ECHAHID HAMMA LAKHDAR UNIVERSITY - EL-OUED
Under the Supervision of the DGRSDT and in collaboration with the CRTI
International Pluridisciplinary PhD Meeting (IPPM'20)
1st Edition, February23-26, 2020
Theme: Modern Technology and Fineness Life

Fixed point in uniform spaces

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Operator theory and EDP: foundations and applications

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Abstract

In this work, we expand some definitions of functional analysis usually defined on metric spaces to non-metric topologiqual spaces. To do so, we introduce some notions associated with non-metrizable spaces, wich extanded the topological structures of these spaces, for make them fairly close of those of the metric spaces. It concerns some topological vector spaces of uniform structure; particularly those generated by family of semi-norms. Specially, we study the extension of the Banach contraction principle on such spaces.

Key words

Vectorial topological space, uniforme space, contraction principle, fixed point theorem.

Introduction

The fixed point theory is one of the most important in analysis, which is the subject of numerous research works. Indeed, it offers a very effective set of tools in the study of the problems formulated by systems of differential or integral equations, resulting from the modeling of various problems of physics, chemistry and biology, ... One of the most famous results in this theory is the Banach contraction principle, which can only have meaning in a metric space.
In this field, several research works have been devoted to the generalization of this principle in a topological vector space devoid of metric structure. Our aim in this study is to contribute to these generalization efforts.

Study tools

let $E = C(\mathbb{R})$, $X \subset E$, $A = \{\text{the compacts of } \mathbb{R}\}$, $0 \in H \subseteq \mathbb{R}$.
We introduce a generalization of notions, wich defined in [4].

Seminorms on E

The assumptions on the family of seminorms $\mathfrak{p} = \{p_\alpha : \alpha \in A\}$ defined on E such as

$$p_\alpha(f) = \sup\{e^{-\lambda t}|f(t)| : t \in \alpha\}, \qquad \alpha \in A$$

1. $\forall h \in H$, $\forall \alpha \in A$

$$\begin{cases} p_\alpha(Tx - Ty) \leq K_\alpha.\dfrac{[h]}{h}.p_{j(\alpha+h)}(x-y) & \text{si} \quad h > 1, \\ p_\alpha(Tx - Ty) \leq K_\alpha.h.p_{j(\alpha+h)}(x-y) & \text{si} \quad 0 < h \leq 1, \\ p_\alpha(Tx - Ty) \leq K_\alpha.p_{j(\alpha)}(x-y) & \text{si} \quad h = 0, \\ p_\alpha(Tx - Ty) \leq K_\alpha.\dfrac{h}{[h]}.p_{j(\alpha+h)}(x-y) & \text{si} \quad h < 0. \end{cases}$$

Where j is an application defined on subset $S \subset A$ such as

$$\begin{array}{rccc} j & : & S & \longrightarrow & S \\ & & k & \longrightarrow & j(k) \end{array}$$

with: $\forall h \in H$, then $k + h \in S$ and equality $j(k+h) = j(k) + j(h)$ is true.

2. $c_\alpha(h) = \sup\{p_{(j^n(\alpha)+\sum_{s=1}^{n} j^s(h))}(x-y), n = 1, 2, 3, ...\} < \infty$

3. For all $h \in H \setminus \{0\}$, we notice:

$$M_h = \min\left\{\frac{[h]}{h}, \frac{h}{[h]}, |h|\right\}$$

Such as $0 < M_h < 1$.
And

$$h_\alpha \doteqdot \inf\left\{h \in \mathbb{R}^* : p_\alpha(Tx - Ty) \leq K_\alpha.M_h.p_{j(\alpha+h)}(x-y)\right\}$$

Under the above hypotheses, we propose to proof the following result.

Theorem 1. *Let X a j-bounded set and $T : X \to E$ is a contractive function such as:*

(a) $\sum_{n=0}^{\infty} 2^{2n} K_\alpha.K_{j(\alpha)}...K_{j^n(\alpha)} < \infty$ for all $\alpha \in A$;

(b) $\forall x \in \partial X$, $Tx \in X$. Then, T has a unique fixed point in X.

Proof. We will demonstrate this theorem, based on the idea in article [2, 4]. Which uses an iterative diagram, and the principle of convergence of Cauchy sequences. □

Application

We will try to apply the previous result to a problem of initial value, whose equations are a mathematical model, which represent many technological processes.
This problems can be formulated as follows:

$$\begin{aligned} \dot{x}(t) &= F\left(\max_{0\leq t\leq a} x(t), \min_{0\leq t\leq a} x(t), x(U_1(t)), ..., x(U_m(t))\right), \qquad t > 0 \\ & \quad \dot{x}(t) = \varphi(t), \qquad t \leq 0 \end{aligned} \tag{1}$$

we put

$$(Tf)(t) = \begin{cases} b + \displaystyle\int_0^t F(\max_{\tau\in[0,a]} f(\tau), \min_{\tau\in[0,a]} f(\tau), f(U_1(\tau)), ..., f(U_m(\tau)))d\tau, & t \geq 0; \\ \varphi(t), & t \leq 0. \end{cases}$$

has a unique fixed point.

Conclusion

Our work is devoted to the extension of the Banach contraction principle on spaces with out a distance. In particular those with uniform structures induced by a family of semi-norms.
This type of result is very useful in applied functional analysis, which is the case of the most problems (of differential equations or integral equations) defined in non-metric topological spaces.

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