(\xi) = \Phi(\xi), \ \ \xi \in [1-\tau, 1], \ z'(2) = \int_1^2 \frac{|z(s)|}{13 + |z(s)|} ds. \tag{3.32}$$

Set

$$\Psi(\xi, z_\xi) = \frac{e^\xi}{(e^\xi+5)^2} \frac{|z_\xi|}{(1+|z_\xi|)}, \qquad (\xi, z) \in J * C([1-\tau, 1], R),$$

$$I(z) = \frac{|z|}{15+|z|}, \qquad \overline{I}(z) = \frac{|z|}{17+|z|}, \qquad z \in R.$$

and

$$\int_1^2 \hbar(s, z(s)) ds = \int_1^2 \frac{|z(s)|}{13 + |z(s)|} ds, \qquad (\xi, z) \in J * R.$$

Let $x, z \in C([\frac{1}{2} - \tau, \frac{1}{2}], R)$ *and* $\xi \in J$. *Then we have*

$$\begin{aligned} |\Psi(\xi, x) - \Psi(\xi, z)| &= \frac{1}{(e^\xi+5)^2} |\frac{x}{1+x} - \frac{z}{1+z}| \\ &\leq \frac{1}{(e^\xi+5)^2} \frac{|x-z|}{|(1+x)(1+z)|} \\ &\leq \frac{|x-z|}{(e^\xi+5)^2} \leq \frac{1}{36}|x-z|. \end{aligned}$$

and

$$\begin{aligned} |\hbar(\xi, x) - \hbar(\xi, z)| &= \frac{1}{13} |\frac{x}{1+x} - \frac{z}{1+z}| \\ &\leq \frac{1}{13} \frac{|x-z|}{|(1+x)(1+z)|} \\ &\leq \frac{|x-z|}{13} \leq \frac{1}{13}|x-z|. \end{aligned}$$

Let $x, z \in R$, *then we have*

$$\begin{aligned} |I_\kappa(x) - I_\kappa(z)| &= |\frac{x}{15+x} - \frac{Y}{15+y}| \\ &\leq \frac{1}{15}|x-z| \end{aligned}$$

and

$$\begin{aligned} |I_\kappa(x) - I_\kappa(z)| &= |\frac{x}{17+x} - \frac{Y}{17+y}| \\ &\leq \frac{1}{17}|x-z|. \end{aligned}$$

Hence the hypotheses (A1)-(A3) holds, with $L_1 = \frac{1}{36}, L_2 = \frac{1}{13},\ l = \frac{1}{15},\ l^* = \frac{1}{17}$, *and* $\nu = \frac{3}{2}, w = 1, \xi_1 = \frac{4}{3}, \mathtt{T} = 2, \mathtt{a} = 1.$
We shall check that condition (3.27).Further

$$\begin{aligned} &\left[L_2 \mathtt{T}(\mathtt{T}-\mathtt{a})\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right) + L_1\left(\frac{w+1}{\Gamma(\nu+1)} + \frac{1+2w}{\Gamma(\nu)}\right)\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)^\nu + wl + 2wl^*\mathtt{T}\left(\log\frac{\mathtt{T}}{\mathtt{a}}\right)\right] \\ &= \left[\frac{2}{13}(\log 2) + \frac{1}{36}\left(\frac{2}{\Gamma\left(\frac{5}{2}\right)} + \frac{3}{\Gamma(\frac{3}{2})}\right)(\log 2)^{\frac{3}{2}} + wl + \frac{4}{17}(\log 2)\right] \\ &= 0.414779517 < 1. \end{aligned}$$

Note that $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$.
Then all hypotheses of theorem (3.8) are fulfilled, and consequently the boundary value problem (3.29)-(3.32) has a unique solution on $[1-\tau, 2]$.

Conclusion and Perspectives

The objective of this thesis is to present the results of the existence and uniqueness of solutions for a new class of initial and boundary value problems of impulsive fractional differential equations with finite delay and Caputo-Hadamard derivative. The results are based on some fixed point theorems : Banach contraction principle, Scheafer fixed point theorem , alternative nonlinear of Leray Schauder and Mönch fixed point theorem combined with the Kuratowski's measure of noncompactness.Finally, we give illustrative examples.

In the future, we will study stability the solutions of this problems.

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