Application of Hamilton equations to Dynamic Systems

1st Soumia AYACH HAMRAOUI, 2nd Kamel KHOUNFAIS, 3rd Noureddine MENECEUR

1,2- Faculty of Sciences Department of Physics 20th August 1955 Skikda University, BP-26 Skikda , ALGERIA.
3 - Faculty of Technology Mechanical Engineering Department EL-Oued University, B.P 789 EL-Oued, ALGERIA.
Correspondent author: soumiahamraoui@yahoo.fr

Abstract— ***In this presentation, instead of the Lagrangian formulation of the dynamics, we will apply the Hamiltonian formulation to be able to model and control system swidely used in applications in robotics.***
In the first part, we will see that using a Legendre transformation, we can associate with Lagrange's equations, which second-order equations, a system of first-order differential equations called canonical equations of Hamilton.
To validate Hamilton's efficiency of formalization, the case of a manipulator with flexible joints was treated.

Keywords—Lagrangian, formulation, Legendre transformation, Hamiltonian formulation, dynamical systems.

I. Introduction

Following the Newtonian synthesis, the 18th century was marked by an astonishing adventure. Under the impulse of Euler and Lagrange, then Hamilton, we discovered the true structure of mechanics governed by a variational principle. According to this principle, the observed evolution of any physical system is that which minimizes a physical quantity called action, noted S [1]
The contributions of this article can be summarized as follows. In Section 2, we present a brief overview of Lagrangian's formalism. In Section 3, first, we define the Legendre transformation, then we deduce the Hamiltonian function as Legendre's transformation of the Lagrangian function. Section 4 focuses on the mathematical comparison of the two proposed formalisms applied to a robotic arm with three degrees of freedom. Our conclusions are summarized in section 5 pointing out a series of aspects that remain to be clarified elsewhere.

II. Lagrangian Formulation

II. 1.Holonomic system

In the Newtonian description, the movement of a mechanical system of N particles, each with position vector $\mathbf{r}_i(t)$ and mass m_i, is given by the equation

$$m_i\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i, \quad i = 1, \dots, N, \tag{1}$$

Here, $\boldsymbol{F}_i$ is the sum of all the forces acting on the particle i.

The resolution of the system of Eqs (1), requires the knowledge of all forces, as well as the initial conditions $\boldsymbol{r}_i(t_0)$and$\dot{\boldsymbol{r}}_i(t_0)$. However, it often happens that the system is subject to constraints. In this case, two types of difficulties appear. First, the Eq (1) are not all independent. Secondly, the forces of constraints are not given a priori. In the case of holonomic constraints, expressed by m equations of the form

$$f_l(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N, t) = 0 \qquad l = 1, \dots., m \tag{2}$$

the first difficulty is solved by the introduction of generalized coordinates q_k (k=1,..,n),

$$\boldsymbol{r}_i(q_1, \dots, q_n, t), \tag{3}$$

n=3N-m is the degree of freedom of the system. To overcome the second difficulty, we will apply the principle of least action. According to this principle, the physical trajectory is selected by the fact that the action [1, 2]

$$S = \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\, dt, \tag{4}$$

where L = T-U is the Lagrangian of the system, has a stationary value.

$$\delta S = \sum_{k=1}^{n} \int_{t_2}^{t_1} \left(\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt = 0. \tag{5}$$

Using $\delta\dot{q}_k = \delta(dq_k/dt)$ and $\delta q_k(t_1) = \delta q_k(t_2) = 0$, we are getting:

$$\delta S = \sum_{k=1}^{n} \int_{t_2}^{t_1} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right) \delta q_k dt = 0. \tag{6}$$

As the integral must be zero for all variations δq_k. For conservative forces, Eq. (6) gives:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0, \tag{7}$$

For non-conservative forces, Eq.(6) gives:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = -Q_k, \tag{8}$$

Where Q_k are the components of the generalized force defined by [3],

$$Q_k = \sum_{i=1}^{N} \boldsymbol{f}_i \frac{\partial \boldsymbol{r}_i}{\partial q_k}\,, \tag{9}$$

and $\boldsymbol{f}_i$ is the non-conservative part of the force$\boldsymbol{F}_i$.

II.2.Nonholonomic system

Constraints are called nonholonomic if they are expressed by equations of the form [1,3] :

$$f_j(q_1, \dots., q_n, \dot{q}_1, \dots \dot{q}_n, t) = 0, \qquad j = 1, \dots, r, \tag{10}$$

r is a the number of constraint equations. We know that the holonomic constraints ensure the independence of the generalized coordinates q_k. However, for non-holonomic systems, q_k are not independent of each other.

The Lagrange multiplier method is an elegant way to take into account these non-holonomic constraints. The Lagrangian of the system under the new constraints is[4]

$$\bar{L}(q_k,\dot{q}_k,\lambda_j,t) = L(q_k,\dot{q}_k,t) + \sum_{j=1}^{r} \lambda_j(q_k,\dot{q}_k,t)\, f_j(q_k,\dot{q}_k,t), \qquad (11)$$

where λ_j are the Lagrange multipliers which are to be determined in the same way as $q_k(t)$. The application of the principle of least action, for the lagrangian (11) of the system under the new constraints, gives

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial}{\partial q_k}\left(\sum_{j=1}^{r}\lambda_j f_j\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_k}\left(\sum_{j=1}^{r}\lambda_j f_j\right)\right) = -Q_k, \qquad (12)$$

We introduce the conjugate moment

$$P_k = \frac{\partial L}{\partial \dot{q}_k}. \qquad (13)$$

III. Hamiltonian Formulation

In the Hamiltonian formulation, the state of a system is described by the variables (q_k, P_k, t), instead of $(q_k, \dot{q}_k, t)$ .To pass variables $(q_k,\dot{q}_k,t)$ to variables (q_k,P_k,t) , the easiest way is to use the mathematical method known as the Legendre transformation.

A Legendre Transformation

Let f be a function of the variable x. We call Legendre transform of the function f, the function F of the variable P defined by $P = \partial f(x)/\partial x$ and such that [1, 4]

$$F(P) = \max\{Px - f(x)\} \qquad (14)$$

We no realize the transformation of Legendre which transforms $(q_k, \dot{q}_k, t)$ to (q_k, P_k, t), where $P_k = \partial L/\partial \dot{q}_k$.

The transition to these new variables introduces the function of Hamiltone, defined by [1, 5]

$$H(q_k,P_k,t) = \sum_{k=1}^{n} P_k\dot{q}_k - L(q_k,\dot{q}_k,t). \qquad (15)$$

B Derivation of Hamilton's equations

In the hamiltonian formulation, the coordinates q_k and the conjugate moments p_k are independent. It remains to formulate the equations of movement with this new magnitude.

If we takein to account (15), the action S may be written as

$$S = \int_{t_1}^{t_2}\left[\sum_{k=1}^{n} P_k\dot{q}_k - H(q_k,P_k,t).\right]dt = 0. \qquad (16)$$

Let's write now, to find Hamilton's equations, that the action is stationary

$$\delta S = \int_{t_1}^{t_2}\left[\sum_{k=1}^{n}\left(dq_k\delta P_k + P_k\delta(dq_k)\right) - \left(\frac{\partial H}{\partial q_k}\delta q_k + \frac{\partial H}{\partial P_k}\delta P_k\right)dt\right] = 0. \qquad (17)$$

With $\delta q_k = 0$ at the endpoints, by integrating the $P_k\delta(dq_k)$ term, we obtain

$$\int_{t_1}^{t_2}\sum_{k=1}^{n} P_k\delta(dq_k) = \int_{t_1}^{t_2}\sum_{k=1}^{n} P_k d(\delta q_k) = -\int_{t_1}^{t_2}\sum_{k=1}^{n} dP_k\delta q_k \qquad (18)$$

and therefore, Eq.(17) becomes

$$\int_{t_1}^{t_2}\left[\sum_{k=1}^{n}\left(dq_k - \frac{\partial H}{\partial P_k}dt\right)\delta P_k - \left(dP_k + \frac{\partial H}{\partial q_k}dt\right)\delta q_k\right] = 0. \qquad (19)$$

The variations δP_k and δq_k are independent, then the terms in parentheses must independently vanish if we are to have $\delta S = 0$, we obtain Hamilton's equations,

$$\dot{q}_k = \frac{\partial H}{\partial P_k}, \qquad \dot{P}_k = -\frac{\partial H}{\partial q_k}. \qquad (20)$$

The Hamilton equations (20) constitute a system of 2n first order differential equations called Hamiltonian canonical equations. They are symmetrical in $q_k$and P_k.

IV. Application

Consider a dynamic system, consisting of two rigid segments (figure-1), of mass m_k and moment of inertia I_k.(k=1,2). The segment 1 is mobile without friction along a horizontal axis under the action of a force **F**. The segment 2 is fixed at point A by a rooted connection ensures rotation without friction is subjected to a torque **T [6]**.

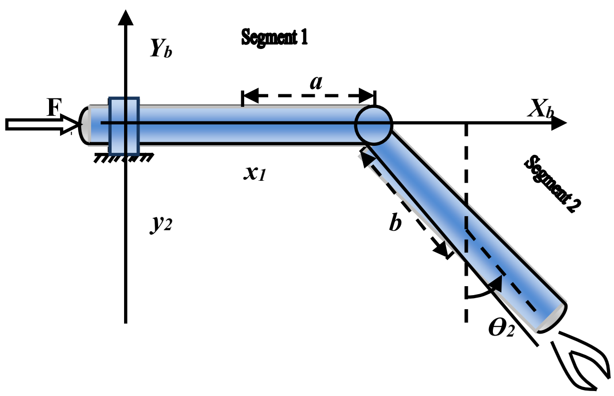

Fig-1: Modeling a manipulator robot

The coordinates that describe the dynamics of the system, are x_k, y_k, andθ_k. (k=1, 2). This system is subject to the following constraints:

$$f_1(x_k, y_k, \theta_k, t) = y_1 = 0, \qquad (21)$$

$$f_2(x_k, y_k, \theta_k, t) = \theta_1 = 0, \qquad (22)$$

$$f_3(x_k, y_k, \theta_k, t) = x_2 - x_1 - b\sin\theta_2 - a = 0, \quad (23)$$

$$f_4(x_k, y_k, \theta_k, t) = y_2 + b\cos\theta_2 = 0. \quad (24)$$

The constraints of Eqs. (21) and (22), express the fact that the (body 1) can move only horizontally. The constraints of Eqs. (23) and (24) express the relationship between the Cartesian coordinates of the centers of mass of the two bodies because of their articulation.

A.Lagrangian formulation:

Coordinates, x_k y_k and Θ_k are no longer independent. In this case, the differential equations of the movement are obtained by Eq. (12), which admit the following vectorial writing:

$$\frac{\partial \overline{L}_s}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial \overline{L}_s}{\partial \dot{q}_k}\right) = -Q_k, \quad (25)$$

where $\overline{L}_s$ is the lagrangian of the system under the new constraints,

$$\overline{L}_s = L_s + \sum_{j=1}^{4} \lambda_j f_j(x_k, y_k, \theta_k, t), \quad (26)$$

L_s are the Lagrangian of the system that uses degrees of freedom, given by:

$$L_s(x_k, y_k, \theta_k, \dot{x}_k, \dot{y}_k, \dot{\theta}_k, t) = T_s - U_s, \quad (27)$$

T_s and U_s are the kinetic and potential energy respectively, given by

$$T_s = \frac{1}{2}\sum_{k=1}^{2}\left[m_k(\dot{x}_k^2 + \dot{y}_k^2) + I_k\dot{\theta}_k^2\right],\ U_s \sum_{k=1}^{2} m_k g y_k. \quad (28)$$

Bearing these expressions in the function of Lagrange, we obtain:

$$L_s = \sum_{k=1}^{2}\left[\frac{1}{2}\left(m_k(\dot{x}_k^2 + \dot{y}_k^2) + I_k\dot{\theta}_k^2\right) - m_k g y_k\right], \quad (29)$$

and the Lagrangian of the system under the new constraints is

$$\overline{L}_s = \sum_{k=1}^{2}\left[\frac{1}{2}\left(m_k(\dot{x}_k^2 + \dot{y}_k^2) + I_k\dot{\theta}_k^2\right) - m_k g y_k\right] + \sum_{j=1}^{4} \lambda_j f_j(x_k, y_k, \theta_k, t). \quad (30)$$

The values of λ_j must be determined by the solution of the n differential equations and the constraints imposed on the coordinates.

$$\frac{\partial L_s}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{q}_k}\right) + \sum_{j=1}^{4} \lambda_j \frac{\partial f_j}{\partial q_k} = -Q_k, \quad (31)$$

Here, q_k represents the coordinates x_k y_k and Θ_k. Q_k are the component of the generalized force, given by Eq. (9),

$$Q_{x_k} = (2-k)F, Q_{y_k} = m_k g, Q_{\theta_k} = (k-1)T, \quad (32)$$

insert the Eqs. (21), (22), (23), (24), (30) and (32) in (31), the equations of motion are written:

$$m_k\ddot{x}_k = (-1)^k\lambda_k + F^{2-k}\ ,\ m_k\ddot{y}_k = -m_k g + \lambda_{k^2} \quad (33)$$

$$I_k\ddot{\theta}_k = (-b)^{k-1}\left[\lambda_{k+1}\cos((k-1)\theta_k) + \lambda_{2k}\sin((k-1)\theta_k)\right] + T^{k-1}. \quad (34)$$

Combining (24) and (25), with (33) and (34), we obtain

$$\lambda_k = (2-k)m_k g. \quad (35)$$

These values express the binding forces applied to both bodies. By eliminating λ_3 and λ_4 between Esq. (33) and (34), we obtain:

$$\begin{pmatrix} m_1\ddot{x}_1 \\ I_2\ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ b\cos\theta_2 & b\sin\theta_2 \end{pmatrix}\begin{pmatrix} m_2\ddot{x}_2 \\ m_2\ddot{y}_2 \end{pmatrix} = \begin{pmatrix} F \\ T - m_2 g b\sin\theta_2 \end{pmatrix} \quad (36)$$

By drifting twice the constraints (23) and (24), we obtain:

$$\begin{pmatrix} m_2\ddot{x}_2 \\ m_2\ddot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & b\cos\theta_2 \\ 0 & b\sin\theta_2 \end{pmatrix}\begin{pmatrix} m_2\ddot{x}_1 \\ m_2\ddot{\theta}_2 \end{pmatrix} + m_2 b\dot{\theta}_2^2\begin{pmatrix} -\sin\theta_2 \\ \cos\theta_2 \end{pmatrix} \quad (37)$$

Substituting (37) in (36), we finally obtain the model of the system in the desired form:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u, \quad (38)$$

With

$$M(q) = \begin{pmatrix} m_1 + m_2 & m_2 b\cos\theta_2 \\ m_2 b\cos\theta_2 & I_2 + m_2 b^2 \end{pmatrix}, \quad (39)$$

$$C(q,\dot{q}) = \begin{pmatrix} 0 & -m_2 b\dot{\theta}_2\sin\theta_2 \\ 0 & 0 \end{pmatrix}, \quad (40)$$

$$g(q) = \begin{pmatrix} 0 \\ m_2 g b\sin\theta_2 \end{pmatrix}, \qquad u = \begin{pmatrix} F \\ T \end{pmatrix}. \quad (41)$$

The Eq. (33) of motion in Lagrangian formalism are second order. and present a dissymmetry between the generalized coordinates q_k and their generalized velocity $\dot{q}_k$.

B. Hamiltonian formulation

Introduce the function of Hamilton, defined by (15):

$$H_s(x_k, y_k, \theta_k P_{x_k}, P_{y_k}, P_{\theta_k}, t) = \sum_{k=1}^{2}\left[P_{x_k}\dot{x}_k + P_{y_k}\dot{y}_k + P_{\theta_k}\dot{\theta}_k\right] - \overline{L}_s. \quad (42)$$

Express $P_k = \partial\overline{L}_s/\partial\dot{q}_k$ and inserting the result in to equation (42), we get

$$H_s = \sum_{k=1}^{2}\left[\frac{P_{x_k}^2 + P_{y_k}^2}{2m_k} + \frac{P_{\theta_k}^2}{2I_k} + m_k g y_k\right] - \sum_{j=1}^{4} \lambda_j f_j(x_k, y_k, \theta_k, t) \quad (43)$$

And the Hamilton equations of the system are given below:

$$\dot{x}_k = \frac{\partial H_s}{\partial P_{x_k}} = \frac{P_{x_k}}{m_k}, \quad \dot{y}_k = \frac{\partial H_s}{\partial P_{y_1}} = \frac{P_{y_k}}{m_k}, \tag{44}$$

$$\dot{\theta}_k = \frac{\partial H_s}{\partial P_{\theta_k}} = \frac{P_{\theta_k}}{I_k}, \quad \dot{P}_{x_k} = -\frac{\partial H_s}{\partial x_k} = (-1)^k \lambda_3 \tag{45}$$

$$, \quad \dot{P}_{y_k} = -\frac{\partial H_s}{\partial y_k} = -m_k g + \lambda_{k^2} \tag{46}$$

$$\dot{P}_{\theta_k} = -\frac{\partial H_s}{\partial \theta_k} = (-b)^{k-1}\big[\lambda_{k+1}\cos\big((k-1)\theta_k\big) + \lambda_{2k}\sin\big((k-1)\theta_k\big)\big]. \tag{47}$$

The equations of motion in Hamiltonian formalism are first-order with respect to time and without asymmetry between the coordinates q_k and conjugate moment P_k

V. CONCLUSION

Lagrangian formalism has the particularity to be built from scalar quantities (energies) and to lead to invariant form motion equations during a change of generalized coordinates. Despite this advantage, this formalism suffers from a major disadvantage, namely that the Lagrangian is not an intrinsic quantity, in the sense that it is defined to an arbitrary close. More precisely, if we add to the Lagrangian the quantity $\dot{q}_k f(q_k)$, the Euler Lagrange equations remain unchanged. In addition, the Lagrange equations are second-order and present a dissymmetry between the generalized coordinates and their generalized velocity. By else, where Hamilton's formalism makes it possible to obtain an equations system of first order and without a symmetry between the coordinates.

REFERENCES

[1] H Goldstein, Mécanique classique, (Presses Universitaires de France, 1964).

[2] L. Landau, E. Lifchitz, Physique Théorique, Tome-1 Mécanique, (Editions Mir Moscou, 1969).

[3] J. S. Torok, Analytical mechanics with an introduction to dynamics system, (A Wiley-lnterscience Publication, 2000).

[4] B.A. Kuperschmidt, Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalism, in: Geometric Methods in Mathematical Physics, Lecture Notes in Math. 775 (Springer, Berlin, 1980) 162-218.

[5] A. Fasano, S. Marmi, Analytical mechanics, (Oxford Press,2002).

[6] B. Siciliano, O. Khalid, Handbook of robotic, (Springer, 2008)